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Abstract. Mathematical principles used into the derivation of models of asymptotic approximation for some singular and regular perturbation problems are presented. Their physical meaning is discussed. As examples the Navier–Stokes and Cahn–Hillard models and some of their models of asymptotic approximation are analyzed. The Cahn–Hillard case was worked when the author was visiting IMA and was inspired by G. Pego’s lecture to IMA Program in Phase Transitions and Free Boundaries, September 1990. Comparison between the equations and boundary conditions of the given model and its approximants is done. The relevance of the simple model examples for the associated complicated models is analyzed.

Models of asymptotic approximation of a perturbation model. The renewed interest in the theory of asymptotic approximation has at least four purposes: the first, theoretical and the others, computational. Thus the models of asymptotic approximation relate several models competing for the description of one and the same phenomenon (e.g. fluid flow). Next, due to the fact that all the terms in an equation of asymptotic approximation have the same order, in the very complicated concrete situations the use of models of asymptotic approximation is the only way to obtain reliable numerical results by means of regular computers (e.g. meteorology, hydrology, transonic flows, flows near the boundary layer separation). The number of these models may be very large (e.g. 10). Next, the methods of asymptotic analysis allow to carry far fields conditions to a finite distance in this way rendering the given problem suitable for computation (e.g. in aerospace problems). Finally the methods of singular perturbations of boundary layer type determine the optimal grid in regions of steep variation of the solution.

The theory of the asymptotic approximation has two main parts: asymptotic analysis and perturbation theory. Perturbation theory represents the set of the applications of asymptotic analysis to cases where the asymptotic variable is the small parameter of the problems while the traditional works of asymptotic analysis especially refer to the independent variable (s) as the asymptotic variable.

Let us recall a few definitions. Let \( f : D \subset C \to C, \ z_0 \in \overline{D} \). The asymptotic behavior of \( f \) for \( z \to z_0 \) is given by the values \( f(z) \) for \( z \to z_0 \). It is a local study. On the set of functions having the same domain of definition define the Euler order relations \( \mathcal{O} \) and \( o : f = \mathcal{O}(g) \) for \( z \to z_0 \) in \( D \) if \( f/g \) is bounded in a neighborhood of \( z_0 \). \( f = o(g) \) for \( z \to z_0 \) in \( D \) if \( \lim_{z \to z_0} |f(z)/g(z)| = 0 \). If \( f = o(g) \) we say that \( f \) is much smaller than \( g \) and \( g \) is much greater than \( f \) and write \( f \ll g, g \gg f \). It seems to us that the notation which contains \( = \) is inadequate; maybe a more natural writing would be \( f \ll g, f \gg o(g) \). There are frequently used three order relations derived from \( \mathcal{O} \) and \( o \). Thus \( f = \mathcal{O}(g) \) for \( z \to z_0 \) in \( D \) iff \( f = O(g) \), \( f \neq o(g) \) for \( z \to z_0 \) in \( D \). If \( f = \mathcal{O}(g) \) we say that \( f \) and \( g \) are of the

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same order. \( f \sim g \) for \( z \to z_0 \) in \( D \) iff \( \lim_{z \to z_0} |f(z)/g(z)| = 1 \). \( f, g \) with \( f \sim g \) are called asymptotically equal. \( f \sim g \) iff \( f = g[1 + o(1)] \). The set \( \{\delta_i\}_{i=1}^{N} \) of functions \( \delta_i : D \to C, \delta_{i+1} \ll \delta_i \) (\( \forall \) \( i \in \mathbb{N} \)) is called an asymptotic sequence. \( N \) may be finite or infinite. Two sequences \( \{\delta_i\}, \{\delta''_i\} \) are equivalent if \( \delta'_i \sim \delta''_i \) (i.e. \( \delta'_i = \delta''_i + o(\delta''_i) \)) (\( \forall \) \( i \in \mathbb{N} \)). \( f \) is called asymptotically equal to \( g \) term by term or asymptotically equivalent to \( g \) with respect to the asymptotic sequence \( \{\delta_i\} \) if \( f \sim \sum_{i=1}^{N} f_i, g \sim \sum_{i=1}^{N} g_i, f_i - g_i = o(\delta_i), (\forall \ i = 1, \mathbb{N}) \). In this case also we write \( f \sim g \).

In this way, given a function, denote by \( \mathcal{O}(f) \) the class of all functions \( g = \mathcal{O}(f) \) for \( z \to z_0 \) in \( D \). Then the figure holds. By \( \tilde{f} \) we denoted the class of all the functions asymptotically equivalent to \( f \) and \( \tilde{f} \) stands for the class of all the functions asymptotically equivalent to \( f \) with respect to a given asymptotic sequence. An asymptotic series is a sum \( \sum_{i=1}^{N} a_i \delta_i \) where \( \delta_{i+1} \ll \delta_i \) for \( z \to z_0 \) in \( D \). An asymptotic expansion of \( f \) with respect to the asymptotic sequence \( \{\delta_i\} \) is an asymptotic series \( \sum_{i=1}^{N} a_i \delta_i \) such that \( f \sim \sum_{i=1}^{N} a_i \delta_i \) hence \( f - \sum_{i=1}^{N} a_i \delta_i = o(\delta_N) \) (i.e. \( \lim_{z \to z_0} \frac{f(z) - \sum_{i=1}^{N} a_i \delta_i(z)}{\delta_N(z)} = 0 \)) and \( \lim_{z \to z_0} \frac{f(z) - \sum_{i=1}^{k} a_i \delta_i(z)}{\delta_{i+1}(z)} = a_{k+1} < \infty \). By definition \( a_1 = \lim_{z \to z_0} f(z)/\delta_1(z) \). The first term \( a_1 \delta_1 \) is called the leading term or the asymptotic representation of \( f \). The sum of an asymptotic series is a class \( \tilde{f} \); because \( \sim \) in the definition of an asymptotic expansion is the relation of equivalence of functions asymptotically equal term by term it follows that all the results in asymptotical analysis are up to such an equivalence.
When \( f(\varepsilon, x) \) and \( \varepsilon \) is the asymptotic variable and \( x \) is the asymptotic parameter all these definitions change accordingly mainly placing the modulus by the norm of the function. In physical problems \( \varepsilon \) is the parameter while \( x \) is the independent variable (time and/or space). \( x \) is called fixed for \( \varepsilon \to 0 \) if \( x = O(1) \) for \( \varepsilon \to 0 \) (\( x \) is connected with \( \varepsilon \) only by its order). In physical problems the following types of asymptotic behaviors are studied: \( \varepsilon \to 0 \), \( \varepsilon \to \infty \); \( t \to 0 \), \( t \to \infty \), \( t \to t_0 < \infty \) (singularity of the solution); \( x \to 0 \), \( x \to \infty \), \( x \to x_0 < \infty \) (singularity of the solution); combinations of these; \( \varepsilon \to \varepsilon_0 \) or/and \( x \to x_0 \) or/and \( t \to t_0 \) where \( \varepsilon_0, x_0, t_0 \in D \) hence are regular points. The complete asymptotic behavior of the solution is realized by the aid of several asymptotic expansions. Matching refers to: functions, Taylor series, asymptotic series with respect to the same asymptotic sequence; asymptotic series with respect to different asymptotic sequences but having the same asymptotic argument; asymptotic series depending on various asymptotic arguments. The asymptotic expansions which are uniformly valid with respect to the asymptotic parameter and one written by means of several asymptotic expansions are called composite asymptotic expansions. Among the set of asymptotic expansions of a function some are contained into others. An asymptotic expansion of \( f \) which is not contained into any other asymptotic expansion of \( f \) is called a significant asymptotic expansion.

In our lecture we give examples of such significant expansions, obtained by the aid of the method of matched asymptotic expansions by the \( n \times m \) matching principle for the Prandtl example model, Navier–Stokes model, Cahn–Hillard model and Navier–Stokes–Fourier model in dynamics of atmosphere.

By a "mathematical model" or shortly "model" we mean a boundary-or/and initial-value problem for a system of differential equations. The aim of the perturbation theory is to yield, for small parameters \( \varepsilon \in \mathbb{R} \), models of asymptotic approximations of a given perturbation problem \( M_\varepsilon \). Denote by \( u(\varepsilon, x) \) the solution of \( M_\varepsilon \) where \( x \in \Omega \) stands for the independent variable and \( \Omega \subset \mathbb{R}^n \) is a domain. Assume that the asymptotic behavior of \( u \) with respect to \( \varepsilon \) as \( \varepsilon \to \infty \) is given by the expansion

\[
(1) \quad u(\varepsilon, x) \sim U_0(\varepsilon, x) + U_1(\varepsilon, x) + \cdots + U_{N_1}(\varepsilon, x) , \quad \varepsilon \to 0 ,
\]

for any \( x \in \Omega \). This asymptotic expansion is, therefore, uniformly valid with respect to \( x \) throughout \( \Omega \) and we say that \( M_\varepsilon \) is a regular perturbation problem. This takes place if \( u \) has the same order over the entire \( \Omega \), namely the order of \( U_0(\varepsilon, x) \) as \( \varepsilon \to 0 \). If \( u \) has different orders with respect to \( \varepsilon \) in various subdomains of \( \Omega \) its asymptotic behavior is no longer described by the single expansion (1) but also at least by another expansion. For instance, assume that for \( x \in \Omega \setminus \Omega_1 \) the asymptotic behavior of \( u \) with respect to \( \varepsilon \) is described by (1) while for \( x \in \Omega_1 \subset \Omega \) this asymptotic behavior is given by

\[
(2) \quad u(\varepsilon, x) \sim U_0(\varepsilon, x) + \cdots + U_{N_2}(\varepsilon, x) , \quad \varepsilon \to 0 .
\]

Recall that the series in (1) and (2) may be finite or infinite, convergent or divergent. They approximate locally (near \( \varepsilon = 0 \)) the behavior of \( u \) (and give its order) as \( \varepsilon \to 0 \),
and not \( u \) as a function of \( \varepsilon \). The sum in (1) and (2), may differ from \( u \). However, \( \varepsilon \) is supposed to be a **variable** (like \( x \), the only difference being that with respect to \( \varepsilon \) there is no differentiation in \( M_\varepsilon \)) called **asymptotic variable**. Correspondingly, in asymptotic theory \( M_\varepsilon \) is a continuum of problems and not a single problem where \( \varepsilon \) has a well-determined value.

