MAXIMUM ENTROPY PRINCIPLES FOR DISORDERED SPINS

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Summary. We transform nonstationary independent random fields with exponential Radon-Nikodym factors and study the asymptotics of the transformed processes. As applications we deduce conditional limit theorems for such random fields, and we study a Curie-Weiss-type mean-field model of a quenched mixed magnetic crystal. This model has quenched site disorder and frustration but non-random coupling constants. We find a continuous phase transition with critical exponents equal to those of the classical Curie-Weiss theory.

1 Introduction

This paper is a sequel to the study of large deviations of nonstationary processes, or disordered spins, initiated in [Se1] and [Se2]. We transform the original process by a family of exponential Radon-Nikodym factors, prove that the collection of transformed processes is tight, and characterize the limit points. The limit points are solutions to variational principles, hence the term 'maximum entropy'. In our case the original processes are independent and the modifying Radon-Nikodym factors are functions of the empirical distribution. Thus this is a way of introducing dependence among the previously independent variables.

In Section 2 we remind the reader of the setting in [Se1] and state the general theorems about tightness, limit points, and large deviations of the transformed processes. Proofs appear at the end of the paper in Section 5, and the middle sections 3 and 4 are devoted to two applications. Regarding the two tasks mentioned above, tightness and characterization of limit points, our results on the latter are far more complete than on the former. We

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use only rather soft large deviation techniques and consequently get convergence only in a very weak sense. It is clear that there is room for improvement here. Our methods do give precise information about the asymptotics of empirical averages, which is useful for studying models from statistical mechanics.

Section 3 generalizes the following well-known fact: A fair coin-tossing process, conditioned on consistently producing tails on no fewer than, say, $3/4$ of the tosses, behaves asymptotically like a biased coin with probabilities $(1/4, 3/4)$. More precisely, if $P = \left( \frac{1}{2} \delta_0 + \frac{1}{2} \delta_1 \right)^{\otimes N}$ is the law of an infinite sequence of fair coins and $S_n$ is the number of tails in the first $n$ tosses, then the conditioned laws

$$P(\cdot \mid S_n \geq \frac{3}{4}n)$$

converge to $(\frac{1}{4} \delta_0 + \frac{3}{4} \delta_1)^{\otimes N}$ as $n \to \infty$. This convergence holds not only in the weak topology but also in an information-theoretic sense as proved by Csiszár, who termed this phenomenon ‘asymptotic quasi-independence’ [Cs2]. We prove analogous results for nonstationary processes, which means that the coin does not have to be the same for each toss.

Section 4 presents an application to the statistical mechanics of quenched disorder. We study a mean-field model of a quenched mixed magnetic crystal where atoms of two different types, A and B, are randomly distributed on the lattice sites and their spins interact in a Curie-Weiss-type mean-field fashion. With the results of section 2 we characterize the limiting Gibbs measures and study the specific magnetization. The model exhibits a continuous phase transition with nonzero spontaneous magnetization at temperatures below critical, and with critical exponents equal to those of the classical Curie-Weiss theory. We find no “spin glass” phase where an overlap order parameter would be strictly positive without the accompanying spontaneous magnetization.

However, the interaction of the A- and B-atoms does produce interesting effects not visible in a pure A- or B-model. The parameters of the model are chosen so that a pure A-model is a ferromagnetic Curie-Weiss with critical temperature 1, and a pure B-model is an antiferromagnetic Curie-Weiss, hence stays in the paramagnetic phase at arbitrarily low temperatures and has no phase transition. But by strengthening the AB-interaction we can induce ferromagnetic ordering in the quenched model at arbitrarily high temperatures despite the resistance to alignment by the antiferromagnetic BB-coupling. Furthermore, the spontaneous magnetization of the B-atoms can be strictly greater than that of the A-atoms.

To our knowledge, maximum entropy principles have not been studied previously for nonstationary processes. Our development owes much to the elegant treatment of uniformly ergodic Markov chains by Bolthausen and Schmuck [BS], as will become clear to the reader who compares the arguments. For further related results see [B], [CC], [CCC], [HR], [M], [Sc], and their references.
2 The setting and the general results

Our setting is that of [Se1] with some changes in notation: At each site $i$ of $\mathbb{Z}^d$ we have a spin $\sigma_i$ with values in a Polish space $\mathcal{S}$, and $\Omega = \mathcal{S}^{\mathbb{Z}^d}$ denotes the space of spin configurations. $\mathbb{Z}^d$ acts on $\Omega$ by translations: $(\theta_j \sigma)_i = \sigma_{i+j}$. Additionally, there is a Polish space $\Sigma$ of *quenched variables*, as they are called in the physics literature. (In [Se1] we called them simply *parameters.*) Generic elements of $\Sigma$ are denoted by $\mathbf{x}$ and $\mathbf{y}$. We assume that $\mathbb{Z}^d$ acts on $\Sigma$ by some homeomorphisms $\theta_i$ that satisfy $\theta_i \circ \theta_j = \theta_{i+j}$. All infinite-volume limits are taken along the sequence of cubes $V_n = [-n, n]^d \cap \mathbb{Z}^d$.

We assume given a measurable map $\mathbf{x} \mapsto p^\mathbf{x}$ from $\Sigma$ into the space $\mathcal{M}_1(\mathcal{S})$ of probability measures on $\mathcal{S}$ (all measurable structures are Borel and all topologies on measures are generated by bounded continuous functions), and form the product measure $P^\mathbf{x} = \otimes_{i \in \mathbb{Z}^d} p^{\theta_i \mathbf{x}}$. The goal is to prove asymptotic results for the spin process $(\sigma_i)$ in terms of the empirical distribution and empirical field, defined by

$$L_n = \frac{1}{|V_n|} \sum_{i \in V_n} \delta_{\sigma_i}$$

and

$$R_n = \frac{1}{|V_n|} \sum_{i \in V_n} \delta_{\theta_i \sigma}.$$

The laws of these two measure-valued random variables under the probabilities $P^\mathbf{x}$ were studied in [Se1]. Here we add some new ingredients: Assume given a measurable function $F : \mathcal{M}_1(\mathcal{S}) \to [-\infty, c]$, where $c$ is a finite constant. Define probability measures $\gamma_n^\mathbf{x}$ on $\Omega$ by setting

$$\gamma_n^\mathbf{x}(f) = \frac{1}{Z_n^\mathbf{x}} P^\mathbf{x}(f \mid V_n \mid F(L_n))$$

for bounded Borel functions $f$ on $\Omega$, where $Z_n^\mathbf{x} = P^\mathbf{x}(e \mid V_n \mid F(L_n))$ is the appropriate normalizing factor. In case $P^\mathbf{x}(F(L_n) = -\infty) = 1$, set $\gamma_n^\mathbf{x} = P^\mathbf{x}$. Define the measures $\zeta_n^\mathbf{x}$ by $\zeta_n^\mathbf{x}(f) = \gamma_n^\mathbf{x}(R_n(f))$. Our motivation for studying these measures comes from two examples:

1. Taking $F = \log 1_C$ for a set $C$ amounts to studying the spin process conditioned on the events $\{L_n \in C\}$. This is interesting when the event $\{L_n \in C\}$ represents a deviation, that is, its $P^\mathbf{x}$-probability vanishes as $n \to \infty$.

2. In statistical mechanical language, $\gamma_n^\mathbf{x}$ is a Gibbs measure for the spins $\sigma_i$ located on the sites of $V_n$ whose interaction energy is $-|V_n|F(L_n(\sigma))$ and whose noninteracting state is described by the measure $P^\mathbf{x}$.

The goal of this section is to characterize the limit points of the measures $\gamma_n^\mathbf{x}$ and $\zeta_n^\mathbf{x}$ as $n \to \infty$. Sections 3 and 4 present applications corresponding to items (1) and (2) above. Proofs for this section are presented in Section 5.

We first develop the terminology and assumptions needed for the results. Throughout, $\pi$ denotes an element of the space $\mathcal{M}_0(\Sigma)$ of invariant probability measures on $\Sigma$. Most
of the functions, sets, and measures that appear in our definitions and theorems depend on \( \pi \), but to keep the notation simple this dependence is suppressed, as it is fairly easy to detect from the context. We say that \( x \) is generic for \( \pi \) if both

\[
(2.2) \quad \pi = \lim_{n \to \infty} \frac{1}{|V_n|} \sum_{i \in V_n} \delta_{\theta_i x}
\]

and

\[
(2.3) \quad \pi(y : p^y \in \cdot) = \lim_{n \to \infty} \frac{1}{|V_n|} \sum_{i \in V_n} \delta_{P_{\theta_i x}}
\]

hold in the weak topologies of probability measures on \( \Sigma \) and \( M_1(\Omega) \), respectively. The left-hand side in (2.3) is the law of the random measure \( y \mapsto p^x \) under \( \pi \). (2.3) follows from (2.2) in case \( x \mapsto p^x \) (and hence \( x \mapsto P^x \)) is continuous. In general, \( \pi \)-a.e. \( x \) is generic for \( \pi \) whenever \( \pi \) is ergodic. In [Se1], conditions (2.2) and (2.3) were termed quasiregularity and \( P \)-regularity, respectively.

Given \( \pi \in M_\emptyset(\Sigma) \), an entropy for \( \nu \in M_1(\Sigma) \) is defined by

\[
(2.4) \quad K(\nu) = \sup_{f \in b^S} \left\{ \nu(f) - \int_{\Sigma} \log p^y(e^f) \pi(dy) \right\}.
\]

\( b^S \) is the Borel field of \( S \), and \( b^A \) denotes the space of bounded \( A \)-measurable functions for any \( \sigma \)-field \( A \).

The upper and lower semicontinuous regularizations of \( F \) are defined by

\[
F^u(\nu) = \inf_{G: G \text{ open}, G \ni \nu} \left\{ \sup_{\rho, \rho \in G} F(\rho) \right\}
\]

and

\[
F^l(\nu) = \sup_{G: G \text{ open}, G \ni \nu} \left\{ \inf_{\rho, \rho \in G} F(\rho) \right\}.
\]

The following assumption characterizes the setting in which our limit theorems for a fixed \( x \) are valid.

**Assumption A.** \( p^x(F(L_n) = -\infty) < 1 \) for large enough \( n \), \( x \) is generic for \( \pi \), and

\[
(2.5) \quad r_1 \equiv \inf_{\nu \in M_1(S)} \{ K(\nu) - F^u(\nu) \} = \inf_{\nu \in M_1(S)} \{ K(\nu) - F^l(\nu) \} < \infty.
\]

In case \( F \) is bounded and continuous, Assumption A requires only that \( x \) be generic for \( \pi \). Let

\[
(2.6) \quad \mathcal{K}_1 = \{ \nu \in M_1(S) : K(\nu) - F^u(\nu) = r_1 \}.
\]

Assumption A and the properties of \( K \) given in Theorem 2.6 of [Se1] imply that \( \mathcal{K}_1 \) is a nonempty compact set. To each \( \nu \in \mathcal{K}_1 \) is associated a \( \pi \)-a.s. unique map \( y \mapsto \psi^\nu_y \) from \( \Sigma \) into \( M_1(\Sigma) \), with which we form the product measure \( \Psi^\nu_y(d\sigma) = \bigotimes_{i \in \Sigma^d} \psi^\theta_{i} \nu(d\sigma_i) \) on \( \Omega \). Define \( j_\infty(\nu) \in M_\emptyset(\Omega) \) by \( j_\infty(\nu) = \int \Psi^\nu_y \pi(dy) \). It will turn out that \( j_\infty \) is a homeomorphism from \( \mathcal{K}_1 \) onto a set \( \mathcal{K}_\infty \) of invariant measures on \( \Omega \), and \( \nu \mapsto \Psi^\nu_y \) is continuous in an appropriate sense. But the precise characterization of \( \psi^\nu_y \) is somewhat technical, so we postpone it till after the central results (see the paragraph following Theorem 2.14). For now we ask the reader to accept on faith that the integral in (2.8) below makes sense.
2.7. Theorem. Assume Assumption A. Then the set $\{\zeta_n^\infty\}_{n=1}^\infty$ is tight. Let $Q$ be any limit point. Then $Q$ is shift-invariant, and there is a probability measure $\Lambda$ on $\mathcal{K}_1$ such that

$$Q = \int_{\mathcal{K}_1} j_\infty(\nu) \Lambda(d\nu),$$

and $\Lambda$ is a limit point of the laws $\gamma_n^\infty(L_n \in \cdot)$.

