HÖLDER CONTINUITY FOR SOLUTIONS OF CERTAIN DEGENERATE PARABOLIC SYSTEMS

By

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1. Introduction. We consider the parabolic system

\[ u_t - \text{div} \left( |\nabla u|^{p-2} \nabla u \right) + b(x, u, \nabla u) = 0 \]

where \( u = u(x, t); \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^N, \nabla = \text{grad}_x \) and \( x \) varies in an open set \( \Omega \subset \mathbb{R}^N \). Naturally we assume \( 1 < p < \infty \). We assume that \( b(x, t, u, Q) \in C^1(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^N \times M^{Nn} \to \mathbb{R}^N) \) satisfies the following controllable growth condition

\[ |b_x(x, t, u, Q)| + |b_u(x, t, u, Q)| |Q| + |b_{Q_i}(x, t, u, Q)Q^i_\alpha| \leq c(1 + |Q|^{p-1}) \]

for all \( x \in \mathbb{R}^n, t \in \mathbb{R}, u \in \mathbb{R}^N \) and \( Q \in M^{Nn} \), and for some \( c \).

Suppose that \( u \) is a solution to (1) which means

\[ u \in C^0[0, T; L^2(\Omega \to \mathbb{R}^N)] \cap L^p[0, T; W^{1, p}(\Omega \to \mathbb{R}^N)] \]

and \( u \) satisfies

\[ \int_{\Omega_T} -u\phi_t + |\nabla u|^{p-2} \nabla u \cdot \nabla \phi + b(z, u, \nabla u)\phi dz = 0 \]

for all \( \phi \in C_0^\infty(\Omega_T \to \mathbb{R}^N) \) where \( z = (x, t) \) and \( \Omega_T = \Omega \times (0, T) \).

The main result in this paper is the following theorem.

**Theorem 1.** Suppose that \( u \in L^{r_0}_{\text{loc}}(\Omega_T), r_0 > \frac{n(2-p)}{p} \) then

\[ u \in C^0_{\text{loc}}(\Omega_T) \]

for some \( \alpha > 0 \).

Note that from the definition of solution \( u \in L^2_{\text{loc}}(\Omega_T) \). Thus the requirement that \( r_0 > \frac{n(2-p)}{p} \) becomes restrictive only when \( \frac{n(2-p)}{p} \geq 2 \), i.e., when \( p \leq \frac{2n}{n+2} \).

When \( p \geq 2 \), E. DiBenedetto [5] proved that solutions for equations (1) with a natural growth condition on \( b \) are Hölder continuous. Essentially his proof lies on the truncation idea of DeGiorgi and a scaling and it seems not applicable to systems. In case \( 1 < p < 2 \) E. DiBenedetto and C. Ya-zhe[10] proved Hölder continuity of solutions for equations. Also

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when \( p > \frac{2n}{n+2} \), E. DiBenedetto and C. Ya-zhe[11] proved H"older continuity for solutions of parabolic systems up to boundary.

On the other hand, E. DiBenedetto and A. Friedman[6] proved that \( \nabla u \in C^\alpha_{loc} \) for homogeneous systems when \( p \geq \frac{2n}{n+2} \). Independently, M. Wiegner[9] proved that \( \nabla u \in C^\alpha_{loc} \) when \( p \geq 2 \). When \( 1 < p < 2 \), H. Choe[1] proved that \( \nabla u \in C^\alpha_{loc} \) for homogeneous systems.

Here we prove an inequality of Poincaré type. Once we have a Poincaré inequality, a Campanato type growth estimate for \( u \) follows from the \( L^\infty \) estimate of \( \nabla u \). Thus theorem 1 follows from the isomorphism theorem of Da Prato [4].

We assume \( u, u_t, \nabla u, \nabla^2 u \) belong to a suitable \( L^q \) space. Justification for this appears in [6] when \( p \geq 2 \) and in [1] when \( 1 < p < 2 \).

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2. \( L^\infty \) bound for \( \nabla u \). In this section we prove that \( u \) is bounded by following Moser iteration idea and we construct a weak Harnack inequality for \( |u| \). In this section we define a cylinder \( Q_R \) by \( Q_R = \{(x, t); |x - x_0| < R, \ t_0 - R^p < t < t_0\} \) where \( (x_0, t_0) \in \mathbb{R}^n \times \mathbb{R} \) is a generic point. Also we define \( \Lambda_R = \{t; t_0 - R^p < t < t_0\} \). A similar theorem appears in [1].