By definition, the problems satisfied by \( V_i \) and \( U_i \) are called **models of i-th order asymptotic approximation** for \( \varepsilon \to 0 \) of \( M_\varepsilon \) in \( \Omega \setminus \Omega_1 \) and \( \Omega_1 \) respectively.

Roughly speaking, the series in (1) is asymptotic if \( U_{i+1} \ll U_i \) as \( \varepsilon \to 0 \) and \( u - \sum_{i=0}^{N_1} U_i \ll U_{N_1} \). Let \( \delta_0(\varepsilon), \delta_1(\varepsilon), \ldots, \delta_{N_1}(\varepsilon) \) be functions of order of \( U_0(\varepsilon, x), \ldots, U_{N_1}(\varepsilon, x) \) as \( \varepsilon \to 0 \) i.e. \( \lim_{\varepsilon \to 0} |U_0(\varepsilon, x)/\delta(\varepsilon)| = A > 0 \). The sequence \( \{\delta_i(\varepsilon)\}_{i=0,N_1} \) is called the asymptotic sequence since \( \delta_{i+1}(\varepsilon) \ll \delta_i(\varepsilon) \) as \( \varepsilon \to 0 \) and we may say that (1) represents the asymptotic expansion of \( u \) with respect to this sequence for \( \varepsilon \to 0 \). If the functions \( U_i(\varepsilon, x) \) have separate variables, \( U_i(\varepsilon, x) = U_i(x) \delta_i(\varepsilon) \), (1) becomes

\[
(1') \quad u(\varepsilon, x) \sim U_0(x)\delta_0(\varepsilon) + U_1(x)\delta_1(\varepsilon) + \cdots + U_{N_1}(x)\delta_{N_1}(\varepsilon), \quad \varepsilon \to 0
\]

and, if \( \delta_1(\varepsilon) = \varepsilon^{i+1} \), (1') reads as a Taylor sum

\[
(1'') \quad u(\varepsilon, x) \sim U_0(x) \cdot 1 + U_1(x)\varepsilon + \cdots + U_{N_1}(x)\varepsilon^{N_1}, \quad \varepsilon \to 0
\]

This expansion is valid if \( u(\varepsilon, x) = O(1), \varepsilon \to 0 \), \( x \in \Omega \setminus \Omega_1 \) i.e. when \( u \) has a bounded limit as \( \varepsilon \to 0 \) (because \( u = O(1) \) means \( \lim_{\varepsilon \to 0, x = O(1)} \frac{u(\varepsilon, x)}{1} = \text{a bounded function of } x \)). Similar reasoning holds for (2).

If (in \( \Omega \setminus \Omega_1 \)) the asymptotic expansion of \( u \) is of form (1)', in finding this expansion the most important thing is to "guess" the associated asymptotic sequence \( \{\delta_i(\varepsilon)\} \) since the coefficients are given by

\[
U_0(x) = \lim_{\varepsilon \to 0}(u(\varepsilon, x)/\delta_0(\varepsilon)), \quad U_i(x) = \lim_{\varepsilon \to 0, x = O(1)} \frac{u(\varepsilon, x) - \sum_{k=0}^{i-1} U_k(x)\delta_k(\varepsilon)}{\delta_i(\varepsilon)}.
\]

The limits are taken for fixed values of \( x \) (\( x = O(1) \) as \( \varepsilon \to 0 \)). The limit taken for fixed values of the independent variable is called **global limit**.

In \( M_\varepsilon, \varepsilon \) and \( x \) are independent as variables but **their order is related** if it is wanted that every term occurring in \( M_\varepsilon \) have a well-determined order, relative to \( \varepsilon \) as \( \varepsilon \to 0 \). A similar situation occurs for a function \( f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \to f(x, y) \) which we want to calculate \( \lim_{x \to x_0, y \to y_0} f(x, y) \). Although \( x \) and \( y \) are independent variables, we can come to \((x_0, y_0)\) following various paths corresponding to different rates at which \((x, y)\) tends to
\((x_0, y_0)\). These paths, described by functions \(y(x)\), realize just the order of \(y\) with respect to \(x\) as \(x \rightarrow x_0\). Let us also take another situation. Consider the algebraic equation

\[ \varepsilon x^2 + x - 1 = 0 \]

for small \(\varepsilon > 0\). Neglecting the first term (and getting the first order approximation of (3))

\[ X_0 - 1 = 0 \]

it implicitly means to have assumed that \(x\) is fixed \((x = \mathcal{O}(1) \varepsilon \rightarrow 0)\), hence \(x \gg \varepsilon\). When \(x \gg 1\), the first term in (3) has undetermined order and, in order to operate any simplification, we need to specify the order of \(x\). If this order is \(x = \mathcal{O}(\varepsilon^{-1})\) as \(\varepsilon \rightarrow 0\) then the first two terms in (3) are dominant over the third. In this case the first-order approximation of (3) is

\[ \varepsilon x_0^2 + x_0 = 0. \]

Equation (3) is singular perturbation algebraic equation because the complete asymptotic behavior of its solutions is described by two series with respect two different asymptotic sequences: the solution \(x_1\) has the expansion

\[ x_1 \sim 1 - \varepsilon + 2\varepsilon^2 + \cdots, \varepsilon \rightarrow 0 \]

with respect to the asymptotic sequence \(1, \varepsilon, \varepsilon^2, \ldots\) (its leading term satisfying equation (4)) while the other solution develops as

\[ x_2 \sim -\frac{1}{\varepsilon} - 2 + \cdots, \varepsilon \rightarrow 0 \]

with respect to the asymptotic sequence \(\frac{1}{\varepsilon}, 1, \varepsilon, \varepsilon^2, \ldots\) (its leading term satisfying equation (5)). In deriving these expansions the reasonings done to deduce the first order approximation of \(x_2\) is repeated for \(x - X_0, x - X_0 - X_1, \ldots, x - x_0, x - x_0 - x_1, \ldots\) where \(X_0, X_1, \ldots, x_0, x_1, \ldots\) are the terms of the asymptotic expansion of \(x_1\) and \(x_2\). For easier calculation in the case of \(x_2\) we introduce the new fixed variable \(\tilde{x} = \varepsilon x, \tilde{x} = \mathcal{O}(1)\) as \(\varepsilon \rightarrow 0\). In the variable \(\tilde{x}\) (3) reads as

\[ \tilde{x}^2 + \tilde{x} - \varepsilon = 0 \]

and its solution, corresponding to \(x_2\), has the following expansion (with respect to the asymptotic sequence \(1, \varepsilon, \varepsilon^2, \ldots\))

\[ \tilde{x}_2(\varepsilon) \sim -1 - 2\varepsilon + \cdots, \varepsilon \rightarrow 0. \]

In conclusion, the asymptotic treatment of (3) involves two different cases: \(1^0\) \(x = \mathcal{O}(1), \varepsilon \rightarrow 0\) and \(2^0\) \(\tilde{x} = \mathcal{O}(1), \varepsilon \rightarrow 0\), corresponding to two different fixed variables.
Prandtl's model example. Let us treat now the differential Prandtl model example of singular perturbations

\begin{align}
(8) & \quad \mu \cdot \dot{u} + k \cdot u + cu = 0, \quad t \in (0, \infty) \\
(9) & \quad \lim_{t \to 0} u(t) = 0, \\
(10) & \quad \lim_{t \to \infty} u(t) = 0,
\end{align}

governing the motion of a mass \( m \) subjected to friction forces (of coefficient \( k \)) and elastic forces (of modulus \( c \)). When \( m \) is small, in order to derive a first order approximation equation we may reason as for equation (3) and correspondingly we may neglect \( \mu \cdot \dot{u} \) for \( \mu \) small i.e. when \( u \) varies slowly. When \( \mu \) is very large, the order of \( \mu \cdot \dot{u} \) is undetermined such that we must perform some change of variables as in the algebraic case. However, in this differential case the change of variable will not concern the unknown \( u \) but the independent variable \( t \). Indeed, \( \mu \) large means very fast variation of \( u \) in a small region of \( (0, \infty) \) near a point \( t_0 \in (0, \infty) \). This region is called mathematical boundary layer. Apart from very simple equations, generally there is no mathematical possibility to find \( t_0 \), therefore the position of this boundary layer. The only suggestions come from experiment or rough calculations. This is the first point of genuine applied mathematics i.e. of interaction of mathematics with other field (e.g. mechanics, chemistry, experiments (physical or numerical)). Since the mathematical boundary layer is thin it means that \( t - t_0 \) is small. It may be of order \( m \) or smaller (e.g. of order \( m^2 \)). This is why in order to get a new fixed independent variable we introduce \( \tilde{t} = (t - t_0)/\eta(m) \) where \( \eta(m) \to 0 \) as \( m \to 0 \) such that \( \tilde{t} = O(1) \) as \( m \to 0 \). Now everything is prepared for a coherent asymptotic analysis which avoids the step by step estimation of various orders in the asymptotic expansion of solutions of (3). In order to derive the asymptotic behavior of the solution of (8)-(10) we apply the matched inner-outer asymptotic expansion method. In the region outside the mathematical boundary layer, called the outer region and defined by \( t - t_0 = O(1) \) for \( m \to 0 \), analytical dependence of the equation of \( m \) suggests an analytical dependence of the solution on \( m \). Correspondingly we choose the following outer expansion