2.9. Theorem. Assume Assumption A and that $\pi$ is ergodic, and fix a subsequence $\{n_j\}$. Then the following are equivalent:

1. There exists a limit $II = \lim_{j \to \infty} \gamma_{n_j}^\infty(R_{n_j} \in \cdot)$ in the weak topology of $\mathcal{M}_1(\mathcal{M}_1(\Omega))$.
2. There exists a limit $A = \lim_{j \to \infty} \gamma_{n_j}^\infty(L_{n_j} \in \cdot)$ in the weak topology of $\mathcal{M}_1(\mathcal{M}_1(S))$.
3. There exists a limit $Q = \lim_{j \to \infty} \zeta_{n_j}^\infty$ in the weak topology of $\mathcal{M}_1(\Omega)$.

When this happens, $II = \Lambda(j_\infty \in \cdot)$, and (2.8) is the ergodic decomposition of $Q$.

An important special case is that where $\pi$ is an IID measure on a configuration space. This leads us to formulate the following assumption:

Assumption B. $\Sigma = \mathcal{X}^\mathbb{Z}^d$ for a Polish space $\mathcal{X}$, $\mathbb{Z}^d$ acts on $\Sigma$ by translations, and $p^x = p^{x_0}$ so that $x \mapsto p^x$ is $\mathcal{B}_n^\Sigma$-measurable. Furthermore, assume (2.5) and that $P^x(F(L_n) = -\infty) < 1$ for large enough $n$, $\pi$-a.s.

The notation $\mathcal{B}_n^\Sigma$ above denotes the $\sigma$-field of $\Sigma$ generated by the $\mathcal{X}$-valued coordinate variables $(x_i : i \in V_n)$. Under Assumption B we have $\Psi^\Sigma_\nu(d\sigma) = \otimes_{i \in \mathbb{Z}^d} \psi_i^\Sigma(d\sigma_i)$ for $\nu \in \mathcal{K}_1$, so if $\pi$ is IID, the $j_\infty(\nu)$ are also IID measures. We get the following addition to Theorem 2.7 (see [Al] on exchangeable measures and their mixing measures):

2.10. Theorem. Assume Assumptions A and B and suppose that $\pi$ is IID. Then each limit point $Q$ of $\{\zeta_n^\infty\}_{n=1}^\infty$ is exchangeable with mixing measure $\Lambda$ given by Theorem 2.9.

More precise information on the convergence of the $\zeta_n^\infty$ seems to depend on the particular model at hand. The Hewitt-Savage 0-1 law implies that, under an IID $\pi$, any convergence that can be seen on a set of positive $\pi$-probability happens almost everywhere:

2.11. Theorem. Assume Assumption B. Fix a subsequence $\{n_j\}$. Then there is a Borel function $g : \Sigma \times \mathcal{M}_1(\Omega) \to \{0,1\}$ such that $g(x,Q) = 1$ iff $Q$ is a limit point of $\{\zeta_{n_j}^\infty\}_{j=1}^\infty$. For each fixed $Q$, $x \mapsto g(x,Q)$ is invariant under finite permutations of the coordinates of $x$. Suppose $\pi$ is IID. Then $x \mapsto g(x,Q)$ is $\pi$-a.s. constant for each fixed $Q$. There is a set $\Sigma_0 \subset \Sigma$ such that $\pi(\Sigma_0) = 1$ and for any $x,y \in \Sigma_0$, $\{\zeta_{n_j}^\infty\}_{j=1}^\infty$ and $\{\zeta_{n_j}^\infty\}_{j=1}^\infty$ have the same limit points.

Next we discuss the convergence of the measures $\gamma_n^\infty$. It turns out fruitful to consider the maps $\gamma_n^\infty : y \mapsto \gamma_n^\infty$ instead of individual measures. Let $\mathcal{M}_\pi^\Sigma$ be the space of $\pi$-a.s. defined Borel-measurable maps $\varphi : \Sigma \to \mathcal{M}_1(\Omega)$. We put a Polish topology on $\mathcal{M}_\pi^\Sigma$. 
by identifying $\varrho'$ with the measure $\pi \otimes \varrho' \equiv \varrho^y(d\sigma)\pi(dy) \in M_1(\Omega \times \Sigma)$. In other words, this topology is generated by the seminorms

$$p_f(\varrho') = \int_E \varrho^y(f^y)\pi(dy),$$

where $f$ is a bounded continuous function on $\Omega \times \Sigma$ and $f^y$ is its $y$-section. Write $\pi\varrho'$ for the marginal of $\pi \otimes \varrho'$, $\varrho' = \int \varrho^y\pi(dy)$. Let us say $\varrho'$ is invariant if $\varrho'^{\theta_iy} = \varrho^y \circ \theta_i^{-1}$ for all $i$, $\pi$-a.s(2.12). Theorem. Assume Assumption B and that $\pi$ is exchangeable. Then $\{\gamma_n^\nu\}_{n=1}^\infty$ is relatively compact. Let $\gamma'$ be any limit point. Then $\gamma'$ is invariant. There is a measure $\Lambda$ on $K_1$ such that

$$(2.13) \quad \gamma' = \int_{K_1} \Psi\nu \Lambda(d\nu),$$

and $\Lambda$ is a limit point of the laws $\pi\gamma'(L_n \in \cdot)$.

For an IID $\pi$ the representation (2.13) of $\gamma'$ is unique, because $\Lambda$ is the unique measure that represents the exchangeable $\pi \otimes \gamma'$ as a mixture of the IID measures $\pi \otimes \Psi\nu$ over $\nu \in K_1$. An analogue of Theorem 2.9 holds, but we leave the details to the reader. One would expect that under suitable circumstances a limit point $Q$ of Theorem 2.7 is an expectation of a limit point $\gamma'$, which is the content of this uniqueness theorem:

**2.14. Theorem.** Assume Assumption B and that $\pi$ is IID. Suppose that for some fixed subsequence $\{n_j\}$ and $Q \in M_0(\Omega)$, $\zeta_{n_j}^x \to Q$ as $j \to \infty$ for all $x$ on a set of positive $\pi$-measure. Then $\zeta_{n_j}^x \to Q$ for $\pi$-a.e. $x$, there exists a $\gamma' \in M_\pi$ such that $\gamma^x_{n_j} \to \gamma'$ as $j \to \infty$, and $Q = \pi\gamma'$.

These are the general results on the limit points of $\gamma_{n}^x$ and $\zeta_{n}^x$. We now turn to the precise characterization of $\psi^\nu_{\gamma'}$. Define the probability measure $\varphi$ on $S \times \Sigma$ by $\varphi(ds,dy) = p^y(ds)\pi(dy)$. By Theorem 2.13 of [Se1],

$$(2.15) \quad K(\nu) = \inf_\rho H(\rho | \varphi),$$

where the infimum is over $\rho \in M_1(S \times \Sigma)$ with marginals $\nu$ and $\pi$, and the relative entropy $H(\rho | \varphi)$ (see [DS] or [V]) is defined by

$$H(\rho | \varphi) = \begin{cases} \rho(\log \frac{d\rho}{d\varphi}) & \text{if } \rho \ll \varphi, \\ \infty & \text{otherwise.} \end{cases}$$

Since $H(\cdot | \varphi)$ is lower semicontinuous, strictly convex, and has compact sublevel sets, there is a unique minimizer $\rho$ whenever $K(\nu) < \infty$. For such a $\nu$ we denote the unique
minimizer by \( \psi_\nu \). Let \( \psi_\nu^Y \) denote a conditional distribution of \( \psi_\nu \) on \( S \), given \( y \in \Sigma \). Then define, as already indicated, \( \Psi^Y_\nu (d\sigma) = \otimes_{i \in \mathbb{Z}} \psi^\nu_{\sigma_1} (d\sigma_1) \) and

\[
(2.16) \quad j_\infty (\nu) = \int \Psi^Y_\nu \pi(dy).
\]

Under Assumption B, write \( \pi_o \) for the marginal of \( \pi \) on \( \mathcal{X} \) and let \( \varphi_o (ds, dx) = p^\mu (ds) \pi_o (x) \) on \( S \times \mathcal{X} \). We can replace (2.15) by

\[
(2.17) \quad K(\nu) = \inf_\rho H(\rho | \varphi_o),
\]

where the infimum is now over \( \rho \in \mathcal{M}_1 (S \times \mathcal{X}) \) with marginals \( \nu \) and \( \pi_o \). (Justification follows in Section 5.) Denote the unique minimizer again by \( \psi_\nu \). Now we can take \( \psi_\nu^Y = \psi_\nu^{Y_0} \) to be \( \mathcal{B}_0^\mathcal{X} \) measurable and \( \Psi^Y_\nu (d\sigma) = \otimes_{i \in \mathbb{Z}} \psi^\nu_{\sigma_1} (d\sigma_1) \). It becomes obvious that \( \pi \otimes \Psi^Y_\nu \) and \( j_\infty (\nu) \) are IID whenever \( \pi \) is.

Underlying our results is a large deviation theory for the laws \( \gamma^x_\nu (\mathbf{R}_n \in \cdot) \). For that, we need to extend the entropy defined by (2.4) to \( Q \in \mathcal{M}_\Theta (\Omega) \): Set

\[
(2.18) \quad K_n (Q) = \sup_{f \in \mathcal{B}_n^\Theta} \left\{ Q(f) - \int_\Sigma \log P^Y (e^f) \pi(dy) \right\},
\]

and

\[
(2.19) \quad k(Q) = \lim_{n \to \infty} \frac{1}{|V_n|} K_n (Q).
\]

\( \mathcal{B}_n^\Theta \) is the \( \sigma \)-field on \( \Omega \) generated by the spins in the cube \( V_n \). The limit in (2.19) exists by Theorem 2.8 in [Se1]. By Theorems 2.1 and 2.2 of [Se1], \( K \) and \( k \) are the rate functions that govern the large deviations of the laws \( P^x (\mathbf{L}_n \in \cdot) \) and \( P^x (\mathbf{R}_n \in \cdot) \), respectively, whenever \( x \) is generic for \( \pi \). Let \( Q_0 \in \mathcal{M}_1 (S) \) denote the single-spin marginal of a probability measure \( Q \in \mathcal{M}_\Theta (\Omega) \). Let

\[
r_\infty = \inf_{Q \in \mathcal{M}_1 (\Omega)} \{ k(Q) - F^u (Q_0) \},
\]

and for all \( Q \in \mathcal{M}_1 (\Omega) \), set

\[
I_\infty (Q) = k(Q) - F^u (Q_0) - r_\infty
\]

in case \( Q \) is invariant and \( I_\infty (Q) = \infty \) otherwise. Let

\[
\mathcal{K}_\infty = \{ Q \in \mathcal{M}_\Theta (\Omega) : k(Q) - F^u (Q_0) = r_\infty \}
\]

and

\[
\mathcal{K}_\infty' = \{ g' \in \mathcal{M}_\pi' : k(\pi g') - F^u ((\pi g')_0) = r_\infty \}.
\]
2.20. Lemma. Assume Assumption A. We have that \( r_1 = r_\infty \). \( I_\infty \) is a tight rate function, meaning that \( I_\infty \) is a lower semicontinuous function from \( \mathcal{M}_1(\Omega) \) into \([0, \infty]\) with compact sublevel sets \( \{I_\infty \leq b\} \) for all real \( b \).