**Theorem 2.** Suppose that \( u \in L^r_{loc}(\Omega_T) \), then \( u \in L^\infty_{loc}(\Omega_T) \), where \( r_0 > \frac{n(2-p)}{p} \) when \( 1 < p < 2 \) and \( r_0 = p \) when \( p \geq 2 \). Moreover we have that for all \( Q_{2R_0} \subset \Omega_T \)

\[
\sup_{Q_{R_0}/2} |u| \leq c \left[ \left( \int_{Q_{R_0}} |u|^{r_0} \, dz \right)^{\frac{1}{r_1}} + 1 \right]
\]

for some constant \( c \) depending only on \( n, N \) and \( p \), where \( r_1 = r_0 - \frac{N}{p}(2-p) \) when \( 1 < p < 2 \) and \( r_1 = 2 \) when \( p \geq 2 \).

**Proof.** Since \( u \in L^r_{loc}(\Omega_T) \), we can assume \( r_0 \geq 2 \). Let \( \rho < R \) and \( \psi \) be the standard cutoff function such that

\[
\psi = 1 \quad \text{in} \quad Q_\rho
\]
\[
\psi = 0 \quad \text{in a neighborhood of parabolic boundary of} \quad Q_R
\]
\[
0 \leq \psi \leq 1, \quad |\psi_t| \leq \frac{c}{(R - \rho)^p}, \quad |\nabla \psi| \leq \frac{c}{R - \rho}.
\]

Let \( \alpha_0 = r_0 - 2 \). We apply \( u|u|^\alpha \psi^p \) as a test function to (1) where \( \alpha \geq \alpha_0 \geq 0 \). So we have

\[
\sup_t \int |u|^{\alpha + 2} \psi^p \, dx + (\alpha + 1)^{2-p} \int \left| \nabla \left( \psi |u|^{\frac{\alpha + p}{p}} \right) \right|^p \, dz
\]

\[
\leq c \int |u|^{\alpha + 2} \psi^{p-1} |\psi_t| \, dz + c(\alpha + 1)^{2-p} \int |u|^{\alpha + p} |\nabla \psi|^p \, dz
\]

\[
+ c \int (1 + |\nabla u|)^{p-1} |u|^{\alpha + 1} \psi^p \, dz.
\]
Using Young’s inequality on (3) we have

\[
\sup_t \int |u|^{\alpha+2} v^p dx + \int |\nabla(\psi |u|^{\frac{a+p}{p}})|^p dz
\leq \frac{c}{(R - \rho)^p} \int_{Q_R} |u|^{\alpha+2} v^p dz + \frac{c}{(R - \rho)^p} \int_{Q_R} |u|^{\alpha+p} dz + c|Q_R|.
\]

(4)

We assume \(2 \leq p\). By Hölder’s inequality and Sobolev’s inequality on (4) we have

\[
\sup_{t_0 - \rho^p \leq t \leq t_0} \int_{B_{\rho}} |u|^{\alpha+2} dx + \int_{\Lambda_{\rho}} \left[ \int_{B_{\rho}} |u|^{(\alpha+p)\frac{n}{N-p}} dx \right]^\frac{n-p}{n} dt
\leq \frac{c}{(R - \rho)^p} \int_{Q_R} |u|^{\alpha+p} dz + c \frac{|Q_R|}{(R - \rho)^p}.
\]

(5)

Hence using (5) and Hölder inequality we have

\[
\int_{Q_{\rho}} |u|^{(\alpha+p)(1+\frac{\rho+2}{N})} dz \leq \left[ \sup_t \int_{B_{\rho}} |u|^{\alpha+2} dx \right]^\frac{\rho}{n} \left[ \int_{\Lambda_{\rho}} \left( \int_{B_{\rho}} |u|^{(\alpha+p)\frac{n}{N-p}} dx \right)^\frac{n-p}{n} dt \right]^{1+\frac{\rho}{n}}
\]

\[
\leq c \left[ \frac{1}{(R - \rho)^p} \int_{Q_R} |u|^{\alpha+p} dz + \frac{|Q_R|}{(R - \rho)^p} \right]^{1+\frac{\rho}{n}}.
\]

(6)