\begin{align}
(11) & \quad u(m, t) \sim U_0(t) + U_1(t)m + U_2(t)m^2 + \cdots, m \to 0
\end{align}

in the region \( \Omega \setminus \Omega_1 = \{ t \in (0, \infty) \mid t - t_0 = O(1), m \to 0 \} \). In our case \( t_0 = 0 \) (this may also be proved mathematically). Introducing (11) into (8) and (10) (condition (9) is left aside because it is not in the outer region) we get

\begin{align*}
(k \dot{U}_0 + cU_0) + m \left( \dot{U}_0 + k \dot{U}_1 + cU_1 \right) + \cdots & \sim 0, \\
\lim_{t \to \infty} U_0(t) + m \lim_{t \to \infty} U_1(t) + \cdots & \sim 0,
\end{align*}

which implies

\begin{align}
\left\{ \begin{array}{ll}
k \dot{U}_0 + cU_0 = 0, & t \in (\delta(m), \infty) \\
\lim_{t \to \infty} U_0(t) = 0,
\end{array} \right.
\end{align}
\[
\begin{align*}
\begin{cases}
k\dot{U}_1 + cU_1 &= -\ddot{U}_0, \\
\lim_{t \to \infty} U_1(t) &= 0.
\end{cases}
\end{align*}
\]

(13) (resp. (13)) represent the models of outer asymptotic approximation of (8)–(10) as \( m /\rightarrow 0 \) of first (second) order. \( \delta(m) (= O(m), m /\rightarrow 0 \) is an arbitrary function. Since \( \delta(m) \) does not influence at all the solutions of (12) and (13) and \( \delta(m) /\rightarrow 0 \) as \( m /\rightarrow 0 \) in (12) and (13) usually \( \delta(m) \) is replaced by 0. The solution of (12) and (13) are (up to a multiplicative constant \( A \), because problems (12) and (13) are linear)

\[
U_0(t) = Ae^{-\hat{t}t}, \quad U_1(t) = -A\frac{c^2}{k^3} te^{-\hat{t}t}.
\]

In the boundary layer \( \Omega_1 = \{ t ∈ (0, \infty) | t = O(m), m /\rightarrow \infty \} = \{ t ∈ (0, \infty) | 0 < t < \delta(m) \} \)
let us first take the change of variable \( t /\rightarrow \hat{t} = \frac{t}{m} k \) (hence \( \eta(m) = m \)). Then (8), (9) read

\[
\begin{align*}
k^2m^{-1} \dddot{\tilde{u}}_{\hat{t}\hat{t}} + k^2m^{-1} \ddot{\tilde{u}}_{\hat{t}} + c\tilde{u} &= 0, \\
\lim_{\hat{t} /\rightarrow 0} \tilde{u}(\tilde{t}) &= 0.
\end{align*}
\]

Condition (10) was left aside since \( t = \infty \) is outside the boundary layer. In (14) and (15), \( \dddot{\tilde{u}}_{\hat{t}\hat{t}}, \ddot{\tilde{u}}_{\hat{t}}, \tilde{u} = O(1) \) for \( m /\rightarrow 0 \) and \( \hat{t} = O(1) \) is \( m /\rightarrow 0 \) such that the same analyticity arguments may motivate the following form of the asymptotic behavior of \( u \) in \( \Omega_1 \)

\[
\tilde{u}(\varepsilon, \hat{t}) \sim \tilde{u}_0(\hat{t}) + m\tilde{u}_1(\hat{t}) + \cdots m /\rightarrow 0.
\]

(16) is called the inner asymptotic expansion of \( u \) and it is valid in \( \Omega_1 \).

The thickness of the boundary layer \( \delta(m) \) is not a precise function because \( t \) and \( \varepsilon \) are not connected by a function but only by an order relation. So, \( \delta(m) \) may be any function in the class of equivalence of functions asymptotically equal to \( m \) as \( m /\rightarrow 0 \). The only thing which may be said about \( \delta(m) \) is that it tends to 0 as fast as \( m \); it is not its form which counts but its order as \( m /\rightarrow 0 \). Unlike this “asymptotic thickness” there is a “mechanical thickness” defined for instance as that \( \delta_{\text{mec}}(m) = t_\ell(m) \) where \( [U_0(t_\ell) - \bar{u}_0(t_\ell k/m)]/\bar{u}_0(t_\ell k/m) = 0.01 \) (i.e. that the value of \( t \) at which the relative difference between the first outer and inner approximations of \( u \) is 1%). The “layer” having \( t = 0 \) and \( t = \delta(m) \) as lower and upper frontiers is called the “mechanical boundary layer”. Similarly the thermal boundary layer is that region in the space where that relative difference in temperature is very small; one may speak about a chemical, atmospheric, turbulent etc. boundary layer. Their thickness varies from one phenomenon to the other (e.g. for the dynamical boundary layer on the flat plate it is of about 4mm while the atmospheric boundary layer may reach 1km). But, anyhow, every boundary layer has a smaller scale as compared with the scale of the phenomenon.
Remark that the boundary layer may be not only spatial but also temporal as in case of our example (8)–(10). It corresponds to a local time scale within which the phenomenon has almost sudden changes and is an initial layer.

Introducing (16) into (14), (15) and matching the obtained series by the null series we get

$$\begin{cases} \bar{u}_0 \ddot{\bar{t}} + \bar{u}_0 = 0, & \bar{t} \in (0, \infty) \\ \lim_{\bar{t} \to 0} \bar{u}(\bar{t}) = 0, & \end{cases}$$

(17)

$$\begin{cases} k^2 \bar{u}_1 \ddot{\bar{t}} + k^2 \bar{u}_1 = -cu_0, & \bar{t} \in (0, \infty) \\ \lim_{\bar{t} \to 0} \bar{u}_1(\bar{t}) = 0. & \end{cases}$$

(18)

Since in (15), (16) the equations are second order, another boundary condition is necessary for the uniqueness of their solution. The physical problem (8)–(10) does not yield anyone so recourse is made to the matching of the asymptotic series (11) and (16). The asymptotic series (11) and (16) will be matched since they represent the asymptotic expansions of the same function (the solution $u$). In this way some information from (11) at the upper frontier of the boundary layer will be used for (16). However, this matching is not a link between the values in the two series but a relation of order between them. There is not a precise place to relate their values. We suppose that there exists a common domain of validity for the inner and outer expansion (the overlapping domain) in which the $n$-terms inner asymptotic expansion of the $m$- terms outer asymptotic expansion of $u$ coincides with the $m$-terms outer asymptotic expansion of the $n$-terms inner asymptotic expansion of $u$. In other words we use the $n \times m$ matching principle. In the overlapping domain, “near” the upper frontier of the boundary layer, $t$ ceases to be of order $\varepsilon$ and becomes of order $1$. It follows that $\bar{t} = \frac{t}{m} k \to \infty$ as $m \to 0$. This is why the equations in (17) and (18) were defined on $(0, \infty)$ and it is $\bar{t} = \infty$ which will occur in the $n \times m$ principle. If we want to complete only the problem (17) then it is enough to use the $1 \times 1$ matching principle. So, the 1-term outer expansion of $u$ is $U_0(t) = U_0 \left( \frac{mt}{k} \right)$. Its 1-term inner expansion (which is taken for $\bar{t} = O(1), m \to 0$) is $\lim_{\bar{t} \to 0} \bar{u}_0(\bar{t}) = \bar{u}_0 \left( \frac{tk}{m} \right)$. Its 1-term outer expansion (which is taken for $t = O(1), m \to 0$) is $\lim_{t \to 0} \bar{u}_0 \left( \frac{tk}{m} \right) = \lim_{\bar{t} \to \infty} \bar{u}_0(\bar{t})$. In conclusion the $1 \times 1$ matching principle reads

$$\lim_{t \to 0} U_0(t) = \lim_{\bar{t} \to \infty} \bar{u}_0(\bar{t}).$$

(19)

For our specific example it leads to

$$\lim_{\bar{t} \to \infty} \bar{u}_0(\bar{t}) = A.$$  

(20)