\( \mathcal{K}_\infty = \{I_\infty = 0\} \) and \( \mathcal{K}_\infty^c \) are nonempty and compact in their respective topologies. Finally, \( j_\infty \) is a homeomorphism from \( \mathcal{K}_1 \) onto \( \mathcal{K}_\infty \), whose inverse is the projection \( Q \mapsto Q_0 \), and similarly \( \nu \mapsto \Psi_\nu \) is a homeomorphism from \( \mathcal{K}_1 \) onto \( \mathcal{K}_\infty^c \), with inverse \( \varphi \mapsto (\pi \varphi)_0 \).

Let \( \overline{E} \) and \( E^c \) denote the weak closure and interior, respectively, of a set \( E \) of probability measures.

2.21. Theorem. Under Assumption A, we have for any Borel set \( E \subset \mathcal{M}_1(\Omega) \),

\[
\limsup_{n \to \infty} \frac{1}{|V_n|} \log \gamma_n^\chi(\mathbf{R}_n \in E) \leq - \inf_{Q \in \overline{E}} I_\infty(Q). \tag{2.22}
\]

If \( F \) is continuous, we also have the lower bound

\[
\liminf_{n \to \infty} \frac{1}{|V_n|} \log \gamma_n^\chi(\mathbf{R}_n \in E) \geq - \inf_{Q \in E^c} I_\infty(Q). \tag{2.23}
\]

When both the upper and the lower bound (2.22)–(2.23) hold, we say that a large deviation principle (LDP) holds, or that \( I_\infty \) is a rate function for the laws \( \gamma_n^\chi(\mathbf{R}_n \in \cdot) \).

If \( F \) is continuous, the laws \( \gamma_n^\chi(\mathbf{L}_n \in \cdot) \) satisfy a LDP with rate function \( I_1(\varphi) = K(\varphi) - F(\varphi) - r_1 \), with \( \mathcal{K}_1 = \{I_1 = 0\} \).

2.24. Remark. It is worth noting that the rate function \( I_\infty \) is not an entropy of the type (2.18)–(2.19) with \( P^\chi \) replaced by \( \gamma_n^\chi \), except in special cases. Let

\[
I_n^\prime(Q) = \sup_{f \in \mathcal{B}^\infty_n} \left\{ Q(f) - \int_{\Sigma} \log \gamma_n^\chi(e^f) \pi(dy) \right\}
\]

for \( Q \in \mathcal{M}_\varnothing(\Omega) \). Then

\[
I_n^\prime(Q) = K_n(Q) - |V_n|Q(F(L_n)) + \int \log Z_n^\chi \pi(dy).
\]

As \( n \to \infty \), the limit \( \mathbf{L}_n(\sigma) \to \kappa(\sigma) \) exists in \( \mathcal{M}_1(S) \) for \( Q \)-a.e. \( \sigma \) by the ergodic theorem, and \( |V_n|^{-1} \int \log Z_n^\chi \pi(dy) \to -r_\infty \) by Varadhan’s theorem [DS, 2.1.10]. So

\[
I_\infty^\prime(Q) = \lim_{n \to \infty} \frac{1}{|V_n|} I_n^\prime(Q)
\]

exists and is given by

\[
I_\infty^\prime(Q) = k(Q) - Q(F(\kappa)) - r_\infty.
\]

Thus if \( F \) is continuous \( I_\infty^\prime(Q) = I_\infty(Q) \) holds whenever \( \kappa = Q_0 \) \( Q \)-a.s., so in particular for ergodic \( Q \), and in case \( F(\varphi) = \nu(f) \) for some \( f \in C_b(S) \) it holds for all \( Q \). See Prop. 3.2 in [Or] for an example of this phenomenon in the context of the Curie-Weiss model.
We close this section with two simple examples to get some feeling for the content of the theorems.

2.25. **Example.** Suppose \( f \in C_b(S) \) and set \( F(\nu) = \nu(f) \). Define \( \mu^x_1, \mu_1 \in M_1(S) \) by \( d\mu^x_1 = p^x(e^{tf})^{-1} e^t d\mu^x \) and \( \mu_1 = \int \mu^x_1 \pi(dx) \). Then \( K_1 = \{ \mu_1 \} \) and the unique limit point \( \gamma^* \) of \( \gamma^x_1 = \otimes \mu^x_1 \). Indeed, \( \gamma^x_1 \) and \( \gamma^x \) coincide on \( B^\Omega_n \).

2.26. **Example.** Suppose \( f \in C(S) \), \( g \in C_b(\mathbb{R}) \), and set \( F(\nu) = g(\nu(f)) \). For \( \beta \in \mathbb{R} \) let \( d\mu^x_\beta = p^x(e^{\beta f})^{-1} e^{\beta f} d\mu^x \) and \( \mu_\beta = \int \mu^x_\beta \pi(dx) \). Let \( a(x) = \text{ess inf} \ f \) and \( b(x) = \text{ess sup} \ f \) with respect to \( p^x \)-measure. Since \( \mu^x_\beta(f) \to a(x) \) as \( \beta \to -\infty \), \( a \) is measurable, and similarly for \( b \). By dominated convergence applied to \( \mu_\beta(f) \) it makes sense to adopt the convention \( \mu_\infty(f) = \pi(b) \) and \( \mu_{-\infty}(f) = \pi(a) \), and this is consistent with having

\[
\mu^x_\infty = p^x(\cdot | f = b(x)) \quad \text{and} \quad \mu^x_{-\infty} = p^x(\cdot | f = a(x))
\]

whenever these measures are well-defined. Assume

\[
\pi\{x : f \text{ is } p^x \text{-a.s. constant}\} < 1.
\]

(Otherwise \( K_1 = \{ p \} \) where \( p = \int p^x \pi(dx) \) and we are done.) Then

\[
\frac{d}{d\beta} \mu_\beta(f) = \int (\mu^x_\beta(f^2) - \mu^x_\beta(f)^2) \pi(dx) > 0
\]

for real \( \beta \), so by dominated convergence \( \beta \mapsto \mu_\beta(f) \) is continuous and strictly increasing from \([-\infty, \infty]\) onto \([\pi(a), \pi(b)]\).

**Claim.** Suppose \( \pi(a) \leq c \leq \pi(b) \) and let \( \beta \) be the unique number satisfying \( \mu_\beta(f) = c \). Then \( \mu_\beta \) is the unique minimizer of \( K(\nu) \) subject to \( \nu(f) = c \), provided this infimum is \( < \infty \).

Suppose first that \( c = \pi(a) \) and let \( \nu \) be such that \( \nu(f) = c \) and \( K(\nu) < \infty \). Then, for \( \pi \text{-a.e. } x, \psi^x_\nu \ll p^x \) which implies \( \psi^x_\nu(f) \geq a(x) \), but then \( \int \psi^x_\nu(f) \pi(dx) = \pi(a) \) forces \( \psi^x_\nu(f = a(x)) = 1 \) \( \pi \text{-a.s.} \). It follows that \( p^x(f = a(x)) > 0 \) \( \pi \text{-a.s.} \) so the measures \( \mu^x_{-\infty} \) are defined. By Cor. 3.1 in [Csi1], \( \pi \otimes \mu_\beta \) must be the minimizer for \( \mu_\beta \) in (2.15), hence by the conditional entropy formula (see (4.4.8) in [DS] or (10.2) in [V])

\[
K(\mu_\beta) = \int H(\mu^x_\beta | p^x) \pi(dx).
\]

The conclusion now follows from this general fact: For any probability measure \( \alpha \) and event \( E \) such that \( \alpha(E) > 0, \alpha(\cdot | E) \) uniquely minimizes \( H(\rho | \alpha) \) subject to \( \rho(E) = 1 \). \( c = \pi(b) \) is dealt with analogously.

Now let \( \pi(a) < c < \pi(b) \) and \( \nu \) again as above. In particular \( \beta \) is finite. By the finite-volume variational principle of classical statistical mechanics [Is, p. 46],

\[
H(\psi^x_\nu | p^x) - \beta \psi^x_\nu(f) \geq H(\mu^x_\beta | p^x) - \beta \mu^x_\beta(f)
\]
for all \( x \) with equality iff \( \psi_\nu^x = \mu_\beta^x \). Integrating against \( \pi \) gives

\[
K(\nu) - \beta \nu(f) \geq K(\mu_\beta) - \beta \mu_\beta(f)
\]

with equality iff \( \nu = \mu_\beta \). This proves the Claim.

It follows that the elements of \( K_1 \) must be \( \mu_\beta \)'s for some values of \( \beta \), and \( K^-_\infty \) consists of elements \( \gamma_\beta \) defined by \( \gamma_\beta^x = \otimes_i \mu_\beta^x \) for each \( \mu_\beta \) in \( K_1 \). Further conclusions depend on the specific \( g \) in question. For example, if \( g \) is concave and differentiable on \( [\pi(a), \pi(b)] \),

\[
K_1 = \begin{cases} 
\{ \mu_\infty \} & \text{if } \beta < g'(\mu_\beta(f)) \text{ for all } \beta, \\
\{ \mu_{-\infty} \} & \text{if } \beta > g'(\mu_\beta(f)) \text{ for all } \beta, \\
\{ \mu_{\beta_o} \} & \text{otherwise, where } \beta_o = g'(\mu_{\beta_o}(f)) \text{ uniquely determines } \beta_o.
\end{cases}
\]

### 3 Conditional limit theorems

In this section we investigate how the nonstationary independent random fields studied in [Se2] behave under conditioning. Such a field is specified by a configuration \( \mu = (\mu_i : i \in \mathbb{Z}^d) \) of probability measures on \( S \). Put the product measure \( P^\mu = \otimes_i \mu_i \) on \( \Omega \) so that the spins \( (\sigma_i) \) become independent random variables with laws \( \mathcal{L}(\sigma_i) = \mu_i \). Let \( \Sigma = M_1(S)^{2d} \) be the space of measure configurations \( \mu \).

Let \( \pi \in M_{(\Sigma)} \) with marginal \( \pi_o \in M_1(M_1(S)) \). As in Section 2 let \( \phi_o(ds, d\mu) = \mu(ds) \pi_o(d\mu) \), and set \( p = \int \mu \pi_o(d\mu) \). Throughout, \( C \) denotes a convex subset of \( M_1(S) \) that satisfies

\[
\inf_{\nu \in C} K(\nu) = \inf_{\nu \in C^o} K(\nu) < \infty.
\]

That such sets are plentiful is not hard to see: Suppose \( A \) is any convex set with \( \inf_{\nu \in A} K(\nu) < \infty \). Let \( A^\delta \) denote the (open or closed) \( \delta \)-fattening of \( A \). Then the decreasing function \( \delta \mapsto \inf_{\nu \in A^\delta} K(\nu) \) is continuous at all but countably many \( \delta > 0 \), and (3.1) is satisfied by \( C = A^\delta \) for any continuity point \( \delta \).

The strict convexity of relative entropy and (2.15) imply that \( K \) is strictly convex, hence by (3.1) there is a unique \( \nu_* \in \bar{C} \) such that \( K(\nu_*) = \inf_{\nu \in C} K(\nu) \). The construction done in Section 2 gives a \( \pi_o \)-a.s. unique \( M_1(S) \to M_1(S) \) map \( \mu \mapsto \psi^\mu \) such that \( \nu_* = \int \psi^\mu \pi_o(d\mu) \) and \( K(\nu_*) = \int H(\psi^\mu | \mu) \pi_o(d\mu) \). Set \( \Psi^\mu = \otimes_{i \in \mathbb{Z}^d} \psi_i^\mu \).