We define \(\rho_{\nu} = \frac{R}{2}(1 + 2^{\nu}),\ \nu = 0, 1, 2, \cdots\), and \(Q_{\nu} = Q_{\rho_{\nu}}\). We now define \(\alpha_{\nu}\) by

\[
\alpha_{\nu+1} = \left(1 + \frac{p}{n}\right) \alpha_{\nu} + \frac{2p}{n}, \quad \alpha_0 = 0.
\]

Then we see that \(\alpha_{\nu} = 2(\theta^\nu - 1)\) where \(\theta = 1 + \frac{p}{n}\). Also note that

\[
\lim_{\nu \to \infty} \frac{\theta^\nu}{\alpha_{\nu} + p} = \frac{1}{2}.
\]

We define \(\phi_{\nu}\) by

\[
\phi_{\nu} = \int_{Q_{\nu}} u^{\alpha_{\nu} + p} dz,
\]

then we can write (6) as

\[
\phi_{\nu+1} \leq c^{\nu} \phi_{\nu}^\theta + c^\nu.
\]

(7)
where \( c \) depends only on \( n, N \) and \( p \). Iterating (7) we prove theorem 2 when \( p \geq 2 \).

Now we assume \( 1 < p < 2 \). From (4) we get

\[
\sup_t \int_{\hat{B}_p} |u|^{\alpha_2 + 2} \, dx + \int \left[ \int_{\hat{B}_p} |u|^{(\alpha + p) \frac{n}{n-p}} \, dx \right] \frac{n-p}{n} \, dt
\]

\[
\leq \frac{c}{(R - \rho)^p} \int_{Q_R} |u|^{\alpha+2} \, dz + c \frac{|Q_R|}{(R - \rho)^p}
\]

and

\[
\int_{Q_R} |u|^{(\alpha+p)(1+\frac{2}{n})} \, dz \leq \left[ \sup_t \int_{\hat{B}_p} |u|^{\alpha_2 + 2} \, dx \right] \frac{\rho}{n} \left[ \int \left( \int_{\hat{B}_p} |u|^{(\alpha + p) \frac{n}{n-p}} \, dx \right) \frac{n-p}{n} \, dt \right]
\]

\[
\leq c \left[ \frac{1}{(R - \rho)^p} \int_{Q_R} |u|^{\alpha+2} \, dz + c \frac{|Q_R|}{(R - \rho)^p} \right]^{1+\frac{2}{n}}
\]

Defining \( \alpha_\nu \) by

\[
\alpha_{\nu+1} + 2 = \alpha_\nu \theta + p + \frac{2p}{n}, \quad \alpha_0 = r_0 - 2,
\]

we can have

\[
\phi_{\nu+1} \leq c^\nu \phi_\nu + c^\nu
\]

for some constant \( c \) depending only on \( n, N \) and \( p \). We note that

\[
\lim_{\nu \to \infty} \frac{\alpha_\nu + 2}{\theta^\nu} = r_0 - \frac{n}{p}(2-p).
\]

Iterating (10) we prove theorem 2 when \( 1 < p < 2 \). \( \Box \)

We define a new cylinder \( S_R \) by \( S_R = B_R(x_0) \times (t_0 - R^2, t_0) \). We denote \( \lambda = \sup_{S_R} |u| \).

Now we recall the following theorem due to Choe (See Theorem 4 in [1]) which proves that

\( \nabla u \in L^\infty_{\text{loc}} \) for \( 1 < p < 2 \).

**Theorem 3.** Suppose \( S_{2R_0} \subset \Omega_T \) and \( 1 < p < 2 \). Then there exists a constant \( c \) depending only on \( \lambda, p, q_0, N \) and \( n \) such that

\[
\sup_{S_{R_0/2}} |\nabla u| \leq c \left( \left[ \int_{S_{R_0}} |\nabla u|^{q_0} \, dz \right]^{\frac{1}{q_0}} + 1 \right)
\]
where \( q_0 > \frac{n}{2}(2 - p) \) and \( q_1 = \frac{2}{2q_0 - n(2 - p)} \).

We note that theorem 3 implies Proposition 3.1' in [11]. Suppose that
\[
p > \frac{2n}{n + 2}
\]
then we see
\[
p > \frac{n}{2}(2 - p)
\]
and we can take \( q_0 = p \) and \( q_1 = \frac{2}{2p + n(p - 2)} \) in theorem 3. Consequently we have the following Corollary.