$\lim_{\bar{t} \to \infty} \bar{u}_0(\bar{t}) = A.$
(17) and (20) from the model of the first inner asymptotic approximation as \( m \to 0 \) of the given model (8)-(10). It has the solution \( \bar{u}_0(t) = A(1 - e^{-t}) \). In order to complete the problem (18) we apply the 2 \( \times \) 2 matching principle: the 2-terms outer expansion of \( u \) is \( U_0(t) + mU_1(t) = U_0 \left( \frac{m \dot{k}}{k} \right) + mU_1 \left( \frac{m \dot{k}}{k} \right) \). Its 2-terms inner expansion is

\[
\lim_{t \to 0 \atop \text{fixed}} U_0 \left( \frac{m \dot{k}}{k} \right) + mU_1 \left( \frac{m \dot{k}}{k} \right) + m \lim_{m \to 0 \atop \text{fixed}} \frac{\partial}{\partial m} \left[ U_0 \left( \frac{m \dot{k}}{k} \right) + mU_1 \left( \frac{m \dot{k}}{k} \right) \right] = \lim_{t \to 0} U_0(t) + \]

\[
m \lim_{m \to 0 \atop \text{fixed}} \left[ \frac{\dot{k}}{k} \frac{\partial \bar{u}_0}{\partial t} \left( \frac{m \dot{k}}{k} \right) + U_1 \left( \frac{m \dot{k}}{k} \right) + \frac{m \dot{k}}{k} \frac{\partial \bar{u}_1}{\partial t} \left( \frac{m \dot{k}}{k} \right) \right] = A + m(-ctA/k^2) = A (1 - ct/k).
\]

The 2-terms inner expansion of \( u \) is \( \bar{u}_0(t) + m\bar{u}_1(t) = \bar{u}_0 \left( \frac{tk}{m} \right) + m\bar{u}_1 \left( \frac{tk}{m} \right) \). Its 2-terms outer expansion is

\[
\lim_{m \to 0 \atop \text{fixed}} \left[ \bar{u}_0 \left( \frac{tk}{m} \right) + m\bar{u}_1 \left( \frac{tk}{m} \right) \right] + m \lim_{m \to 0 \atop \text{fixed}} \frac{\partial}{\partial m} \left[ \bar{u}_0 \left( \frac{tk}{m} \right) + m\bar{u}_1 \left( \frac{tk}{m} \right) \right] = A + \]

\[
\lim_{m \to 0 \atop \text{fixed}} \left[ m\bar{u}_1 \left( \frac{tk}{m} \right) \right] + m \lim_{m \to 0 \atop \text{fixed}} \left[ -\frac{tk}{m^2} \frac{\partial \bar{u}_0}{\partial t} \left( \frac{tk}{m} \right) + \bar{u}_1 \left( \frac{tk}{m} \right) - \frac{m tk}{m^2} \frac{\partial \bar{u}_1}{\partial t} \left( \frac{tk}{m} \right) \right] = A + \lim_{m \to 0 \atop \text{fixed}} \left[ m\bar{u}_1 \left( \frac{tk}{m} \right) \right] + \]

\[
m \lim_{m \to 0 \atop \text{fixed}} \left[ \bar{u}_1 \left( \frac{tk}{m} \right) - \frac{tk}{m} \frac{\partial \bar{u}_1}{\partial t} \left( \frac{tk}{m} \right) \right].
\]

Consequently, the \( 2 \times 2 \) matching principle leads to the following equality of two functions of \( t \) (called intermediate expansions)

\[
A - Act/k = A + \lim_{m \to 0 \atop \text{fixed}} \left[ m\bar{u}_1 \left( \frac{tk}{m} \right) \right] + m \lim_{m \to 0 \atop \text{fixed}} \left[ \bar{u}_1 \left( \frac{tk}{m} \right) - \frac{tk}{m} \frac{\partial \bar{u}_1}{\partial t} \left( \frac{tk}{m} \right) \right]
\]

which implies

\[
(21) \quad \lim_{m \to 0 \atop \text{fixed}} \left[ m\bar{u}_1 \left( \frac{tk}{m} \right) \right] + m \lim_{m \to 0 \atop \text{fixed}} \left[ u_1 \left( \frac{tk}{m} \right) - \frac{tk}{m} \frac{\partial u_1}{\partial t} \left( \frac{tk}{m} \right) \right] = -Actk^{-1}.
\]

This condition together with (18) form the problem of the second order inner asymptotic approximation of problem (8)-(10) for \( m \to 0 \). The solution of (18), (21) is \( \bar{u}_1(t) = -Aict^{-2}(t + \dot{t}e^{-t}) \) (because the solution of (18) is \( \bar{u}_1(t) = -Aict^{-2}[d - 1 + \dot{t} + (1 - d + \dot{t})e^{-t}] \) and imposing condition (21) it follows \( d = 1 \).

The limit \( \lim_{m \to 0 \atop \text{fixed}} \), equivalent with \( \lim_{m \to 0 \atop t = \theta(m),m \to 0} \), defined on the subdomain \( \Omega_1 \) of \( \Omega \) (where the given independent variable is not fixed) is called a local limit. The variable \( \dot{t} \) which is fixed in this (inner) subdomain is a local (or inner) coordinate.

Comparison of the given equation (8) \( m\dot{u} + k\dot{u} + cu = 0 \) with its first inner asymptotic approximation (17) (written in the variable \( t \)) \( m\dot{u}_0 + k\dot{u}_0 = 0 \) shows that intuitive reasonings may be very hard in complicate cases.

**The complete asymptotic behavior of the solution** \( u \) **of (8)-(10).** The asymptotic behavior of the solution \( u \) of (8)-(10) up to \( \Theta(m) \), \( m \to 0 \) terms reads

\[
(22) \quad u(m,t) \sim \begin{cases} t(1 - mtc^2/k^3)e^{-ct/k} , & t = \Theta(1) , m \to 0 \\
A(1 - e^{-t} - mck^{-2}\tilde{t} - mck^{-2}\tilde{t}e^{-t}) , & t = \Theta(m) , m \to 0 ,
\end{cases}
\]

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or, in the form of a uniformly valid composite asymptotic approximation

\[ u(m, t) \sim A(1 - mt^2/k^3) e^{-ct/k} + A(-1 - mck^{-2}\hat{t}) e^{-\hat{t}} \]

obtained by adding (11) with (16) and subtracting from the sum the intermediate series \( A(1 - ctk^{-1}) \). This subtraction avoids to account two times for the intermediate series. Indeed the intermediate series is contained for \( t = \mathcal{O}(1), m \to 0 \) in the outer expansion and for \( t = \mathcal{O}(m), m \to 0 \), in the inner expansion. (We say that the asymptotic expansion \( D_1 \) is contained into asymptotic expansion \( D_2 \) when \( D_2 \) written in the independent variable of \( D_1 \) has an asymptotic expansion just \( D - 1 \).) An asymptotic expansion is significant if it is not contained in others. The matched inner-outer asymptotic expansion method is applied to those problems whose solution has two significant asymptotic expansion, namely, the inner and the outer ones. They have an overlapping domain of common validity and the solution is completely described asymptotically by means of these inner and outer expansions only.

For \( t = \mathcal{O}(1), m \to 0 \), \( u \) is of order of 1. For \( t = \mathcal{O}(m), m \to 0 \) \( u \) is also of order of 1 but considering \( \hat{t} \) as fixed. If the inner first order asymptotic approximation of \( u \) is written in the given independent variable we get the function \( A \left( 1 - e^{-k\hat{t}} \right) \equiv \tilde{u}_0 \left( \frac{kt}{m} \right) \) whose order of magnitude is not well-determined in the absence of extra informations on the order of \( t \) as \( m \to 0 \). Compare with the algebraic case when this order was \( \varepsilon^{-1} \).

The expression of \( \tilde{u}_0 \left( \frac{kt}{m} \right) \) shows a fast decrease with respect to \( t \) for \( m \) fixed; so the supposed asymptotic behavior of \( u \) for \( t \) and \( m \) small is confirmed. However, the (exact) solution is different from its asymptotic expansion for \( m \to 0 \). Indeed the solution have many other asymptotic expansion for \( m \to \infty, t \to 0, t \to \infty \) etc. each one called asymptotic or formal solution. The value of \( u(m, t)/[U_0(t) + mU_1(t)] \) is closer to 1 closer to zero is \( m \). Similarly \( u \left( m, \frac{mt}{k} \right)/[\tilde{u}_0(\hat{t}) + m\tilde{u}_1(\hat{t})] \to 1 \) as \( m \to 0 \). However for \( m \) not small this assertions are no longer valid. For values of \( m \) near a value \( m_0 \) if we want an asymptotic expansion of \( u \) we denote \( m - m_0 = \varepsilon \) and look for the asymptotic behavior of \( u(m_0 + \varepsilon, t) \) as \( \varepsilon \to 0 \).

Another asymptotic behavior of \( u(m, t) \), namely with respect to \( t \) for \( t \to \infty \), is obtaining supposing

\[ u(m, t) \sim \frac{1}{t} u_0(m) + \frac{1}{t^2} u_1(m) + \cdots , t \to \infty \]

and introducing this series into (8). It follows \( u_0 = u_1 = \cdots = 0 \), so \( u \sim 0 + 0 + 0 + \cdots \) as \( t \to \infty \). Since no assumption on the order of \( m \) was done it was understood that \( m = \mathcal{O}(1), t \to \infty \). If now we consider \( m \to 0 \) then the asymptotic expansion for \( m \to 0 \) from the asymptotic expansion for \( t \to \infty \) is \( u(m, t) \sim 0 + 0 + \cdots \) as \( t \to \infty \) and \( m \to 0 \). Conversely the asymptotic expansion for \( t \to \infty \) from the asymptotic expansion for \( m \to 0 \) is obtained from \( (U_0(t) + mU_1(t) = A(1 - mt/k)e^{-ct/k} \) for \( t \to \infty \) and is also \( u(m, t) \sim 0 + 0 + \cdots \) as \( m \to 0 \) and \( t \to \infty \). In particular \( \lim_{m \to 0} \lim_{t \to \infty} u(m, t) = \lim_{t \to \infty} \lim_{m \to 0} u(m, t) = 0 \). Unlike this
behavior, we have

$$\lim_{t \to 0} \lim_{m \to 0} u(m, t) = \lim_{t \to 0} U_0(t) = A \neq \lim_{m \to 0} \lim_{t \to 0} u(m, t) = \lim_{m \to 0} 0 = 0$$

which shows that $u$, as a function of $(m, t)$, is not continuous at the point $(0,0)$.