#### 3.2. Theorem

Whenever \( \mu \) is generic for \( \pi \),

\[
P^\mu(\mathbf{R}_n(\cdot) | L_n \in C) \to \int \Psi^\mu \pi(d\mu)
\]

as \( n \to \infty \) in the weak topology of \( M_1(\Omega) \). In particular, if \( \pi \) is IID, the limit is \( \nu_*^{\otimes \mathbb{Z}^d} \).
3.3. Theorem. Suppose \( \pi \) is IID. Then \( P(\cdot \mid L_n \in C) \) converges to \( \Psi^\ast \) in the sense that

\[
\lim_{n \to \infty} \int P_\mu(f(\cdot, \mu) \mid L_n \in C) \pi(d\mu) = \int \int f(\sigma, \mu) \Psi^\mu(d\sigma) \pi(d\mu)
\]

for any bounded continuous function \( f \) on \( \Omega \times \Sigma \).

Proof. To make the connection with Section 2, let \( P^\mu = \mu_0 \) and define \( F(\mu) = \log 1_C(\mu) \). Then \( \gamma^\ast_n = P(\cdot \mid L_n \in C) \) and \( \zeta^\mu_n = P^\mu(1_N(\cdot) \mid L_n \in C) \). Assumptions A and B of Section 2 are automatically satisfied under (3.1): \( F^\mu = \log 1_C^\ast \) and \( F_\ell = \log 1_{C^\circ} \), so (2.5) is equivalent to (3.1). And

\[
P^\mu(F(L_n) > -\infty) = P^\mu(L_n \in C) \geq P^\mu(L_n \in C^\circ),
\]

which is eventually \( > 0 \), for by the LDP of Theorem 2.3 in [Se2] and (3.1),

\[
\liminf_{n \to \infty} \frac{1}{|V_n|} \log P^\mu(L_n \in C^\circ) \geq -\inf_{\nu \in C^\ast} K(\nu) > -\infty.
\]

Apply Theorems 2.7, 2.12, and 2.14. \( \square \)

Let us make some comparisons with the expected process. Suppose \( \pi \) is IID and that \( \mu \) is generic for \( \pi \) (only (2.2) needs to be checked now). The expected process has law \( P = \int P^\mu \pi(d\mu) = p^\otimes 2^d \). Assume that \( C \) satisfies (3.1) also with \( K \) replaced by the function \( H(\cdot \mid p) \) to guarantee that the laws \( P(\cdot \mid L_n \in C) \) converge. The limits of empirical averages under the unconditioned measures \( P \) and \( P^\mu \) are the same, given by \( P \) and \( p \). So for example at position level \( L_n \to \nu \) a.s. under both \( P^\mu \) and \( P \), \( p \) is the unique zero for both the rate function \( K \) of \( P^\mu(L_n \in \cdot) \) and the rate function \( H(\cdot \mid p) \) of \( P(L_n \in \cdot) \), and only the rate of convergence is different \( (K(\nu) \geq H(\nu \mid p) \text{ for all } \nu, \text{ see Theorem 3.9 in [Se2]}) \).

However, \( P(\cdot \mid L_n \in C) \) is not an average of the laws \( P^\mu(\cdot \mid L_n \in C) \), hence we can no longer expect a common limit. We have \( P(\cdot \mid L_n \in C) \to \alpha^\otimes 2^d \), where \( \alpha \) minimizes \( H(\cdot \mid p) \) over \( \alpha \in C \). There is no a priori reason why \( \alpha \) and \( \nu \) should coincide, and in fact it is easy to construct examples where this does not happen. To assure the reader, here is one:

3.4. Example. Let \( S = \{1, 2, 3, 4\} \), \( \mu_1 = \frac{1}{4}(1, 3, 0, 0) \), \( \mu_2 = \frac{3}{4}(0, 0, 3, 1) \), and \( \pi_0 = \frac{3}{4} \delta_{\mu_1} + \frac{1}{4} \delta_{\mu_2} \), which gives \( p = \frac{1}{16}(3, 9, 3, 1) \). Let \( C = \{ \nu: \nu(2) \leq \frac{3}{16}, \nu(4) \geq \frac{3}{16} \} \). Noting firstly that for some \( \psi^\mu \ll \mu \) we have \( \nu_\ast = \frac{3}{4} \psi^\mu \| + \frac{1}{4} \psi^\mu \| \) and

\[
(3.5) \quad K(\nu_\ast) = \frac{3}{4} H(\psi^\mu \| \mu_1) + \frac{1}{4} H(\psi^\mu \| \mu_2),
\]

and secondly that \( \nu_\ast \in C \) forces \( \psi^\mu(2) \leq \frac{1}{4} \) and \( \psi^\mu(4) \geq \frac{3}{4} \), we minimize (3.5) by letting \( \psi^\mu \) be the image of \( \mu \) under the permutation (12)(34). Then \( \nu_\ast = \frac{1}{16}(9, 3, 1, 3) \) and \( K(\nu_\ast) = \frac{1}{2} \log 3 \). But this does not minimize \( H(\cdot \mid p) \) over \( C \), for \( \alpha_\ast = \frac{1}{16}(5, 3, 5, 3) \) gives \( H(\alpha_\ast \mid p) = \frac{5}{8} \log \frac{5}{3} < \frac{1}{2} \log 3 \). (To see that \( \alpha_\ast \) in fact is the minimizer, let \( \alpha \in \mathcal{M}_1(S) \) and
write \( H(\alpha \mid p) = H(\alpha \mid \alpha_\ast) + \alpha (\log \frac{d\mu}{d\nu_\ast}) \). Elementary reasoning shows that the second term is \( \geq H(\alpha_\ast \mid p) \), and this suffices. This reasoning follows Theorem 2.2 in [Cs1].

To get a clearer picture of the issue, let us reformulate these minimization problems on the bigger space \( S \times \mathcal{M}_1(S) \): Let \( \psi_\ast \) minimize \( H(\rho \mid \varphi_\circ) \) over probability measures \( \rho \) on \( S \times \mathcal{M}_1(S) \) with \( \rho_S \in C \) and \( \rho_{\mathcal{M}_1(S)} = \pi_\circ \), and let \( \eta_\ast \) minimize \( H(\rho \mid \varphi_\circ) \) over \( \rho \) with \( \rho_S \in C \). Then \( \nu_\ast \) and \( \alpha_\ast \) are the \( S \)-marginals of \( \psi_\ast \) and \( \eta_\ast \), respectively, and depending on the particular case, may or may not coincide. In Example 3.7 below \( \nu_\ast = \alpha_\ast \) but \( \psi_\ast \neq \eta_\ast \), so \( \psi_\ast = \eta_\ast \) is a stricter requirement. Here is the precise sense:

3.6. Proposition. The following are equivalent:

1. \( \psi_\ast = \eta_\ast \).
2. \( \nu_\ast = \alpha_\ast \) and \( K(\nu_\ast) = H(\alpha_\ast \mid p) \).
3. \( \pi_\circ = \int \varphi_\circ^s \alpha_\ast(ds) \), where \( \varphi_\circ^s \) is a conditional distribution of \( \varphi_\circ \) on \( \mathcal{M}_1(S) \), given \( s \in S \).
4. \( \eta_\ast(\mathcal{M}_1(S)) = \pi_\circ \).

Proof. (1) \( \iff \) (4) is immediate since \( \eta_\ast \) minimizes entropy without the constraint \( \eta_\ast(\mathcal{M}_1(S)) = \pi_\circ \). The minimizing \( \eta_\ast \) must be \( \eta_\ast(ds, d\mu) = \alpha_\ast(ds) \varphi_\circ^s(d\mu) \) (look at the conditional entropy formula, (4.4.8) in [DS] or (10.2) in [V]) and consequently \( H(\alpha_\ast \mid p) = H(\eta_\ast \mid \varphi_\circ) \). Hence (3) \( \iff \) (4), and together with \( K(\nu_\ast) = H(\psi_\ast \mid \varphi_\circ) \) this gives (1) \( \implies \) (2). Let \( \psi_\ast \) be a conditional distribution of \( \psi_\ast \) on \( \mathcal{M}_1(S) \), given \( s \in S \). (2) and the conditional entropy formula give

\[
K(\nu_\ast) = H(\nu_\ast \mid p) + \int H(\psi_\ast^s \mid \varphi_\circ^s) \nu_\ast(ds)
= H(\alpha_\ast \mid p) + \int H(\psi_\ast^s \mid \varphi_\circ^s) \alpha_\ast(ds),
\]

hence (2) again forces \( \int H(\psi_\ast^s \mid \varphi_\circ^s) \alpha_\ast(ds) = 0 \) and we conclude that

\[
\psi_\ast = \nu_\ast(ds) \psi_\ast^s(d\mu) = \alpha_\ast(ds) \varphi_\circ^s(d\mu) = \eta_\ast. \quad \square
\]

As a final point of comparison with the expected process, notice that the measure \( P(\cdot \mid L_n \in C) \) not only has identical marginals \( P(d\sigma_i \mid L_n \in C) \) for \( i \in V_n \), but also this marginal itself lies in \( \overline{C} \). Here is the easy argument for the reader's convenience: For any \( f \in \mathcal{B} \),

\[
P(f(\sigma_1) \mid L_n \in C) = P(L_n(f) \mid L_n \in C) \leq \sup_{\nu \in \overline{C}} \nu(f),
\]

which suffices by the separation theorems of locally convex spaces. The first equality above relies on invariance under permutations which we do not have in our setting. We can make a similar claim only on the average, that is,

\[
\int P^\mu(d\sigma_1 \mid L_n \in C) \pi(d\mu) \in \overline{C}.
\]

This follows since

\[
P^\mu(f(\sigma_1) \mid L_n \in C) = P^\mu(f(\sigma_{\tau_1}) \mid L_n \in C)
\]
for any permutation \( \tau \) on \( V_n \), and \( \pi \) is invariant under permutations (still assuming \( \pi \) IID), so
\[
\int P^\mu(f(\sigma_1) \mid L_n \in C) \pi(d\mu) = \int P^\mu(L_n(f) \mid L_n \in C) \pi(d\mu) \leq \sup_{\nu \in C} \nu(f).
\]
In particular, it is not clear how a result corresponding to Csiszár's 'convergence in information' could be derived in our setting (see p. 790 in [Cs2]).

To close this section we return to the coin-tossing example mentioned in the introduction, this time with two different coins.

3.7. Example. Let \( \frac{1}{2} < a < 1 \) and suppose we have two biased coins, with probabilities \( \alpha = (1 - a, a) \) and \( \beta = (a, 1 - a) \) on the space \( S = \{0, 1\} \). We choose a sequence of coins independently, each time picking an \( \alpha \)-coin with probability \( z \) and a \( \beta \)-coin with probability \( 1 - z \), and once the choices are made, flip the coins to run our coin-tossing process. Suppose \( 1 > t > m(z) \equiv za + (1 - z)(1 - a) \), so that the long run frequency of tails (\( \equiv \) the value \( 1 \in S \)) is below \( t \), for almost every realization of the coin choices. Given a typical sequence \( \mu = (\mu_1, \mu_2, \mu_3, \ldots) \) of coin choices (\( \mu_i = \alpha \) or \( \beta \) for each \( i \)), how does the coin-tossing process behave upon conditioning on \( S_n \geq nt \)?