**Corollary.** Suppose \( S_{2R_0} \subset \Omega_T \) and \( \frac{2n}{n+2} < p < 2 \). Then there exists a constant \( c \) depending only on \( n, N \) and \( p \) such that
\[
\sup_{S_{R_0/2}} |\nabla u| \leq c \left( \int_{S_{R_0}} |\nabla u|^p dz \right)^{\frac{1}{p}} + 1.
\]

Now we consider the case \( p \geq 2 \). Again by following Moser iteration we prove a weak Harnack inequality for \( \nabla u \). In this case we recall that E. DiBenedetto and C. Ya-zhe proved a similar theorem using a more complicated iteration (see Proposition 3.1 in [11]).

**Theorem 4.** Suppose \( S_{2R_0} \subset \Omega_T \) and \( p \geq 2 \). Then there exists a constant \( c \) depending only on \( n, N \) and \( p \) such that
\[
\sup_{S_{R_0/2}} |\nabla u| \leq c \left( \int_{S_{R_0}} |\nabla u|^p dz \right)^{\frac{1}{2}} + 1.
\]

**Proof.** Differentiating (1) with respect to \( x_\gamma \) we have
\[
(u^i_{x_\gamma})_t - (a^{ij}_{\alpha\beta} |\nabla u|^{p-2} u^j_{x_\gamma x_\alpha})_{x_\beta} + b^j_{x_\gamma} (x, t, u, \nabla u)
+ b^i_{u_j} (z, u, \nabla u) u^j_{x_\alpha} + b^i_{Q_j} (x, t, u, \nabla u) u^j_{x_\alpha x_\gamma} = 0,
\]
where
\[
a^{ij}_{\alpha\beta} = \delta^{ij} \delta_{\alpha\beta} + (p - 2) \frac{u^i_{x_\beta} u^j_{x_\alpha}}{|\nabla u|^2}
\]
and \( \delta \) is the Kronecker delta function. We introduce a new cutoff function \( \psi \) such that
\[
\psi = 1 \text{ in } S_\rho
\]
\[
\psi = 0 \text{ in a neighborhood of parabolic boundary of } S_R
\]
\[
0 \leq \psi \leq 1, |\psi_t| \leq \frac{c}{(R - \rho)^2}, |\nabla \psi| \leq \frac{c}{R - \rho}.
\]
Taking $\phi = u, |\nabla u|^\alpha \psi^2$, $\alpha \geq 0$ as a test function to (14) we get

$$\sup_t \int |\nabla u|^{\alpha+2} \psi^2 \, dx + \int |\nabla (|\nabla u|^\frac{\alpha+p}{2} \psi)|^2 \, dz$$

$$\leq c \int |\nabla u|^{\alpha+2} \psi \psi_t \, dz + c \int |\nabla u|^{\alpha+p} |\nabla \psi|^2 \, dz. \quad (15)$$

$$+ c \int (1 + |\nabla u|^{p+\alpha}) \psi^2 \, dz.$$

By Hölder’s inequality and Sobolev’s inequality we have

$$\int_{S_\rho} |\nabla u|^{\alpha+p} (1+\frac{2}{n} \frac{\alpha+p}{n}) \, dz$$

$$\leq c \left[ \sup_t \int |\nabla u|^{\alpha+2} \psi^2 \, dx \right]^{\frac{2}{n}} \left[ \int |\nabla (|\nabla u|^\frac{\alpha+p}{2} \psi)|^2 \, dz \right]$$

$$\leq c \left[ \frac{1}{(R-\rho)^2} \int_{S_R} |\nabla u|^{\alpha+p} \, dz + \frac{|S_R|}{(R-\rho)^2} \right]^{1+\frac{2}{n}}. \quad (16)$$

We set $\rho_\nu = (1 + 2^{-\nu}) \frac{R_\nu}{2}$, $\rho = \rho_{\nu+1}$ and $R = \rho_\nu$. We define $\mu = 1 + \frac{2}{n}$. Also we define $\alpha_\nu$ by

$$\alpha_{\nu+1} + p = \mu \alpha_\nu + p + \frac{4}{n}, \quad \alpha_0 = 0.$$

If we define

$$\Phi(\nu) = \int_{S_{\rho_\nu}} |\nabla u|^{\alpha_\nu+p} \, dz,$$

then (16) can be written as follows

$$\Phi(\nu + 1) \leq c^\nu \Phi(\nu) + c^\nu \quad (17)$$

for some $c$ depending only on $n$ and $p$. We note that

$$\lim_{\nu \to \infty} \frac{\mu^\nu}{\alpha_\nu + p} = \frac{1}{2}.$$

So iterating (17), we prove theorem 4. \[\square\]