Problems of singular perturbations are associated with problems whose solution is not continuous at some point $(\varepsilon, \bar{x})$ from the domain of definition of the problem $\mathcal{D} = \mathbb{R}_+^* \times \Omega$ where $\varepsilon$ is a parameter and $\Omega$ is the domain of variation of the independent variable $\bar{x}$ (which may be the time or/and the space variables).

To complete the asymptotic study of (8)–(10) we must look for the asymptotic behavior of $u(m, t)$ as $m \to \infty$ and $t = \mathcal{O}(1)$, $m \to \infty$. Denote $m = \varepsilon^{-1}$, then $\varepsilon \to 0$ and problem (8)–(10) writes

\begin{align*}
(8') & \quad \ddot{v} + k \varepsilon \dot{v} + c \varepsilon v = 0 \, , \quad t \in (0, \infty) \\
(9') & \quad \lim_{t \to 0} v(\varepsilon, t) = 0 \\
(10') & \quad \lim_{t \to \infty} v(\varepsilon, t) = 0,
\end{align*}

where $v(\varepsilon, t) \equiv u(\varepsilon^{-1}, t)$. Assuming that

$$v(\varepsilon, t) \sim V_0(t) + \varepsilon V_1(t) + \cdots \quad \varepsilon \to 0$$

and introducing this series into (8')–(10') we obtain the following problems of first and second order asymptotic approximation as $\varepsilon \to 0$ of (8')–(10')

\begin{align*}
(24) & \quad \begin{cases}
\dot{V}_0 = 0 \, , \quad t \in (0, \infty) \\
\lim_{t \to 0} V_0(t) = 0 \, , \, \lim_{t \to \infty} V_0(t) = 0
\end{cases} \\
& \quad \begin{cases}
\dot{V}_1 = -k \dot{V}_0 \, , \quad t \in (0, \infty) \\
\lim_{t \to 0} V_1(t) = 0 \, , \, \lim_{t \to \infty} V_1(t) = 0
\end{cases}
\end{align*}

whose solutions are $V_0 = V_1 \equiv 0$. It is easy to see that $v \sim 0 + \varepsilon \cdot 0 + \varepsilon^2 \cdot 0 + \cdots$, $\varepsilon \to 0$ and, consequently, $u \sim 0 + \frac{1}{m} \cdot 0 + \frac{1}{m^2} \cdot 0 + \cdots$, $m \to \infty$.

Asymptotic series deduced are asymptotic expansions of the solution $u$. So far the derivation of (22) is coherent but not yet rigorous. The main unjustified assumptions done were: 1° the asymptotic behavior of $u$ is expressed by the aid of the asymptotic sequence $1, m, m^2, \ldots, m \to 0$. This means that we assumed that, for $t = \mathcal{O}(1)$, $m \to 0$, we have $u = \mathcal{O}(1), u - U_0 = \mathcal{O}(m)m \to 0$ etc.; 2° the conditions (20) and (21). They have no physical basis because the given problem (8)–(10) gives no condition other than at $t = 0$ and $t = \infty$. They have no mathematical basis either because in some cases the $n \times m$ principle

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fails (e.g. when \( \lim_{\bar{t} \to 0} \bar{u}_0(\bar{t}) = \infty \)). There is only the belief that this principle holds because it concerns two series associated with the same function and comes to a kind of commutativity in taking the expansions; 3\(^o\) by differentiating the asymptotic expansions of the solution we get asymptotic expansions for the derivatives of the solutions; 4\(^o\) the problem (8)--(10) has a solution and it is unique. In our particular case this uniqueness is immediate but in more complicated cases care must be taken to not associate the asymptotic expansion of one solution with another solution; 5\(^o\) the choice of some boundary conditions for the first order model of outer asymptotic approximation (the reduced model which in the singular case is different from the limit problem). In order to prove that (22) is indeed the asymptotic expansion of \( u \) as \( m \to 0 \) for \( t \) fixed we must show that \( u - (U_0 + mU_1) = o(m), \ m \to 0 \) i.e. \( \lim_{m \to 0} (u - U_0 - mU_1)/m = 0 \), which, in our case, is immediate because the (exact) solution is \( u(m, t) = A(e^\lambda_1 t - \lambda_2 t) \) where \( \lambda_{1,2} = \frac{-k \pm \sqrt{k^2 - 4mc}}{2m} \). Similarly it is seen that \( u \sim \bar{u}_0(\bar{t}) + m\bar{u}_1(\bar{t}) \) is an asymptotic expansion of \( u \) as \( m \to 0 \) for \( t = o(m), \ m \to 0 \).

For more complicated cases the explicit form of the solution is unknown such that in order to prove that a formal series is an asymptotic expansion of this solution some bounds derived from the equation must be used. Also the equation itself will be written in an integral form.

**Navier–Stokes, Euler and Prandtl models.** The Navier–Stokes model in one of the simplest situations reads

\[
\begin{cases}
u v_x + v(hv)_y = -p_x + \text{Re}^{-1} h\{(hu)_y - v_x\}_y, \\
u v_x + hvv_y - Ku^2 = -hp_y - \text{Re}^{-1} h\{(hu)_y - v_x\}_x, \\
u_x + (hv)_y = 0,
\end{cases}
\]

(25) \( (x, y) \in \Omega \subset \mathbb{R}^2 \)

\( \text{Re} \in \mathbb{R}^+_* \)

\( \text{Re} = \text{Re}^+ \times \mathbb{R}^+_* \times \mathbb{R} \)

\( \text{Re} \in \mathbb{R}^+_* \times \mathbb{R} \)

\( \lim_{y \to 0} u(\text{Re}, x, y) = \lim_{y \to 0} v(\text{Re}, x, y) = 0, \ (\text{Re}, x) \in \mathbb{R}^+_* \times \mathbb{R} \)

\( \lim_{|y| \to \infty} u(\text{Re}, x, y) = U_\infty(x), \ (\text{Re}, x) \in \mathbb{R}^+_* \times \mathbb{R} \)

(26)

(27)

\( \Omega \) is the domain exterior to a fixed rigid impermeable body immersed into the fluid, \((u, v)\) is the fluid velocity, \( f – its \) pressure, \( \text{Re} \) stands for the Reynolds number, \( y = 0 \) is the equation of \( \partial \Omega \), \( x \) is the curvilinear abscissa along \( \partial \Omega \), \( y \) is taken along the normal to \( \partial \Omega \), \( K(x) \) is the curvature of \( \partial \Omega \), \( h = 1 + yK \). Suppose that \( \text{Re} \to \infty \) and put \( \varepsilon = \text{Re}^{-1/2} \); (25)--(27) is a singular perturbation problem of the boundary layer type. Assuming that the outer expansions are \( u(\text{Re}, x, y) \sim U_0(x, y), v(\text{Re}, x, y) \sim V_0(x, y), p(\text{Re}, x, y) \sim P_0(x, y) \) and introducing them into (25)--(27) we obtain the following model of first order outer
asymptotic approximation for \( \text{Re} \to \infty \) of the Navier–Stokes model

\[
\begin{align*}
U_0 U_{0x} + V_0 (hU_0)_y &= -P_{0x} , \quad (x, y) \in \Omega \\
U_0 V_{0x} + hV_0 V_{0y} &= -hP_{0y} + KU_0^2 , \quad (x, y) \in \Omega \\
U_{0x} + (hV_0)_y &= 0 , \quad (x, y) \in \Omega \\
\lim_{y \to 0} V_0(x, y) &= 0 , \\
\lim_{|y| \to \infty} U_0(x, y) &= U_\infty(x).
\end{align*}
\]

(28)

This is the Euler model of aerodynamics. Remark that only one condition from (26) occurs and so, near the boundary \( y = 0 \) one boundary condition was lost in passing from the given model to the outer one. Hence, the mathematical boundary layer is near \( y = 0 \). In it we assume \( u(\text{Re}, x, y) \equiv \tilde{u}(\varepsilon, x, y) \sim \tilde{u}_0(x, y) \), \( v(\text{Re}, x, y) \equiv \tilde{v}(\varepsilon, x, y) \sim \varepsilon \tilde{v}_0(x, y) \), \( p(\text{Re}, x, y) \equiv \tilde{p}(\varepsilon, x, y) \sim \tilde{p}_0(x, y) \) where \( \tilde{y} = y/\varepsilon \). Changing variables \((x, y) \to (x, \tilde{y})\) in (25), (26) introducing the inner expansions in the resulting problem and taking into account the \( 1 \times 1 \) matching principle we obtain the following model of first order inner asymptotic approximation for \( \text{Re} \to \infty \) of the Navier–Stokes model, called the Prandtl model,

\[
\begin{align*}
\begin{cases}
\tilde{u}_0 \tilde{u}_{0x} + \tilde{v}_0 \tilde{u}_{0\tilde{y}} &= -\tilde{p}_{0x} + \tilde{u}_0 \tilde{v}_{0\tilde{y}} , \\
0 &= -\tilde{p}_{0\tilde{y}} , \\
\tilde{u}_{0x} + \tilde{v}_{0\tilde{y}} &= 0 ,
\end{cases}
\end{align*}
\]

(29)

\[
\begin{align*}
\tilde{u}_0(x, 0) = \tilde{v}_0(x, 0) = 0 , \\
\lim_{\tilde{y} \to -\infty} \tilde{u}_0(x, \tilde{y}) = U_0(x, 0) , & \quad \lim_{\tilde{y} \to -\infty} \tilde{v}_0(x, \tilde{y}) = 0 .
\end{align*}
\]

(30)

As \( \tilde{p}_0 \) is constant across the boundary layer, we take \( \tilde{p}_{0x}(x, \tilde{y}) = \tilde{p}_{0x}(x, 0) \approx P_{0x}(x, 0) = -U_0 U_{0x} \) from (28) continued to \( \Omega \cup \partial \Omega \) and taking into account (28).