To be precise, we have \( \pi^* = z\delta_\alpha + (1 - z)\delta_\beta \), the unique zero of \( K \) is \( p = (1 - m(z), m(z)) \), \( C = \{ \nu : \nu(1) \geq t \} \), and by strict convexity of \( K \), \( \nu^* = (1 - t, t) \). To find \( \psi^\alpha \) and \( \psi^\beta \), let
\[
v = v(z, \theta) = z \frac{a\theta}{1 - a + a\theta} + (1 - z) \frac{(1 - a)\theta}{a + (1 - a)\theta},
\]
a strictly increasing function of \( \theta \) with \( v(z, 0) = 0 \) and \( v(z, \infty) = 1 \). There is a unique \( \theta = \theta(z, t) \in (0, \infty) \) such that \( t = v(z, \theta) \). Define the function \( f \) on \( S \) by \( f(0) = 1 \), \( f(1) = \theta \), and set \( d\psi^\mu = \mu(f)^{-1} f \, d\mu \). Then \( v(z, \theta) = \int \psi^\mu(1) \pi^*(d\mu) \), and since this is equal to \( t = \nu^*(1) \) by choice of \( \theta \), we have found \( \psi^* \) such that \( \nu^* = \int \psi^\mu \pi^*(d\mu) \) and \( K(\nu^*) = \int H(\psi^\mu) \pi^*(d\mu) \). (If \( d\psi = f \otimes g \, d\varphi \), with \( g(\mu) = \mu(f)^{-1} \), \( \psi \) must be the minimizer in (2.17) by Cor. 3.1 in [Cs1].)

This gives the answer to the above question: The conditioned process behaves like the sequence \( (\psi^\mu_1, \psi^\mu_2, \psi^\mu_3, \ldots) \), where the new coins have probabilities
\[
\psi^\alpha = \left( \frac{1 - a}{1 - a + a\theta}, \frac{a\theta}{1 - a + a\theta} \right)
\]
and
\[
\psi^\beta = \left( \frac{a}{a + (1 - a)\theta}, \frac{(1 - a)\theta}{a + (1 - a)\theta} \right).
\]
Note that \( \psi^\alpha \) and \( \psi^\beta \) do depend on \( z \), namely through \( \theta = \theta(z, t) \). For example, if \( z = \frac{1}{2} \) and \( t < a \) we have
\[
\beta(1) < \psi^\beta(1) < t < \alpha(1) < \psi^\alpha(1),
\]
so perhaps somewhat surprisingly the lowest entropy is realized by making the \( \psi^\alpha \)-coin even more biased than the \( \alpha \)-coin. Both \( \partial \psi^\alpha(1)/\partial z < 0 \) and \( \partial \psi^\beta(1)/\partial z < 0 \), so picking the \( \alpha \)-coin more and more often decreases the expectations of the new coins.
We claimed above (see the paragraph preceding Proposition 3.6) that in this example \( \alpha_\ast = \nu_\ast \) but \( \psi_\ast \neq \eta_\ast \). That \( \alpha_\ast = \nu_\ast \) follows from the one-dimensionality of the example: Both \( H(\cdot | p) \) and \( K \) are strictly convex functions with common global minimum at \( p \), so over \( C \) they must both be minimized by \( \nu_\ast \). In particular, the process \( P^\mu(R_n(\cdot) | S_n \geq nt) \) and the conditioned expected process \( \bar{P}^\beta(\cdot | S_n \geq nt) \) both converge to \( \nu_\ast^\otimes N \). To see that \( \psi_\ast \neq \eta_\ast \) take \( z = \frac{1}{2} \) again and compute

\[
(\eta_\ast)_{\mathcal{M}_1(S)} = \int \varphi^\ast \nu_\ast(ds) = t(a\delta_\alpha + (1 - a)\delta_\beta) + (1 - t)((1 - a)\delta_\alpha + a\delta_\beta) \neq \pi_\circ.
\]

4 A mean-field mixed magnetic crystal

We consider here a mean-field version of the \( A_pB_{1-p} \) model of a quenched mixed magnetic crystal studied by [Ah]. Earlier work on this mean-field model appears in [Lu]. Our treatment follows the spirit of the treatment of the Curie-Weiss model (of which the present model is a disordered generalization) given in [E] and [Or]. The critical behavior of this model turns out to be the same as that of the Curie-Weiss.

Let \( \mathcal{S} = \{\pm 1\} \) be the single spin space and \( \mathcal{X} = \{0, 1\} \). We have two types of atoms, \( \text{A} \) and \( \text{B} \), and for each site \( i = 1, 2, 3, \ldots \) we flip a coin to place an \( \text{A} \)-atom with probability \( p \) and a \( \text{B} \)-atom with probability \( 1 - p \), where \( 0 < p < 1 \) is a fixed parameter of the model. The \( \text{A} \) and \( \text{B} \)-atoms have opposite types of magnetic ordering: An \( \text{A}\text{A} \)-pair interacts ferromagnetically with coupling constant \( 1 \) and a \( \text{B}\text{B} \)-pair interacts antiferromagnetically with coupling constant \(-1\). A mixed \( \text{A}\text{B} \)-pair interacts with coupling constant \( J > 0 \) which is another parameter of the model. Let \( x_i = 1 \) or 0 according to whether site \( i \) is occupied by an \( \text{A} \)- or a \( \text{B} \)-atom. This gives the quenched variable \( x = (x_i)_{i=1}^\infty \).

The Hamiltonian or interaction energy of \( n \) \( \pm 1 \)-valued spins \( \sigma = (\sigma_1, \ldots, \sigma_n) \) in the sample \( x \) with external field \( h \) is

\[
H_n^x(\sigma) = -\frac{1}{2n} \sum_{1 \leq i, j \leq n} \sigma_i \sigma_j [x_i x_j - (1 - x_i)(1 - x_j) + J(x_i(1 - x_j) + x_j(1 - x_i))] - h \sum_{1 \leq i \leq n} \sigma_i.
\]

This Hamiltonian is of the mean-field type in the sense that every spin interacts with every other spin with equal strength, independently of the distance separating them. The model has frustration in that no spin configuration can simultaneously satisfy all the bonds: In an \( \text{ABB} \)-triple, the \( J \)-coupling tends to align the \( \text{B} \) spins with the \( \text{A} \)-spin, but the antiferromagnetic \( \text{BB} \)-coupling works against aligning the \( \text{BB} \)-pair. We study the thermodynamic equilibrium of the spins for a fixed realization \( x \) of the occupation coin flips. This situation is called quenched disorder in the physics literature.

The space of spin configurations is \( \Omega = \mathcal{S}^N \) and the space of quenched variables \( \Sigma = \mathcal{X}^N \). The a priori measure of a single spin is \( \lambda_\circ = (\delta_{-1} + \delta_{+1})/2 \) on \( \mathcal{S} \), the coin-tossing measure on \( \mathcal{X} \) is \( \pi_\circ = p\delta_1 + (1 - p)\delta_0 \), and we put the products \( \lambda = \lambda_\circ^\otimes N \) and \( \pi = \pi_\circ^\otimes N \) on \( \Omega \) and \( \Sigma \),
respectively. Once an inverse temperature \( \beta > 0 \) is specified, the Gibbs measure \( \mu_n^x \) gives probability

\[
\mu_n^x(\sigma) = \frac{1}{Z_n^x} e^{-\beta H_n^x(\sigma)} \lambda_n(\sigma)
\]

(4.1)

to the configuration \( \sigma = (\sigma_1, \ldots, \sigma_n) \). Here \( \lambda_n \) is the restriction of \( \lambda \) to \( n \) spins.

We shall first describe the results for this model and then present the proofs at the end. Finding the set \( \mathcal{K}_1 \) of minimizing measures (2.6) boils down to solving

\[
s_1 = \xi_1(s_0) \equiv \frac{1}{\beta J_p} \left[ \frac{1}{2} \log \frac{1 + s_0}{1 - s_0} + \beta(1 - p)s_0 - \beta h \right]
\]

(4.2)

\[
s_0 = \xi_0(s_1) \equiv \frac{1}{\beta J(1 - p)} \left[ \frac{1}{2} \log \frac{1 + s_1}{1 - s_1} - \beta ps_1 - \beta h \right]
\]

for \( (s_0, s_1) \) in \([-1, 1] \times [-1, 1]\). These equations reveal a phase transition at the critical temperature

\[
\beta_c = \beta_c(p, J) = \frac{1 - 2p + (1 + 4J^2 p(1 - p))^{1/2}}{2(1 + J^2)p(1 - p)}.
\]

(4.3)

For each \( \beta > 0 \) and real \( h \), we single out a particular solution \( (z_0, z_1) = (z_0(\beta, h), z_1(\beta, h)) \) of (4.2):

- \( \beta \leq \beta_c, \ h = 0 \): The unique solution is \( (z_0, z_1) = (0, 0) \).
- \( \beta > \beta_c, \ h = 0 \): There is a unique solution \( (z_0, z_1) \) in \((0, 1) \times (0, 1)\) and the solution set is \( \{(-z_0, -z_1), (0, 0), (z_0, z_1)\} \).
- \( \beta > 0, \ h > 0 \): There is a unique solution \( (z_0, z_1) \) in \((0, 1) \times (0, 1)\) (and possibly some other solutions which we ignore).
- \( \beta > 0, \ h < 0 \): There is a unique solution \( (z_0, z_1) \) in \((-1, 0) \times (-1, 0)\) (and possibly some other solutions which we ignore).

It is obvious from (4.2) that \( (z_0(\beta, -h), z_1(\beta, -h)) = (-z_0(\beta, h), -z_1(\beta, h)) \) for \( h \neq 0 \). Physically speaking these solutions are magnetizations or expected values of the spins of A- and B-atoms: \( z_i \) for an A-atom and \( z_0 \) for a B-atom. Next we define product measures \( \mu^x \) on \( \Omega \) determined by these expectations, for a given occupation variable \( x = (x_i)_{i=1}^\infty \):

- \( \beta \leq \beta_c, \ h = 0 \): Set \( \mu^x = \lambda \) for all \( x \).
- \( \beta > \beta_c, \ h = 0 \): Set \( \mu^x_\pm = \otimes_{i=1}^\infty \psi_{\pm}^{x_i} \), where \( \psi_{\pm}^{x_i} \in \mathcal{M}_1(S) \) are defined by

\[
\psi_{\pm}^{x_i} = \frac{1 \mp z_x}{2} \delta_{-1} + \frac{1 \pm z_x}{2} \delta_{+1}
\]

(4.4)

for \( x = 0, 1 \). Then put \( \mu^x = (\mu^x_+ + \mu^x_-)/2 \).

- \( \beta > 0, \ h \neq 0 \): Define \( \psi^x \in \mathcal{M}_1(S) \) by

\[
\psi^{x_i} = \frac{1 - z_x}{2} \delta_{-1} + \frac{1 + z_x}{2} \delta_{+1}
\]

(4.5)

for \( x = 0, 1 \) and then \( \mu^x = \otimes_{i=1}^\infty \psi^{x_i} \).
This defines an element \( \mu^* = \mu^{\beta, h, \cdot} \) of \( M_{\pi}^* \). The connection to the \( z_x \) is clear: For example under \( \mu^*_+ \) the expected spin at site \( i \) is \( z_1 \) if the site is occupied by an A-atom and \( z_0 \) if the site is occupied by a B-atom. Recall the Polish topology on \( M_{\pi}^* \) defined in Section 2.