3. Hölder continuity of $u$. In this section we define $\Lambda^-_R = (t_0 - 2R, t_0 - R)$ and $Q^-_R = B_R \times \Lambda^-_R$. We introduce a cutoff function $\eta \in C_0^\infty(B_R)$ such that

$$\eta = 1 \text{ in } B_{\frac{R}{2}}$$

$$0 \leq \eta \leq 1, |\nabla \eta| \leq \frac{c}{R}.$$
Also we define

\[ u_{R,t} = \frac{1}{|B_R|} \int_{B_R} u(x,t) \, dx \]

and

\[ u_R = \frac{1}{|Q_R|} \int_{Q_R} u \, dz. \]

First we prove a lemma which is essential for a Poincaré inequality for solutions of a degenerate parabolic system.

**Lemma 1.** Suppose \( Q_{2R} \subset \Omega_T \), then \( u \) satisfies the following inequality

\[
\sup_{t \in \Lambda_R} \int_{B_R} ds \int_{\Lambda_R} \eta^2 |u(x,t) - u_{R,s}|^2 \, dx \\
\leq cR^{s_1} \int_{Q_{2R}} |\nabla u|^p \, dz + c \int_{Q_R} |u - u_{R,t}|^2 \, dz + cR^{s_2}.
\]

for all \( R < R_0 \) where \( c \) depends only on \( n, N \) and \( p \), and

- \( s_1 = p \) and \( s_2 = n + 2p \) when \( p \geq 2 \)
- \( s_1 = p(p - 1) \) and \( s_2 = n + p + p(p - 1) \) when \( 1 < p < 2 \).

**Proof.** Since \( u \in C^0[0,T; L^2(\Omega)] \), there exists \( \bar{t}(s) \in \Lambda_R \) for all \( s \in \Lambda_R \) such that

\[
\int_{B_R} \eta^p |u(x,\bar{t}) - u_{R,s}|^2 \, dx = \sup_{0 \leq t \leq s} \int_{B_R} \eta^p |u(x,t) - u_{R,s}|^2 \, dx.
\]

We take \((u-u_{R,s})\eta^p X_{[s,\bar{t}]}\) as a test function to (1) where \( X_{[x,\bar{t}]} = \mathbb{R} \rightarrow \mathbb{R} \) is the characteristic function such that \( X_{[s,\bar{t}]}(s) = 1 \) for all \( s \in [s,\bar{t}] \) and \( X_{[s,\bar{t}]}(s) = 0 \) for all \( s \not\in [s,\bar{t}] \). Hence we have

\[
\int u_t \cdot (u - u_{R,s})\eta^p (x)X_{[s,\bar{t}]} \, dz \\
+ \int |\nabla u|^{p-2} \nabla u \cdot \nabla ((u - u_{R,s})\eta^p)X_{[s,\bar{t}]} \, dz \\
+ \int b(x,u,\nabla u)(u - u_{R,s})\eta^p X_{[s,\bar{t}]} \, dz = 0.
\]
Now we assume $p \in [2, \infty)$. Using Young's inequality and the structure condition on $b$ we have

$$
\int |u - u_{R,s}|^2 \eta^p(x, \bar{t}) dx + \int_{B_R \times [s, \bar{t}]} |\nabla u|^p \eta^p dz \\
\leq c \int |u - u_{R,s}|^2 \eta^p(x, s) dx + \frac{\varepsilon}{R^p} \int_{B_R \times [s, \bar{t}]} |u - u_{R,s}|^p \eta^p dz \\
+ c \int_{B_R \times [s, \bar{t}]} |\nabla u|^p dz + cR^{n+p}
$$

(20)

for some $c$ independent of $R$. Integrating (20) with respect to $s$ from $t_0 - 2R^p$ to $t_0 - R^p$, we have

$$
\int_{\Lambda_R} ds \int_{B_R} |u - u_{R,s}|^2 \eta^p(x, \bar{t}) dx \\
\leq c \int_{Q_R} |u - u_{R,t}|^2 \eta^p dz + \frac{c\varepsilon}{R^p} \int_{\Lambda_R} ds \int_{B_R \times [s, \bar{t}]} |u - u_{R,s}|^p \eta^p dz \\
+ cR^p \int_{Q_{2R}} |\nabla u|^p dz + cR^{n+2p}.
$$

(21)

By the choice of $\bar{t}$ we have that for small $\varepsilon$

$$
\int_{\Lambda_R} ds \int_{B_R} |u - u_{R,s}|^2 \eta^p(x, \bar{t}) dx \\
\leq c \int_{Q_R} |u - u_{R,t}|^2 \eta^p dz + cR^p \int_{Q_{2R}} |\nabla u|^p dz + cR^{n+2p}.
$$

(22)

So we have lemma 2 when $p \geq 2$.