Similarly, for \( \text{Re} \to 0 \), in the first approximation the Navier–Stokes model may be replaced by two models: the Stokes model near the wall \( \partial \Omega \) and the Oseen model, far from it.

**Cahn–Hillard model.** Let us now apply the method of inner-outer expansions to the Cahn–Hillard model

\[
\begin{align*}
u_t &= \varepsilon^2 \Delta \mu \\
\mu &= F'(u) - \varepsilon^2 \Delta u
\end{align*}
\]

(31)

\( (\varepsilon, x, t) \in \mathbb{R}^+_t \times \Omega \subset \mathbb{R}^n \times \mathbb{R}^+_t \)

\[
\begin{align*}
\lim_{t \to 0} u(\varepsilon, x, t) = U_0(x) , & \quad (\varepsilon, x) \in \mathbb{R}^+_t \times \Omega \\
n \text{grad} \mu &= 0 , & n \text{ grad } u &= 0 , \quad (x, t) \in \mathbb{R}^+_t \times \partial \Omega
\end{align*}
\]

(32)  \quad (33)
where the unknowns are the concentration and the chemical potential \( u, \mu : \mathbb{R}_+^+ \times \Omega \times \mathbb{R}_+^+ \to \mathbb{R} \); \( n \) is the normal to \( \partial \Omega \); \( F \) is the known bulk free energy density; \( t' \equiv d/du \); \( t = t_f/\varepsilon^2 \) where \( t_f \) is the physical time and \( \varepsilon(>0) \) is a small parameter. Denote by \( \Gamma \) a unknown smooth \( n - 1 \) hypersurface, evolving in time; it is described by the equation \( \varphi(\varepsilon, x, t) = 0 \) and is defined by the property that it splits \( \Omega = \Omega_+ \cup \Omega_+ \cup \Gamma, \Gamma \cap \partial \Omega = \emptyset \), such that \( u > u_0^+ \) in \( \Omega_+ \) and \( u < u_0^- \) in \( \Omega_- \) for all points whose distance from \( \Gamma \) is greater than \( \varepsilon \). Introducing the outer expansion \( u(\varepsilon, x, t) \sim u_0(x, t) + \varepsilon U_1(x, t) + \varepsilon^2 U_2(x, t), \mu(\varepsilon, x, t) \sim M_0(x, t) + \varepsilon M_1(x, t) + \varepsilon^2 M_2(x, t) \) where \( x, t, U_i, M_i = O(1) \) for \( \varepsilon \to 0 \) into (31)--(33) we obtain the following models of first, second and third order outer asymptotic approximation for the Cahn–Hilliard model for \( \varepsilon \to 0 \)

\[
\begin{align*}
\left\{ \begin{array}{l}
U_{0t} = 0, \\
M_0 = F'(U_0),
\end{array} \right. & (x, t) \in \Omega \times \mathbb{R}_+^+ \\
\lim_{t \to 0} U_0(x, t) = u_0(x), (x, t) \in \Omega \times \mathbb{R}_+^+ & n \text{ grad } M_0 = n \text{ grad } U_0 = 0, (x, t) \in \mathbb{R}_+^+ \times \partial \Omega
\end{align*}
\]

\[
\begin{align*}
\left\{ \begin{array}{l}
U_{1t} = 0, \\
M_1 = F''(U_0) U_1,
\end{array} \right. & (x, t) \in \Omega \times \mathbb{R}_+^+ \\
\lim_{t \to 0} U_1(x, t) = 0, (x, t) \in \Omega \times \mathbb{R}_+^+ & n \text{ grad } U_1 = n \text{ grad } M_1 = 0, (x, t) \in \partial \Omega \times \mathbb{R}_+^+
\end{align*}
\]

\[
\begin{align*}
\left\{ \begin{array}{l}
U_{2t} = \Delta M_0, \\
M_2 = F''(U_0) U_2 + \frac{1}{2} F'''(U_0) U_1^2 - \Delta U_0,
\end{array} \right. & (x, t) \in \Omega \times \mathbb{R}_+^+ \\
\lim_{t \to 0} U_2(x, t) = 0, x \in \Omega, & n \text{ grad } U_2 = n \text{ grad } M_2 = 0, (x, t) \in \partial \Omega \times \mathbb{R}_+^+
\end{align*}
\]

Although these models are defined for \( x \in \Omega \), however the approximation they realize is valid only for \( \Omega \setminus \Omega_1 \), where \( \Omega_1 \) is a mathematical inner layer around \( \Gamma \).

In \( \Omega_1 \) the current point is \( x + \varepsilon z \text{ grad } \varphi \) for \( x \in \Omega \setminus \Omega_1 \) and we assume the following asymptotic behavior: \( u(\varepsilon, x + \varepsilon z \text{ grad } \varphi, t) \equiv \tilde{u}(\varepsilon, z, x, t) \sim \tilde{u}_0(z, x, t) + \varepsilon \tilde{u}_1(z, x, t) + \varepsilon^2 \tilde{u}_2(z, x, t), \mu(\varepsilon, x + \varepsilon \text{ grad } \varphi, t) \equiv \tilde{\mu}(\varepsilon, z, x, t) \sim \tilde{\mu}_0(z, x, t) + \varepsilon \tilde{\mu}_1(z, x, t) + \varepsilon^2 \tilde{\mu}_2(z, x, t) \) where \( z, x, t, \tilde{u}_i, \tilde{\mu}_i = O(1) \) for \( \varepsilon \to 0 \) and \( z = \varepsilon^{-1} \varphi(\varepsilon, x, t) \equiv \varepsilon^{-1} \varphi_0(x, t) + \varphi_1(x, t) \). \( \varepsilon z \) is the distance from the point \( x \) to \( \Gamma \). In order to express all the variation in the direction normal to \( \Gamma \) by the dependence on \( z \), it is assumed that each function \( \tilde{\mu}_i \) and \( \tilde{u}_i \), denoted generally by \( \tilde{q}_i \), is such that \( \tilde{q}_i(\varepsilon, z, x, t) \sim \tilde{q}_i(\varepsilon, z, x + \alpha \text{ grad } \varphi, t) \) i.e. \( \text{ grad } \varphi \cdot \text{ grad } x \tilde{q}_i = 0 \). Denote by \( m = \text{ grad } \varphi, k = \Delta \varphi, |\text{ grad } \varphi| = 1 \). Then \( |\text{ grad } \varphi_0 + \varepsilon \text{ grad } \varphi_1| = 0 \) hence \( |\text{ grad } \varphi_0| = 1 \) and \( |\text{ grad } \varphi_1| = 0 \). This last equality implies that \( \varphi_1(x, t) = \varphi_1(t) \) and \( k = \Delta \varphi_0 + \varepsilon \Delta \varphi_1 = \Delta \varphi_0 \). Introducing the inner series into (31)--(33), taking into account these remarks, the fact that the solutions of the outer models are \( U_0 = u_0(x), M_0 = F'(u_0), U_1 = M_1 = 0, U_2 = t\Delta(F'(u_0)) \), \( M_2 = t F''(u_0) \Delta(F'(u_0)) - \Delta u_0 \) and using the \( 3 \times 3 \) matching principle we obtained the following models of first, second and third order
inner asymptotic approximation of the Cahn–Hilliard model for $\varepsilon \to 0$

\begin{align}
&\varphi_0 t = 0, \quad (z, x, t) \in \mathbb{R}_+^+ \times \Omega \times \mathbb{R}_+^+, \quad x + \varepsilon \varepsilon \text{grad} \varphi \in \Omega_1 \\
&\tilde{u}_0 t + \tilde{u}_0 x \varphi_1 t = \tilde{\mu}_0 + x + \varepsilon \varepsilon \text{grad} \varphi \in \Omega_1 \\
&\tilde{\mu}_0 = F'(\tilde{u}_0) - \tilde{u}_0 \varphi_0, \quad (z, x, t) \in \mathbb{R}_+^+ \times \Omega \times \mathbb{R}_+^+, \quad x + \varepsilon \varepsilon \text{grad} \varphi \in \Omega_1 \\
&\lim_{t \to 0} \tilde{u}_0(z, x, t) = U_0(x), \quad (z, t) \in \mathbb{R}_+^+ \times R_*^+ \\
&\lim_{z \to -\infty} \tilde{u}_0(z, x, t) = U_0(x), \quad (x, t) \in \Omega \times \mathbb{R}_+^+ \\
&\lim_{z \to -\infty} \tilde{\mu}_0(z, x, t) = F'(U_0(t)), \quad (x, t) \in \Omega \times \mathbb{R}_+^+
\end{align}