4.6. Theorem. For all \( p, J, \beta, \) and \( h \), \( \mu^* = \lim \mu_n^* \) as \( n \to \infty \).

Considering \( \mu^x \) as the equilibrium state of the spins with occupations \( x \), we can say the following: At high temperature (\( \beta \leq \beta_c \)) and zero field (\( h = 0 \)) we see a completely random state where the spins are independent and identically distributed, even independently of the type of atom (each marginal of \( \mu^x \) equals \( \lambda_o \)), with no preference for \( + \) or \(-\). The state ordered by a positive field (\( h > 0 \)) has a positive magnetization (expected spin \( \psi^x_i(s) = z_{z_i} > 0 \) at each site), although the spins remain independent and there is lack of long range order in the sense that the magnetization (\( z_1 \) or \( z_0 \)) at site \( i \) varies with the type (A or B) of the occupant of \( i \). At \( \beta > \beta_c \), \( h = 0 \) we have a ferromagnetically ordered state where all the spins simultaneously obey either \( \mu^x_+ \) or \( \mu^x_- \), and in both cases they tend to align themselves, without the force of an external field.

The phase transition is also reflected in the large deviations of empirical averages. Let \( I_\infty \) be the rate function of the laws \( \mu^x_+(R_n \in \cdot) \), and \( S \) the rate function of the laws \( \mu_n^x(\frac{1}{n} \sum^n_{i=1} \sigma_i \in \cdot) \) of the average block spin. Put \( z(\beta, h) = p z_1(\beta, h) + (1 - p) z_0(\beta, h) \).

4.7. Theorem.

\[
I_\infty(Q) = 0 \quad \text{iff} \quad \begin{cases} 
Q = \lambda & \text{in case } h = 0 \text{ and } \beta \leq \beta_c, \\
Q = \pi \mu_+^* \text{ or } Q = \pi \mu_-^* & \text{in case } h = 0 \text{ and } \beta > \beta_c, \\
Q = \pi \mu^* & \text{in case } h \neq 0.
\end{cases}
\]

\[
S(t) = 0 \quad \text{iff} \quad \begin{cases} 
t = z(\beta, 0) = 0 & \text{in case } h = 0 \text{ and } \beta \leq \beta_c, \\
t = z(\beta, 0) \text{ or } t = -z(\beta, 0) & \text{in case } h = 0 \text{ and } \beta > \beta_c, \\
t = z(\beta, h) & \text{in case } h \neq 0.
\end{cases}
\]

The specific magnetization is defined by

\[
m(\beta, h) = \lim_{n \to \infty} \frac{1}{n} \mu^x_n(\sum^n_{i=1} \sigma_i),
\]

a limit which exists \( \pi \)-a.s. and is given by

\[
m(\beta, 0) = 0 \text{ for all } \beta > 0, \text{ and} \\
m(\beta, h) = z(\beta, h) \text{ for all } \beta > 0 \text{ and } h \neq 0.
\]

(4.8)

Note that the spin-flip symmetry of the Hamiltonian forces \( \mu^x_n(\sigma_i) = 0 \) if \( h = 0 \). So in the absence of a symmetry-breaking field, the magnetization does not see the phase transition as our model does not incorporate boundary conditions. As functions of \( h \), \( z_1(\beta, h), z_0(\beta, h) \), and \( m(\beta, h) \) are strictly increasing. Set

\[
m(\beta, +) = \lim_{h \to 0^+} m(\beta, h) \quad \text{and} \quad m(\beta, -) = \lim_{h \to 0^-} m(\beta, h).
\]
4.9. Theorem. $m(\beta, \pm) = \pm z(\beta, 0)$.

It follows that $m(\beta, +) = 0 = m(\beta, 0)$ if $\beta \leq \beta_c$ but $m(\beta, +) > 0 = m(\beta, 0)$ if $\beta > \beta_c$. Thus at low temperatures our model exhibits spontaneous magnetization, another aspect of the phase transition.

Suppose for the moment that $h \geq 0$. Since the $A$-atoms are ferromagnetically ordered among themselves and the $B$-atoms antiferromagnetically, it seems plausible that the $+$-tendency of the $A$-atoms is stronger than that of the $B$-atoms, that is, $z_1 \geq z_0$. This turns out not to be necessarily the case, and even more surprisingly, the cut-off point depends only on $p$ and $J$ and not at all on $\beta$ and $h$, provided we are in the regime of nontrivial ($\neq (0, 0)$) solutions:

4.10. Theorem. Suppose $\beta > \beta_c$ or $h > 0$. Then

$$z_1(\beta, h) > z_0(\beta, h) \text{ iff } p \leq 1/2 \text{ or } J < (2p - 1)^{-1},$$

$$z_1(\beta, h) = z_0(\beta, h) \text{ iff } p > 1/2 \text{ and } J = (2p - 1)^{-1},$$

$$z_1(\beta, h) < z_0(\beta, h) \text{ iff } p > 1/2 \text{ and } J > (2p - 1)^{-1}.$$

The interesting phenomenon one looks for in quenched models is spin-glass behavior. The spin glass state should be characterized by nonzero local magnetizations $\mu^x(\sigma_i)$ even while the overall magnetization vanishes (see p. 3 in [FH]). Following standard theory, we look at the order parameter

$$q_n^x = \frac{1}{n} \sum_{i=1}^{n} \left[ \mu_n^x(\sigma_i) \right]^2$$

as $n \to \infty$ (see (2.42) in [FH]). It is convenient to rewrite $q_n^x$ in terms of an empirical average of the overlap between two independent samples with a common occupation variable $x$:

Letting $\tau$ denote the spin of the independent copy,

$$q_n^x = \frac{1}{n} \sum_{i=1}^{n} \mu_n^x(\sigma_i) \mu_n^x(\tau_i) = \mu_n^x \otimes \mu_n^x \left( \frac{1}{n} \sum_{i=1}^{n} \sigma_i \tau_i \right).$$

However, again by the spin-flip symmetry $q_n^x \equiv 0$ for all $x$ and $n$ if $h = 0$, so precisely as with block spin and magnetization we investigate the large deviations of the laws

$$\mu_n^x \otimes \mu_n^x \left( \frac{1}{n} \sum_{i=1}^{n} \sigma_i \tau_i \right) \in .

(4.11)$$

to detect symmetry breaking. Let $V$ be the rate function for these laws.


$$V(t) = 0 \text{ iff } \begin{cases} t = 0 & \text{in case } h = 0 \text{ and } \beta \leq \beta_c, \\ t = \pm (pz_1(\beta, 0)^2 + (1 - p)z_0(\beta, 0)^2) & \text{in case } h = 0 \text{ and } \beta > \beta_c, \\ t = pz_1(\beta, h)^2 + (1 - p)z_0(\beta, h)^2 & \text{in case } h \neq 0. \end{cases}$$
We see that broken symmetry for $q_n^0$ does not happen without ferromagnetic ordering ($V(t) = 0$ for some nonzero $t$ iff $S(t) = 0$ for some nonzero $t$), so a spin-glass phase in this particular technical sense does not occur.

Our final result looks at the behavior of some physical quantities near the critical temperature. A basic fact is that the phase transition is continuous, in that as $\beta \searrow \beta_c$, $m(\beta, \pm) \to 0 = m(\beta_c, \pm) = m(\beta_c, 0)$. This is a consequence of the next lemma.

4.13. Lemma. The quantities $z_0(\beta, 0)$ and $z_1(\beta, 0)$ are continuous functions of $\beta$, and for $\beta \geq \beta_c$ strictly increasing. In particular they decrease to 0 as $\beta \searrow \beta_c$.

A critical exponent $c$ is defined as follows: We write $f \sim x^c$ as $x \searrow 0$ if

$$\lim_{x \searrow 0} \frac{\log f(x)}{\log x} = c.$$  

The zero-field specific heat $C_{h=0}$ is defined by

$$C_{h=0}(\beta) = -\beta^2 \frac{\partial u}{\partial \beta}(\beta, 0),$$

where the specific energy

$$u(\beta, h) = \lim_{n \to \infty} \frac{1}{n} \mu_n^x(H_n^x)$$

exists as a $\tau$-a.s. constant limit. We will argue that $u$ is a continuous function of $(z_0, z_1)$, hence a continuous function of $\beta$ at fixed $h = 0$, with $u(\beta, 0) = 0$ for $\beta \leq \beta_c$ and $u(\beta, 0) < 0$ for $\beta > \beta_c$.

4.14. Theorem. We have the following critical behavior, for $x = 0, 1$:

(4.15) $z_x(\beta, 0) \sim (\beta - \beta_c)^{1/2}$ as $\beta \searrow \beta_c$,

(4.16) $z_x(\beta_c, h) \sim h^{1/3}$ as $h \searrow 0$, and

(4.17) $\frac{\partial z_x}{\partial h}(\beta, 0) \sim |\beta - \beta_c|^{-1}$ as $\beta \to \beta_c^\pm$.

(4.18) $C_{h=0}$ has a discontinuity at $\beta = \beta_c$.

We close with some remarks before turning to the proofs.

4.19. Remark. Traditionally the critical exponents are presented for the spontaneous magnetization $m(\beta, +)$, the magnetization $m(\beta_c, h)$, and the zero-field susceptibility $\chi = \frac{\partial m}{\partial h}(\beta, 0)$. Using $m = pz_1 + (1 - p)z_0$ the above results yield the familiar exponents of the classical Curie-Weiss theory of magnetism (see Section 4-6 in [T]): $m(\beta, +) \sim (\beta - \beta_c)^{1/2}$ as $\beta \searrow \beta_c$, $m(\beta_c, h) \sim h^{1/3}$ as $h \searrow 0$, and $\chi(\beta, 0) \sim |\beta - \beta_c|^{-1}$ as $\beta \to \beta_c^\pm$. The discontinuity of specific heat is also part of the classical Curie-Weiss picture.

If we let $q(\beta)$ denote the positive zero of the rate function $V$ in the case $h = 0$ of Theorem 4.12, we have $q(\beta) \sim (\beta - \beta_c)^1$ as $\beta \searrow \beta_c$.  

4.20. Remark. In (4.17) above note that $z_{\beta}(\beta, h)$ is not continuous as $h$ passes through 0 if $\beta > \beta_c$, so in that case the derivative is one-sided, i.e. $\partial z_{\beta}/\partial h(\beta, 0^{\pm})$. The explicit expression for $\lim_{\beta \searrow \beta_c} C_h(0)(\beta)$ can be found at the end of the section.

4.21. Remark. Even though our model does not have a phase where simultaneously $m = 0$ and $q > 0$, our analysis suggests a way to produce such an effect: Simply take $J < 0$ of a suitable magnitude so that the nonzero magnetizations of A- and B-atoms cancel each other out.

4.22. Remark. We assumed $0 < p < 1$ and $J > 0$ throughout, but the expression (4.3) for $\beta_c$ gives correct information also about the limiting cases:

1. In the case $p = 0$ (all atoms are of type B) we have an antiferromagnetic Curie-Weiss model whose only limiting Gibbs measure is easily seen to be $\lambda$ for all temperatures. Accordingly $\lim_{\beta \searrow 0} \beta_c = \infty$.

2. The opposite case $p = 1$ is the classical Curie-Weiss model, whose $\beta_c = 1$ [E], and we have $\lim_{p \to 1} \beta_c = 1$.

3. If we set $J = 0$ the A- and B-atoms equilibrate independently of each other, which is reflected mathematically in a decoupling of the equations (4.2). The B-atoms always choose $\lambda$ and the A-atoms behave as a dilute Curie-Weiss model with $\beta_c = p^{-1}$. And $\lim_{J \searrow 0} \beta_c = p^{-1}$.

4.23. Remark. The constant $J$ influences the model in seemingly contradictory ways. On the one hand, $\beta_c$ decays like $J^{-1}$ as $J \nearrow \infty$ by (4.3), so we can produce a phase transition at arbitrarily high temperatures (recall that temperature $\sim \beta^{-1}$) by taking $J$ large enough.