In case $p \in (1, 2)$ we estimate the second term of (19) as follows

$$
\int |\nabla u|^p - 2 \nabla u \cdot \nabla ((u - u_{R,s}) \eta^p) X_{[s, \bar{t}]} dz \\
= \int |\nabla u|^p \eta^p X_{[s, \bar{t}]} dz + p \int |\nabla u|^{p-2} u^i_\alpha \eta \eta_{\alpha} (u^i - u^i_{R,s}) \eta^{p-1} X_{[s, \bar{t}]} dz
$$

(23)
and using the fact that \( p(p - 1) < p \) and \( \frac{p}{p-1} > 2 \), we have

\[
\left| \int \nabla_u |^{p-2} u^i_{x_\alpha} \eta_{x_\alpha} (u^i - u^i_{R,s}) \eta^{p-1}_\| x_{[s,i]} dz \right| \\
\leq \int \nabla u |^{p-1} |\nabla \eta|^{p-1} |u - u_{R,s}| X_{[s,i]} dz \\
\leq \frac{\varepsilon}{R^p} \int |u - u_{R,s}|^{p-1} \eta^{p} X_{[s,i]} dz + \frac{c}{R^{p(2-p)}} \int |\nabla u|^{p(p-1)} dz \\
\leq \frac{c\varepsilon}{R^p} \int |u - u_{R,s}|^{2} \eta^{p} X_{[s,i]} dz + \frac{c}{R^{p(2-p)}} \int_{B_R \times [s,i]} |\nabla u|^{p} dz + cR^{N+p(p-1)}. \tag{24}
\]

Applying Young's inequality and the structure condition of \( b \) on (19) and combining (19), (23) and (24) we have

\[
\int_{B_R} |u - u_{R,s}|^{2} \eta^{p}(x, \bar{t}) dz \\
\leq \frac{\varepsilon}{R^p} \int |u - u_{R,s}|^{2} \eta^{p} X_{[s,i]} dz + \frac{c}{R^{p(2-p)}} \int_{B_R \times [s,i]} |\nabla u|^{p} dz \\
+ cR^{n+p(p-1)} + c \int_{B_R} |u - u_{R,s}|^{2} \eta^{p}(x, s) dx. \tag{25}
\]

Integrating (25) with respect to \( s \) and using the choice of \( \bar{t} \) we prove lemma 2 when \( 1 < p < 2. \)

Now we prove a Poincaré’s inequality.

**THEOREM 4.** Suppose \( Q_{2R} \subset \Omega_T \), then we have

\[
\int_{Q_{2R}} |u - u_{R}|^{2} dz \leq cR^2 \int_{Q_{2R}} |\nabla u|^{2} dz \\
+ cR^{s_1} \int_{Q_{2R}} |\nabla u|^{p} dz + cR^{s_2}, \tag{26}
\]

where \( c \) is independent of \( R \).
Proof. Using lemma 2 we have

\[ \int_{Q_R} |u - u_{R \hat{z}}|^2 dz \leq c \int_{Q_R} |u - u_{R,s}|^2 \eta \eta^p dx, \]

\[ \leq \frac{c}{R^p} \int_{Q_R} ds \int_{Q_R} |u - u_{R,s}|^2 \eta \eta^p dx, \]

\[ \leq c \frac{1}{\text{sup}_{t \in \Lambda_R} \Lambda_R} \int_{Q_R} ds \int_{Q_R} |u - u_{R,s}|^2 \eta \eta^p(x, \bar{t}) dx, \]

\[ \leq c \int_{Q_R} |u - u_{R,t}|^2 \eta \eta^p dx + c \int_{Q_R} \nabla u \eta^p dx + c \int_{Q_R} \eta^p dx, \]

\[ \leq c R^2 \int_{Q_R} |\nabla u|^2 dx + c R^p \int_{Q_R} \eta^p dx + R^p \]