\begin{align}
&\varphi_1 = \varphi_1(t), \quad (z, x, t) \in \mathbb{R}_+^+ \times \Omega \times \mathbb{R}_+^+, \quad x + \varepsilon \varepsilon \text{grad} \varphi \in \Omega_1 \\
&\tilde{u}_1 t + \varphi_1 \tilde{u}_1 x = \tilde{\mu}_1 + \Delta \varphi_0 + \tilde{\mu}_1, \quad (z, x, t) \in \mathbb{R}_+^+ \times \Omega \times \mathbb{R}_+^+, \quad x + \varepsilon \varepsilon \text{grad} \varphi \in \Omega_1 \\
&\tilde{\mu}_1 = \tilde{u}_1 F''(\tilde{u}_0) - \tilde{u}_1 \Delta \varphi_0, \quad (z, x, t) \in \mathbb{R}_+^+ \times \Omega \times \mathbb{R}_+^+, \quad x + \varepsilon \varepsilon \text{grad} \varphi \in \Omega_1 \\
&\lim_{t \to 0} \tilde{u}_1(z, x, t) = 0, \quad (z, x) \in \mathbb{R}_+^+ \times \Omega \\
&\lim_{\varepsilon \to 0} \left[ -\frac{\varphi_0}{\varepsilon} \tilde{u}_0 x + \varphi_0 + \varphi_1, x, t \right] - \varphi_0 \tilde{u}_2 x + \varphi_0 + \varphi_1, x, t \right] \left[ \varphi_0 + \varphi_1, x, t \right] - \frac{\varphi_0}{\varepsilon} \tilde{\mu}_1 x + \varphi_0 + \varphi_1, x, t \right] \left[ \varphi_0 + \varphi_1, x, t \right] -
\end{align}

\begin{align}
&\frac{\varphi_0}{\varepsilon} \tilde{\mu}_1 x + \varphi_0 + \varphi_1, x, t \right] \left[ \varphi_0 + \varphi_1, x, t \right] -
\end{align}

\begin{align}
&\tilde{u}_2 t + \varphi_1 \tilde{u}_2 x = \Delta \tilde{u}_0 + \tilde{u}_1 x \varphi_0 + \tilde{\mu}_2, \quad (z, x, t) \in \mathbb{R}_+^+ \times \Omega \times \mathbb{R}_+^+, \quad x + \varepsilon \varepsilon \text{grad} \varphi \in \Omega_1 \\
&\tilde{\mu}_2 = -\Delta \tilde{u}_0 - \tilde{u}_2 x - \tilde{u}_1 x \Delta \varphi_0 + \frac{\tilde{u}_1}{2} F''(\tilde{u}_0) + \tilde{u}_2 F''(\tilde{u}_0), \quad (z, x, t) \in \mathbb{R}_+^+ \times \Omega \times \mathbb{R}_+^+, x + \varepsilon \varepsilon \text{grad} \varphi \in \Omega_1 \\
&\lim_{t \to 0} \tilde{u}_2(z, x, t) = 0, \quad (z, x) \in \mathbb{R}_+^+ \times \Omega \quad (x, t) \in \Omega \times \mathbb{R}_+^+ \\
&\lim_{\varepsilon \to 0} \left[ \frac{\varphi_0}{\varepsilon} \tilde{u}_0 x z + \frac{\varphi_0}{\varepsilon} \tilde{u}_0 x z + \frac{\varphi_0}{\varepsilon} \tilde{u}_1 x z + \tilde{u}_1 - \frac{\varepsilon}{\varepsilon} \tilde{u}_1 x z + \frac{\varepsilon}{\varepsilon} \tilde{u}_2 x z \right] = \frac{\varepsilon}{\varepsilon} (\text{grad} \varphi_0)^2 \text{grad}(\text{grad} U_0) + z \text{grad} \varphi_0 \text{grad} U_1 + U_2, \quad (x, t) \in \Omega \times \mathbb{R}_+^+ \\
&\lim_{\varepsilon \to 0} \left[ \frac{\varepsilon}{\varepsilon} \tilde{\mu}_0 x + \frac{\varepsilon}{\varepsilon} \tilde{\mu}_0 x + \frac{\varepsilon}{\varepsilon} \tilde{\mu}_1 x + \tilde{\mu}_1 - \frac{\varepsilon}{\varepsilon} \tilde{\mu}_2 x + \frac{\varepsilon}{\varepsilon} \tilde{\mu}_2 x \right] = \frac{\varepsilon}{\varepsilon} (\text{grad} \varphi_0)^2 \text{grad}(\text{grad} M_0) + z \text{grad} \varphi_0 \text{grad} M_1 + M_2, \quad (x, t) \in \Omega \times \mathbb{R}_+^+
\end{align}

All the relations involving $\lim_{\varepsilon \to 0}$ were obtained by writing the $3 \times 3$ matching principle in the form: the 3 terms outer expansion of $(\tilde{u}_0 + \varepsilon \tilde{u}_1 + \varepsilon^2 \tilde{u}_2)(z, x, t)$ is equal to the 3 terms inner expansion of $(U_0 + \varepsilon U_1 + \varepsilon^2 U_2)(x + \varepsilon \varepsilon \text{grad} p, t)$ whence the equality of leading terms reads $\lim_{\varepsilon \to 0} [U_0(x + \varepsilon \varepsilon \text{grad} \varphi, t) + \varepsilon U_1(x + \varepsilon \varepsilon \text{grad} \varphi, t) + \varepsilon^2 U_2(x + \varepsilon \varepsilon \text{grad} \varphi, t)] = z = 0(1), \quad x = 0(1)$
\[
\lim_{\varepsilon \to 0} \left[ \tilde{u}_0 \left( \frac{\varphi_0(x)}{\varepsilon} + \varphi_1(x), x, t \right) + \varepsilon \tilde{u}_1 \left( \frac{\varphi_0(x)}{\varepsilon} + \varphi_1(x), x, t \right) + \varepsilon^2 \tilde{u}_2 \left( \frac{\varphi_0(x)}{\varepsilon} + \varphi_1(x), x, t \right) \right], \text{ etc.}
\]

We assumed that \( \tilde{u}_1(\infty, x, t), \tilde{u}_2(\infty, x, t) < \infty \). Up to slight differences (concerning \( \varphi_1, U_1 \) and \( M_1 \)) our results coincide with those of [2]. We obtained \( U_1 = M_1 = 0 \) assuming a certain initial condition at \( t \to 0 \) while in [2] the problem (31)–(33) is studied in a larger context, allowing for \( t \) various scales.

The choice of the form of the (inner) expansion is the point where a lot of physical information may be introduced into the model. It mainly concerns the order of the leading term, its independence of some variables, the thickness of the inner layer. In some cases, when the model of asymptotic approximation is deduced by heuristical bases, it is important to see which are the hypotheses underlying it and allowing to know the order of magnitude of various quantities. In the next section we shall show how to use the freedom in that choice in order to derive a model in the dynamics of atmosphere.

Scales; dimensionless quantities; order relations between dimensionless numbers; physical quantities whose order is unknown. Asymptotic hypotheses. Models of asymptotic approximation for synoptic-scale flows. All the questions will be discussed in order to derive the quasigeostrophic model as a model of fourth order asymptotic approximation for small Rossby numbers of the initial-boundary value problem for the primitive equations in dynamics of atmosphere. The quasigeostrophic model governs sufficiently well the synoptic-scale flow in the middle latitude troposphere outside the planetary boundary layer when the diabatic processes are neglected. This flow takes place in a horizontal layer of extent \( L \) (synoptic scale) and thickness \( h \ll L \) (troposphere thickness) where the mean planetary flow has a mean horizontal velocity \( U \). A characteristic pressure is the climatological mean pressure at the sea level \( P \). Let \( c_v \) be the specific heat, \( R = R^*/\mu \), \( R^* \) the universal constant of gases and \( \mu \), the molar mass of air. Denote by \( f_0 = 2\Omega \sin \theta_0 \) the Coriolis parameter and introduce the following geometrical and physical characteristics of the Earth: its radius \( a \), the gravitational acceleration \( g \) and the angular velocity \( \Omega \). \( \theta_0 \) is the latitude around which is centered the domain of interest. The primitive equations consists of: Euler equations, continuity equation, energy equation and the equation of (classical thermodynamics) state for ideal gases. Let us introduce the following dimensionless variables and parameters: \( t' = tU/L \) (time); \( x' = x/L \), \( y' = y/L \) (horizontal coordinates), \( z' = z/h \) (altitude), \( \alpha = L/a \) (aspect ratio characterizing the shape of the domain), \( \delta = h/L \) (aspect ratio of the domain), \( \chi = f_0U/g \) (number of non-hydrostaticity equal to the Coriolis force-weight ratio), \( \varepsilon = U/(f_0L) \) the Rossby number, \( \nu = R_0/c_v = \gamma - 1 \) (\( \gamma \)- the adiabatic exponent), \( u = Uu' \), \( v = Uv' \) (velocity components in \( x \) and \( y \) directions respectively), \( p = Pp' \) (pressure), \( w = \eta(Uh/L)w' \) (vertical velocity), \( T = (gh/R)T' \) (temperature), \( \rho = (P/gh)\rho' \) (density). Remark two different length scales and only one time scale, corresponding to the horizontal component of the motion. The vertical velocity has an a priori unknown order. Taking into account the generally
admissible values

\[ L = 10^6 m, U = 10 m/s, \nu = 1.4, a = 6 \cdot 10^6 m, h = 10^4 m, g = 10 m/s, f_0 = 10^{-4} s^{-1} \]

it follows \( \varepsilon = 10^{-1}, \delta = 10^{-2}, \alpha = 2 \cdot 10^{-1}, \chi = 10^{-4}, \nu = 0.4 \) which suggested the order relations \( \delta = \Theta(\varepsilon^2), \alpha = \Theta(\varepsilon), \chi = \Theta(\varepsilon^4), \nu = \Theta(1) \) for \( \varepsilon \to 0 \). Also we supposed that

\[ \eta \sim \eta_0 + \varepsilon \eta_1 + \varepsilon^2 \eta_2 + \varepsilon^3 \eta_3 + \cdots \quad \text{for} \quad \varepsilon \to 0. \]