An intuitively plausible explanation is that increasing $J$ increases the effective coupling of the A-atoms, since each AA-pair is coupled not only directly but also via each B-atom. Thus the ordering effect dominates the entropy effect already at higher temperatures. In fact (4.31) shows that for any given $\beta$ and $p$, we are in the ferromagnetically ordered state whenever

$$J^2 > \frac{(1 - \beta p)(1 + \beta(1 - p))}{\beta^2 p(1 - p)}.$$

On the other hand, $J$ inhibits magnetization by the external field. For a fixed $\beta < \beta_c$, $\partial z_{\beta}/\partial h$ decays like $J^{-1}$ for large $J$ by (4.43). This effect is natural since the antiferromagnetic BB-couplings resist the aligning effect of the field.

4.24 Remark. The phase transition can also be parametrized in terms of a critical density $p_c$ of A-atoms, so that the model is in the ferromagnetic state whenever $p > p_c$. For the critical exponent one finds $z_{\beta} \sim (p - p_c)^{1/2}$ as $p \searrow p_c$. We leave the details to the reader.

4.25. Remark. Finally we wish to point out the sense in which our model is a mean-field approximation of the quenched mixed magnetic crystal. An Ising-type Hamiltonian for the model is (see (1) in [Ah])

$$\mathcal{H} = -\sum_{\langle i, j \rangle} \sigma_i \sigma_j [x_i x_j - (1 - x_i)(1 - x_j) + J(x_i(1 - x_j) + x_j(1 - x_i))] - h \sum_{i \in V_n} \sigma_i,$$
where the first sum is over nearest-neighbor pairs \((i,j)\) in the cube \(V_n\) in \(\mathbb{Z}^d\). Let \(s_1\) and \(s_0\) be the mean magnetizations of A- and B-atoms, respectively, and assume that each spin \(\sigma_i\) only interacts with the mean fields produced by the other spins. Then \(\mathcal{H}\) simplifies to

\[
\mathcal{H}' = - \sum_{i \in V_n} \sigma_i [x_ip s_1 - (1 - x_i)(1 - p)s_0 + J(x_i (1 - p)s_0 + (1 - x_i)p s_1) + h].
\]

Now that the spins are decoupled, the Gibbs measure of this Hamiltonian is just a product measure. Consistency requires that \(s_0\) and \(s_1\) be the conditional expectations of the spins under this measure. For \(s_0\) we get

\[
s_0 = \frac{\exp\{\beta(-(1 - p)s_0 + Jps_1 + h)\} - \exp\{-\beta(-(1 - p)s_0 + Jps_1 + h)\}}{\exp\{\beta(-(1 - p)s_0 + Jps_1 + h)\} + \exp\{-\beta(-(1 - p)s_0 + Jps_1 + h)\}} = \tanh(\beta(-(1 - p)s_0 + Jps_1 + h)),
\]

or equivalently (apply \(\tanh^{-1}\) on both sides and rearrange) \(s_1 = \xi_1(s_0)\). Similarly for \(s_1\). Thus we have recovered (4.2), so indeed the magnetizations of our model give a mean-field approximation of the “true” quantities of the model with Hamiltonian \(\mathcal{H}\).

To prove our results we apply the theory of Section 2 by doing the following: Since the quenched variable \(x\) appears explicitly in the Hamiltonian, we must work with the skew model which includes the quenched variable as a deterministic component: The map \(x \mapsto P^x\) is defined by \(P^x = \lambda \otimes \delta_x\), so \(P^x\) is a measure on \(\Omega \times \Sigma\). Define the function \(F\) for probability measures \(\nu\) on \(S \times \mathcal{A}\) by

\[
F(\nu) = \frac{\beta}{2}(-\nu(s)^2 + 2(1 + J)\nu(s)\nu(sx) - 2J\nu(sx)^2) + \beta h\nu(s),
\]

where we write \((s,x)\) for a generic element of \(S \times \mathcal{A}\) and \(\nu(s)\) for the integral \(\int s \nu(ds)\). Then \(-\beta H^x_n(\sigma) = nF(\overline{L}_n)\) where \(\overline{L}_n = n^{-1} \sum_1^n \delta_{\sigma_i,x_i}\) is the empirical distribution of the skew model, and \(\gamma^x_n = \mu^x_n \otimes \delta_x\) for the measure \(\gamma^x_n\) defined by (2.1).

To find the minimizing measures in \(\mathcal{K}_1\) we need only consider \(\nu\) with \(\mathcal{A}\)-marginal \(\pi_o\), by (5.10). Let us parametrize such a measure \(\nu\) by its conditional spin expectations:

\[
\begin{align*}
s_0 &= \nu(s \mid x = 0) = \text{expected spin of a B-atom}, \\
s_1 &= \nu(s \mid x = 1) = \text{expected spin of an A-atom}.
\end{align*}
\]

The parameter \((s_0,s_1)\) ranges over the square \([-1,1] \times [-1,1]\). In these terms \(\nu(s) = ps_1 + (1 - p)s_0\) and \(\nu(sx) = ps_1\), so

\[
(4.26) \quad F(\nu) = \frac{\beta}{2}(p^2 s_1^2 - (1 - p)^2 s_0^2 + 2Jp(1 - p)s_0 s_1) + \beta h(ps_1 + (1 - p)s_0).
\]

Entropy is given by

\[
(4.27) \quad K(\nu) = \frac{1 - p}{2}[(1 - s_0)\log(1 - s_0) + (1 + s_0)\log(1 + s_0)]
+ \frac{p}{2}[(1 - s_1)\log(1 - s_1) + (1 + s_1)\log(1 + s_1)].
\]
Set \( G(\nu) = K(\nu) - F(\nu) \). We wish to argue that the minimum of \( G \) is taken uniquely at \((z_0(\beta, h), z_1(\beta, h))\) if \( h \neq 0 \) or \( \beta \leq \beta_c \) and at \((\pm(z_0(\beta, h), z_1(\beta, h)))\) in the case \( h = 0, \beta > \beta_c \). \( \partial G / \partial s_x = 0 \) iff \( s_{1-x} = \xi_{1-x}(s_x) \) for \( x = 0,1 \), where the functions \( \xi_x \) are defined in (4.2). \( \xi_x \) is strictly increasing, so it has a well-defined inverse \( \eta_1 \) on \([-1,1] \times [-1,1] \). \( \partial G / \partial s_0 = 0 \) iff \( s_0 = \eta_1(s_1) \). (The reader is advised to sketch the graphs of \( \xi_0 \) and \( \eta_1 \) in the square \([-1,1] \times [-1,1] \).) We have \( \partial^2 G / \partial s_0^2 > 0 \) on \([-1,1] \), the minima of \( G \) are necessarily on the graph \( s_0 = \eta_1(s_1) \). Set \( g(s_1) = G(\eta_1(s_1), s_1) \). Differentiating gives

\[
g'(s_1) = \beta Jp(1-p)[\xi_0(s_1) - \eta_1(s_1)],
\]

and we can read off the minima:

**Case I:** \( h = 0 \). Now \( \xi_0(0) = \eta_1(0) = 0 \), the graph of \( \eta_1 \) is increasing and crosses the square from left to right, and \( \lim_{s_1 \to \pm 1} \xi_0(s_1) = \pm \infty \). Furthermore, on \([0,1] \eta_1 \) is strictly decreasing but \( \xi_0 \) strictly increasing, and vice versa on \([-1,0] \). From this it follows that if

\[
\eta_1'(0) \leq \xi_0'(0)
\]

\( g \) takes its unique minimum at \( 0 \), whereas if

\[
\eta_1'(0) > \xi_0'(0)
\]

\( g \) takes its minima at \( \pm z_1 \) for the unique positive number \( z_1 \) that satisfies \( \xi_0(z_1) = \eta_1(z_1) \).

(4.30) is equivalent to

\[
\beta^2(1 + J^2)p(1-p) + \beta(2p - 1) - 1 > 0.
\]

This quadratic has a unique positive root \( \beta_c \) given by (4.3), so (4.29) and (4.30) are equivalent to \( \beta \leq \beta_c \) and \( \beta > \beta_c \), respectively. Note for later use that when \( h = 0 \)

\[
\beta \neq \beta_c \iff \eta_1'(0) \neq \xi_0'(0) \iff \xi_0'(0) \xi_1'(0) \neq 1.
\]

**Case II:** \( h \neq 0 \). Suppose first that \( h > 0 \). Compared to case I this amounts to shifting the graph of \( s_0 = \eta_1(s_1) \) up and the graph of \( s_0 = \xi_0(s_1) \) down. Thus the graphs of \( \eta_1 \) and \( \xi_0 \) no longer intersect at \((0,0) \). But there is a unique point of intersection \((z_1, z_0) \) in \((0,1) \times (0,1) \) where the graph of \( \xi_0 \) crosses \( \eta_1 \) from below to above, and hence \( z_1 \) is a local minimum of \( g \). There can be other local minima with either \( s_0 < 0 \) or \( s_1 < 0 \), but it is easy to check directly from (4.26)–(4.27) that \( G(s_0, s_1) > G(|s_0|, |s_1|) \) unless both \( s_0 > 0 \) and \( s_1 > 0 \). Thus the global minimum of \( G \) is attained at \((z_0, z_1) \). The case \( h < 0 \) follows from this because \( K(\nu) \) and \( F_{h=0}(\nu) \) are invariant under the spin flip \((s_0, s_1) \mapsto (-s_0, -s_1) \).

We have found the measures in \( \mathcal{K}_1 \) in terms of the conditional expected spins \((z_0, z_1) \) specified earlier after (4.3). We emphasize again that the elements of \( \mathcal{K}_1 \) are measures on \( \mathcal{S} \times \mathcal{X} \), and consequently the measures \( j_\infty(\nu) \) for \( \nu \in \mathcal{K}_1 \) live on \( \Omega \times \Sigma \):

- \( \beta \leq \beta_c, h = 0: \mathcal{K}_1 = \{ \lambda_\circ \pi_o \} \) and \( j_\infty(\lambda_\circ \pi_o) = \lambda \circ \pi \).
- \( \beta > \beta_c, h = 0: \mathcal{K}_1 = \{ \nu_+, \nu_- \}, \) where \( \nu_{\pm}(ds, dx) = \psi_{\mp}(ds) \pi_o(dx) \) (recall definition (4.4)), and \( j_\infty(\nu_{\pm}) = \Psi_{\mp} \equiv \pi \otimes \mu_{\pm} \).
- \( \beta > 0, h \neq 0: \mathcal{K}_1 = \{ \nu \}, \) where \( \nu(ds, dx) = \psi(ds) \pi_o(dx) \) (recall definition (4.5)), and \( j_\infty(\nu) = \Psi \equiv \pi \otimes \mu \).
The next step is to prove that the measures $\zeta_n^x = \gamma_n^x(\overline{R}_n(\cdot))$ converge, where $\overline{R}_n$ is the empirical field of the skew model,

$$\overline{R}_n = \frac{1}{n} \sum_{i=0}^{n-1} \delta(\theta_i, \sigma_i, x).$$

Only in the case $\beta > \beta_c, h = 0$ can we have more than one limit point. The possible limit points are $t\Psi_+ + (1-t)\Psi_-$ for $0 \leq t \leq 1$. By the spin-flip symmetry of the Hamiltonian

$$\zeta_n^x(\sigma_1) = \frac{1}{n} \gamma_n^x(\sum_{i=1}^{n} \sigma_i) = 0$$

for all $n$ and $x$. A straightforward computation gives

$$t\Psi_+(\sigma_1) + (1-t)\Psi_-(\sigma_1) = (2t-1)((1-p)z_0 + pz_1)$$

which must equal 0, so $t = 1/2$ since $(1-p)z_0 + pz_1 > 0$. Theorem 2.7 now yields

4.33. Lemma. For all $x$ generic for $\pi$,

$$\lim_{n \to \infty} \zeta_n^x = \begin{cases} 
\lambda \otimes \pi & \text{in case } h = 0 \text{ and } \beta \leq \beta_c, \\
(\Psi_+ + \Psi_-)/2 & \text{in case } h = 0 \text{ and } \beta > \beta_c, \\
\Psi & \text{in case } h \neq 0.
\end{cases}$$

Proof of Theorem 4.6. Apply Theorems 2.12 and 2.14 and Lemma 4.33. □

Proof of Theorem 4.7. $\Omega$-marginals of the measures $j_{\infty}(\nu), \nu \in \mathcal{K}_1$, specified above give the zeroes of the rate function $I_{\infty}$. The contraction $Q \mapsto Q(\sigma_1)$ gives the zeroes of the rate function $S$. □

(4.8) follows for $h = 0$ from the spin-flip symmetry and for $h \neq 0$ from Theorem 4.7.