(27)

where we used a Poincaré type inequality for \( x \) variables only as follows

\[ \int_{Q_R} |u - u_{R,t}|^2 \eta^p dx \leq c R^2 \int_{Q_R} |\nabla u|^2 dx. \]

Proof of Theorem 1. Since \( \nabla u \) is bounded, we have from theorem 4

\[ \int_{Q_R} |u - u_R|^2 dx \leq c R^{n+p+2} \]

when \( p \in [2, \infty) \)

\[ \leq c R^{n+p+(p-1)} \]

when \( p \in (1,2) \)

where \( c \) is independent of \( R \). Hence by isomorphism theorem of Da Prato[4] we prove theorem 1. \( \square \)
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<table>
<thead>
<tr>
<th>#</th>
<th>Author(s)</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>637</td>
<td>Xinfu Chen</td>
<td>Generation and propagation of the interface for reaction–diffusion equations</td>
</tr>
<tr>
<td>638</td>
<td>Philip Korman</td>
<td>Dynamics of the Lotka–Volterra systems with diffusion</td>
</tr>
<tr>
<td>639</td>
<td>Harlan W. Stech</td>
<td>Generic Hopf bifurcation in a class of integro-differential equations</td>
</tr>
<tr>
<td>640</td>
<td>Stephane Laederich</td>
<td>Periodic solutions of non-linear differential difference equations</td>
</tr>
<tr>
<td>641</td>
<td>Peter J. Olver</td>
<td>Canonical Forms and Integrability of BiHamiltonian Systems</td>
</tr>
<tr>
<td>642</td>
<td>S.A. van Gils, M.P. Krupa and W.F. Langford</td>
<td>Hopf bifurcation with non-semisimple 1:1 Resonance</td>
</tr>
<tr>
<td>643</td>
<td>R.D. James and D. Kinderlehrer</td>
<td>Frustration in ferromagnetic materials</td>
</tr>
<tr>
<td>644</td>
<td>Carlos Rocha</td>
<td>Properties of the attractor of a scalar parabolic P.D.E.</td>
</tr>
<tr>
<td>645</td>
<td>Debra Lewis</td>
<td>Lagrangian block diagonalization</td>
</tr>
<tr>
<td>646</td>
<td>Richard C. Churchill and David L. Rod</td>
<td>On the determination of Ziglin monodromy groups</td>
</tr>
<tr>
<td>647</td>
<td>Xinfu Chen and Avner Friedman</td>
<td>A nonlocal diffusion equation arising in terminally attached polymer chains</td>
</tr>
<tr>
<td>648</td>
<td>Peter Gritzmann and Victor Klee</td>
<td>Inner and outer j- Radii of convex bodies in finite-dimensional normed spaces</td>
</tr>
<tr>
<td>649</td>
<td>P. Szmolyan</td>
<td>Analysis of a singularly perturbed traveling wave problem</td>
</tr>
<tr>
<td>650</td>
<td>Stanley Reiter and Carl P. Simon</td>
<td>Decentralized dynamic processes for finding equilibrium</td>
</tr>
<tr>
<td>651</td>
<td>Fernando Reitich</td>
<td>Singular solutions of a transmission problem in plane linear elasticity for wedge-shaped regions</td>
</tr>
<tr>
<td>652</td>
<td>Russell A. Johnson</td>
<td>Cantor spectrum for the quasi-periodic Schrödinger equation</td>
</tr>
<tr>
<td>653</td>
<td>Wenhong Liu</td>
<td>Singular solutions for a convection diffusion equation with absorption</td>
</tr>
<tr>
<td>654</td>
<td>Deborah Brandon and William J. Hrusa</td>
<td>Global existence of smooth shearing motions of a nonlinear viscoelastic fluid</td>
</tr>
<tr>
<td>655</td>
<td>James F. Reineck</td>
<td>The connection matrix in Morse–Smale flows II</td>
</tr>
<tr>
<td>656</td>
<td>Claude Baesens, John Guckenheimer, Seunghan Kim and Robert Mackay</td>
<td>Simple resonance regions of torus diffeomorphisms</td>
</tr>
<tr>
<td>657</td>
<td>Willard Miller, Jr.</td>
<td>Lecture notes in radar/sonar: Topics in Harmonic analysis with applications to radar and sonar</td>
</tr>
<tr>
<td>658</td>
<td>Calvin H. Wilcox</td>
<td>Lecture notes in radar/sonar: Sonar and Radar Echo Structure</td>