The equations of motion read (after dropping the primes)

\[
\rho \left[ \chi \varepsilon \delta^{-1} \frac{d \eta}{dt} + \chi \eta \varepsilon \delta \frac{\cos(\theta_0 + \alpha y)}{\cos \theta_0} \frac{\cos(\theta_0 + \alpha y)}{\sin \theta_0 \cos \theta_0} \right]
\]

\[
(1 + \alpha \delta z) v + \chi \eta \frac{\cos^2(\theta_0 + \alpha y)}{\sin \theta_0 \cos \theta_0} (1 + \alpha \delta z) w \right] = -\frac{\partial p}{\partial x},
\]

\[
\rho \left[ \chi \varepsilon \delta^{-1} \frac{d \eta}{dt} + \chi \eta \varepsilon \delta \frac{\cos(\theta_0 + \alpha y)}{\cos \theta_0} \frac{v}{\cos \theta_0} \right]
\]

\[
(1 + \alpha \delta z) u \right] = -\frac{\cos(\theta_0 + \alpha y)}{\cos \theta_0} \frac{\partial p}{\partial y},
\]

\[
\rho \left[ \chi \varepsilon \delta \frac{d \eta}{dt} - \chi \varepsilon \delta \frac{\cos(\theta_0 + \alpha y)}{\cos \theta_0} \frac{u^2 + v^2}{\cos \theta_0 \cos \theta_0} \right]
\]

\[
(1 + \alpha \delta z) u \right] + \frac{\cos(\theta_0 + \alpha y)}{\cos \theta_0} \frac{1 + \alpha \delta z}{(1 + \alpha \delta z)} \right] =
\]

\[
-\frac{\cos(\theta_0 + \alpha y)}{\cos \theta_0} \frac{1 + \alpha \delta z}{(1 + \alpha \delta z)} \frac{\partial p}{\partial z},
\]

\[
\frac{d p}{dt} + \rho \nabla \cdot \vec{v} = 0,
\]

\[
\frac{d T}{dt} + \nu T \nabla \cdot \vec{v} = 0,
\]

\[ p = \rho T, \]

where \( \frac{d}{dt} = \frac{\cos(\theta_0 + \alpha y)}{\cos \theta_0} \frac{1 + \alpha \delta z}{\cos \theta_0} \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + \frac{\cos(\theta_0 + \alpha y)}{\cos \theta_0} v \frac{\partial}{\partial y} + \eta \frac{\cos(\theta_0 + \alpha y)}{\cos \theta_0} \right] (1 + \alpha \delta z) \right) \frac{\partial}{\partial z}. \]

Applying the regular perturbation method i.e. looking for asymptotic expansion of an unknown function in the form \( Y \sim Y_0 + \varepsilon Y_1 + \varepsilon^2 Y_2 + \varepsilon^3 Y_3 + \cdots, \varepsilon \to 0 \) where \( Y \) may be \( u', v', w', \rho', p' \) or \( T' \), the coefficients \( Y_0, Y_1, \ldots \) being functions of \( t, x, y \) and \( z \) only, we deduced [3] the equations of the first fourth orders of asymptotic approximation as \( \varepsilon \to 0 \) of the primitive equations. On physical bases we also assumed that \( p_0 = p_0(x, y, z), \rho_0 = \rho_0(x, y, z), T_0 = 1, \rho_1 = p_1(x, y, z), \rho_1 = p_1(x, y, z), T_1 = T_1(x, y, z) \). The boundary and initial conditions are not specified since they are obtained formally without any restriction.

The first order approximation equations are

\[
\begin{cases}
\frac{\partial p_0}{\partial x} = 0, \quad \frac{\partial p_0}{\partial y} = 0, \quad \rho_0 = -\frac{\partial p_0}{\partial z}, \quad \eta_0 \frac{\partial w_0}{\partial z} + \frac{\partial v_0}{\partial y} + \frac{\partial u_0}{\partial x} = 0, u_0 \frac{\partial p_0}{\partial x} + v_0 \frac{\partial p_0}{\partial y} + \eta_0 w_0 \frac{\partial p_0}{\partial z} + \\
+ \rho_0 \left( \eta_0 \frac{\partial w_0}{\partial z} + \frac{\partial v_0}{\partial y} + \frac{\partial u_0}{\partial x} \right) = 0, \quad \rho_0 = p_0.
\end{cases}
\]
They imply \( \eta_0 = 0 \). The second order approximations are

\[
\begin{aligned}
\frac{\partial p_1}{\partial x} &= 0, \quad \frac{\partial p_1}{\partial y} = 0, \quad p_1 = -\frac{\partial p_1}{\partial z}, \quad u_0 \frac{\partial T_1}{\partial x} + v_0 \frac{\partial T_1}{\partial y} + \frac{\partial v_1}{\partial y} - y \tan \theta_0 \frac{\partial u_0}{\partial y} - \tan \theta_0 v_0 + \frac{\partial u_1}{\partial x} + \\
&+ \eta_1 \frac{\partial w_0}{\partial z} = 0,
\end{aligned}
\]

\[
\begin{aligned}
u_0 \frac{\partial p_1}{\partial x} + v_0 \frac{\partial p_1}{\partial y} + \rho_0 \left( \frac{\partial v_1}{\partial y} - y \tan \theta_0 \frac{\partial v_0}{\partial y} - \tan \theta_0 \cdot v_0 + \frac{\partial u_1}{\partial x} \right) + \eta_1 \frac{\partial (\rho_0 w_0)}{\partial z} = 0, \quad p_1 = \rho_0 T_1 + \rho_1.
\end{aligned}
\]

Taking into account (37) and (38) it follows \( \eta_1 = 0 \) on the third approximation we have

\[
\begin{aligned}
- \rho_0 v_0 &= -\frac{\partial p_2}{\partial x}, \quad \rho_0 u_0 = -\frac{\partial p_2}{\partial y}, \quad \rho_2 = -\frac{\partial p_2}{\partial z}, \quad u_0 \frac{\partial T_2}{\partial t} + v_0 \frac{\partial T_2}{\partial y} + \frac{\partial v_2}{\partial y} - y \tan \theta_0 \frac{\partial v_1}{\partial y} \\
&- \frac{1}{2} y^2 \frac{\partial v_0}{\partial y} - \\
- \tan \theta_0 v_1 - y v_0 + \frac{\partial u_2}{\partial x} + \eta_2 \frac{\partial w_0}{\partial z} = 0, \quad \frac{\partial p_2}{\partial t} + u_0 \frac{\partial p_2}{\partial x} + v_0 \frac{\partial p_2}{\partial y} + \rho_0 \left( \frac{\partial v_2}{\partial y} - y \tan \theta_0 \frac{\partial v_1}{\partial y} - \frac{1}{2} y^2 \frac{\partial v_0}{\partial y} - \\
- \tan \theta_0 v_1 - y v_0 + \frac{\partial u_2}{\partial x} \right) + \eta_2 \frac{\partial (\rho_0 w_0)}{\partial z} = 0, \quad p_2 = \rho_2 + \rho_1 T_1 + \rho_0 T_2.
\end{aligned}
\]

Taking into account (37) and (38), from (39) we get \( \eta_2 \neq 0 \) and we put it \( \eta_2 = 1 \) (if \( \eta_0 \neq 1 \) we change accordingly \( w_0 \)). Therefore \( \eta = O(\varepsilon^2), \varepsilon \to 0 \), hence the vertical velocity is two orders of magnitude smaller than the horizontal components. Finally the equations of the fourth order asymptotic approximation read

\[
\begin{aligned}
\rho_0 \left( \frac{d_0 u_0}{dt} - 2 \tan^{-1} 2 \theta_0 y v_0 - v_1 \right) - \rho_1 v_0 &= -\frac{\partial p_3}{\partial x}, \quad p_0 \left( \frac{d_0 v_0}{dt} + \tan^{-1} 2 \theta_0 y u_0 + u_1 \right) + \rho_1 u_0 = -\frac{\partial p_3}{\partial y} + \\
y \tan \theta_0 \frac{\partial p_3}{\partial y} .
\end{aligned}
\]

They represent the quasigeostrophic approximation. (We denoted \( \frac{d}{dt} = \frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x} + v_0 \frac{\partial}{\partial y} \).)

The main advantage of decomposing the primitive equations into (37)-(40) is that in each equation of models (1)-(4) the order of all quantities (with respect to \( \varepsilon \)) is the same, enabling an easier computation. The asymptotic study puts also into evidence the order of the quantities from the primitive equations, corresponding to the case of velocity of the quasigeostrophic approximation.

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