</tr>
<tr>
<td>659</td>
<td>Richard E. Blahut</td>
<td>Lecture notes in radar/sonar: Theory of remote surveillance algorithms</td>
</tr>
<tr>
<td>660</td>
<td>D.V. Anosov</td>
<td>Hilbert’s 21st problem (according to Bolibruch)</td>
</tr>
<tr>
<td>661</td>
<td>Stephane Laederich</td>
<td>Ray–Singer torsion for complex manifolds and the adiabatic limit</td>
</tr>
<tr>
<td>662</td>
<td>Geneviève Raugel and George R. Sell</td>
<td>Navier-Stokes equations in thin 3d domains: Global regularity of solutions I</td>
</tr>
<tr>
<td>663</td>
<td>Emanuel Parzen</td>
<td>Time series, statistics, and information</td>
</tr>
<tr>
<td>664</td>
<td>Andrew Majda and Kevin Lamb</td>
<td>Simplified equations for low Mach number combustion with strong heat release</td>
</tr>
<tr>
<td>665</td>
<td>Ju. S. Il'yashenko</td>
<td>Global analysis of the phase portrait for the Kuramoto–Sivashinsky equation</td>
</tr>
<tr>
<td>666</td>
<td>James F. Reineck</td>
<td>Continuation to gradient flows</td>
</tr>
<tr>
<td>667</td>
<td>Mohamed Sami Elbialy</td>
<td>Simultaneous binary collisions in the collinear N–body problem</td>
</tr>
<tr>
<td>668</td>
<td>John A. Jacquez and Carl P. Simon</td>
<td>Aids: The epidemiological significance of two different mean rates of partner-change</td>
</tr>
<tr>
<td>669</td>
<td>Carl P. Simon and John A. Jacquez</td>
<td>Reproduction numbers and the stability of equilibria of SI models for heterogeneous populations</td>
</tr>
<tr>
<td>670</td>
<td>Matthew Stafford</td>
<td>Markov partitions for expanding maps of the circle</td>
</tr>
<tr>
<td>671</td>
<td>Ciprian Foias and Edriss S. Titi</td>
<td>Determining nodes, finite difference schemes and inertial manifolds</td>
</tr>
<tr>
<td>672</td>
<td>M.W. Smiley</td>
<td>Global attractors and approximate inertial manifolds for abstract dissipative equations</td>
</tr>
<tr>
<td>673</td>
<td>M.W. Smiley</td>
<td>On the existence of smooth breathers for nonlinear wave equations</td>
</tr>
<tr>
<td>674</td>
<td>Hitay Özbay and Janos Turi</td>
<td>Robust stabilization of systems governed by singular integro-differential equations</td>
</tr>
<tr>
<td>675</td>
<td>Mary Silber and Edgar Knobloch</td>
<td>Hopf bifurcation on a square lattice</td>
</tr>
<tr>
<td>676</td>
<td>Christophe Golé</td>
<td>Ghost circles for twist maps</td>
</tr>
<tr>
<td>677</td>
<td>Christophe Golé</td>
<td>Ghost tori for monotone maps</td>
</tr>
<tr>
<td>678</td>
<td>Christophe Golé</td>
<td>Monotone maps of $T^n \times \mathbb{R}^n$ and their periodic orbits</td>
</tr>
<tr>
<td>679</td>
<td>E.G. Kalnins and W. Miller, Jr.</td>
<td>Hypergeometric expansions of Heun polynomials</td>
</tr>
<tr>
<td>680</td>
<td>Victor A. Pliss and George R. Sell</td>
<td>Perturbations of attractors of differential equations</td>
</tr>
<tr>
<td>681</td>
<td>Avner Friedman and Peter Knabner</td>
<td>A transport model with micro- and macro-structure</td>
</tr>
<tr>
<td>682</td>
<td>E.G. Kalnins and W. Miller, Jr.</td>
<td>A note on group contractions and radar ambiguity functions</td>
</tr>
<tr>
<td>683</td>
<td>George R. Sell</td>
<td>References on dynamical systems</td>
</tr>
<tr>
<td>684</td>
<td>Shui-Nee Chow, Kening Lu and George R. Sell</td>
<td>Smoothness of inertial manifolds</td>
</tr>
<tr>
<td>685</td>
<td>Shui-Nee Chow, Xiao-Biao Lin and Kening Lu</td>
<td>Smooth invariant foliations in infinite dimensional spaces</td>
</tr>
</tbody>
</table>