4.34. Lemma.

(1) $z_0(\beta, h)$ and $z_1(\beta, h)$ are strictly increasing functions of $h$.

(2) $z_0(\beta, 0)$ and $z_1(\beta, 0)$ are strictly increasing functions of $\beta \geq \beta_c$.

Proof. (1) It suffices to consider $h \geq 0$. Fix $\beta$, let $z^h_z = z_x(\beta, h)$, and write $\xi^h_z$ for the functions defined in (4.2) to specify the $h$. Let $0 \leq k < h$. Then

$$\xi^h_z = \xi^k_z + (k-h)/Jp < \xi^k_z.$$

In particular, $\xi^k_1(\eta^k_1(s_1)) = s_1 = \xi^h_1(\eta^h_1(s_1)) < \xi^k_1(\eta^k_1(s_1))$, so by the monotonicity of $\xi^k_1$

$$\eta^k_1 < \eta^h_1.$$

Now let $t \geq z^h_1$. Since $(\eta^h_1)'$ decreases and $(\xi^k_1)'$ increases on $[0,1)$, $(\xi^h_1)'(z^h_1) > (\eta^h_1)'(z^h_1)$ because $g''(z^h_1) > 0$ (recall (4.28)), and $\eta^h_1(z^h_1) = \xi^h_1(z^h_1)$, we have $\eta^h_1(t) \leq \xi^h_1(t)$. This
together with \((4.35)\) and \((4.36)\) gives \(\eta^h(t) < \xi^h_0(t)\), which implies that \(t\) cannot equal \(z^h_1\). Since this holds for all \(t \geq z^h_1\), it must be that \(z^h_1 < z^h_1\). \((4.36)\) and the monotonicity of \(\eta^h_1\) then give \(z^h_0 = \eta^h_1(z^h_1) < \eta^h_1(z^h_1) = z^h_0\).

(2) Upon replacing \(k\) and \(h\) by \(k'\) and \(\beta\) such that \(\beta_c \leq \beta' < \beta\), the above argument can be repeated with obvious changes. \[\]

**Proof of Theorem 4.9.** Part (1) of Lemma 4.34 implies that magnetization \(m(\beta, h)\) is a strictly increasing function of \(h\). We may let \(h \searrow 0\) and \(h \nearrow 0\) in the equations \(z^h_x = \xi^h_x(z^h_{1-x})\), \(x = 0, 1\), and since the limits of \(z^h_x\) satisfy the equations for \(h = 0\), they are the solutions \((z_0(\beta, 0), z_1(\beta, 0))\) and \((-z_0(\beta, 0), -z_1(\beta, 0))\). \[\]

**Proof of Theorem 4.10.** Assuming \(\beta > \beta_c\) or \(h > 0\) implies that \(z_0, z_1 > 0\). Use \((4.2)\) and the fact that \(\log(1 + t)/(1 - t)\) is strictly increasing: \(z_1 > z_0\) implies

\[
\beta pj^{z}_1 \beta(1 - p)z_0 + \beta h = \frac{1}{2} \log \frac{1 + z_0}{1 - z_0} < \frac{1}{2} \log \frac{1 + z_1}{1 - z_1} = \beta j(1 - p)z_0 + \beta p z_1 + \beta h
\]

\[\Rightarrow \beta p(J - 1)z_1 < \beta(1 + J)(1 - p)z_0 < \beta(1 + J)(1 - p)z_1\]

\[\Rightarrow J(2p - 1) < 1,
\]

so either \(p \leq 1/2\) or \(J < (2p - 1)^{-1}\). Similar reasoning shows that \(z_1 < z_0\) forces \(J(2p - 1) > 1\) and \(z_1 = z_0\) leads to \(J(2p - 1) = 1\). \[\]

To study the laws \((4.11)\) we define a new setting that fits Section 2. The relevant space is now \(\tilde{S} = S \times S \times \mathcal{X}\) with generic variable \((s, t, x)\). For a probability measure \(\rho\) on \(\tilde{S}\) we define

\[
\tilde{F}(\rho) = F(\rho_{sx}) + F(\rho_{tx})
\]

where \(\rho_{sx}\) and \(\rho_{tx}\) are the \((s, x)\)- and \((t, x)\)-marginals of \(\rho\). Then \(\mu^x_n \otimes \mu^n_x\) is the \((\sigma, \tau)\)-marginal of the measure defined by \((2.1)\) with \(F\) replaced by \(\tilde{F}\) and \(P^x\) replaced by \(\tilde{P}^x = \lambda \otimes \lambda \otimes \delta_x\). The entropy is (see \((5.10)\))

\[
\tilde{K}(\rho) = \begin{cases} H(\rho \mid \lambda_o, \lambda_o \otimes \pi_o) & \text{if } \rho_x = \pi_o, \\ \infty & \text{otherwise}. \end{cases}
\]

The rate function \(V\) of the laws \((4.11)\) comes by the contraction

\[
V(x) = \inf_{\rho(s,t)=x} \tilde{I}_1(\rho)
\]

from the rate function \(\tilde{I}_1\) of the empirical distribution of the tilde model.

**4.37. Lemma.**

\[
\tilde{I}(\rho) = 0 \text{ iff } \begin{cases} \rho = \lambda_o \otimes \lambda_o \otimes \pi_o & \text{in case } h = 0 \text{ and } \beta \leq \beta_c, \\ \rho(ds, dt, dx) = \psi^x_\pm(ds) \psi^x_\pm(dt) \pi_o(dx) & \text{in case } h = 0 \text{ and } \beta > \beta_c, \\ \text{(all four combinations allowed)}, \\ \rho(ds, dt, dx) = \psi^x(ds) \psi^x(dt) \pi_o(dx) & \text{in case } h \neq 0. \end{cases}
\]
Proof. The zeroes of \( \tilde{I} \) are the measures that minimize \( \tilde{K} - \tilde{F} \). It suffices to observe that for given \( \rho_{sz} \) and \( \rho_{tz} \) with \( (\rho_{sz})_z = (\rho_{tz})_z = \pi_o \), the entropy \( \tilde{K}(\rho) \) is uniquely minimized by \( \rho(ds,dt,dx) = \rho_{sz}^*(ds) \rho_{tz}^*(dt) \pi_o(dx) \) with entropy \( \tilde{K}(\rho) = H(\rho_{sz} | \lambda_o \otimes \pi_o) + H(\rho_{tz} | \lambda_o \otimes \pi_o) \). Thus we need to minimize

\[
\tilde{K}(\rho) - \tilde{F}(\rho) = H(\rho_{sz} | \lambda_o \otimes \pi_o) - F(\rho_{sz}) + H(\rho_{tz} | \lambda_o \otimes \pi_o) - F(\rho_{tz})
\]

over \( \rho_{sz}, \rho_{tz} \in \mathcal{M}_1(S \times \mathcal{X}) \) with \( \mathcal{X} \)-marginal \( \pi_o \), a task already done above. \( \square \)

Proof of Theorem 4.12. By the usual reasoning (compact sublevel sets), the zeroes of \( V \) are the numbers \( \rho(st) \) for \( \tilde{I}(\rho) = 0 \). For example, in case \( h = 0 \) and \( \beta > \beta_c \),

\[
\rho(st) = p(\{ \pm \}^1 z_1)([\pm]z_1) + (1 - p)(\{ \pm \} z_0)([\pm]z_0) = \pm(p z_1^2 + (1 - p) z_0^2),
\]

where \( \{ \pm \} \) and \( [\pm] \) are signs that have to agree. \( \square \)

Proof of Lemma 4.13. Since \( z_x(\beta, 0) = 0 \) for \( \beta \leq \beta_c \) and the strict increasingness for \( \beta \geq \beta_c \) was proved in Lemma 4.34, we only need to show that, if \( \beta \nless \beta' \) or \( \beta \not\succ \beta' \), \( \beta' \geq \beta_c \), then \( \lim_{\beta \rightarrow \beta'} z_x(\beta, 0) = z_x(\beta', 0) \). But this is immediate upon passing to the limit in the equations (4.2). \( \square \)

Proof of Theorem 4.14. (4.15) Let \( z_x = z_x(\beta, 0) \). Apply the expansion

\[
\frac{1}{2} \log \frac{1 + s}{1 - s} = s + \frac{s^2}{3} + O(s^5)
\]

first to \( z_1 = \xi_1(z_0) \) to get

\[
z_1 = \frac{1}{\beta J_p} ((1 + \beta(1 - p)) z_0 + \frac{z_0^3}{3} + O(z_0^5)),
\]

then to \( z_0 = \xi_0(z_1) \) to get

\[
z_0 = \frac{1}{\beta J(1 - p)} ((1 - \beta p) z_1 + \frac{z_1^3}{3} + O(z_1^5)).
\]

After some algebra

\[
C_1(\beta)(\beta - \beta_c) = C_2(\beta) z_1^2 + O(z_1^4)
\]

with

\[
C_1(\beta) = \frac{(1 + J^2)(\beta - \tilde{\beta})}{\beta^2 J^2},
\]

where \( \tilde{\beta} \) is the negative root of (4.31), and

\[
C_2(\beta) = \frac{(1 - \beta p)^3}{3 \beta^4 J^4 p(1 - p)^3} + \frac{1 - \beta p}{3 \beta^2 J^2 p(1 - p)}.
\]
Since \( C_1(\beta) \to C_1(\beta_c) > 0 \) as \( \beta \searrow \beta_c \) we have \( z_1 \sim (\beta - \beta_c)^{1/2} \). For \( z_0 \) use this and (4.40).

(4.16) The proof is so similar to the one above that we leave it to the reader.

(4.17) Fix \( \beta \neq \beta_c \). First we wish to argue that \( z_1 \) is a differentiable function of \( h \) around \( h = 0, z_1 = z_1(\beta, 0) \) by the implicit function theorem. Let

\[
f(h, s_1) = s_1 - \xi_1(h, \xi_0(h, s_1)),
\]

where we have explicitly indicated the dependence on \( h \). \( f(0, z_1(\beta, 0)) = 0 \) and

\[
\frac{\partial f}{\partial s_1} = 1 - \frac{\partial \xi_1}{\partial s_0} \frac{\partial \xi_0}{\partial s_1}.
\]

If \( \beta < \beta_c \) then \((0, z_1(\beta, 0)) = (0, 0)\) and this derivative is \( \neq 0 \) by (4.32); if \( \beta > \beta_c \) it is \( \neq 0 \) because \((\xi_1'(z_0))^{-1} = \eta_1'(z_1) < \xi_1'(z_1)\) as observed in the proof of Lemma 4.34.

Differentiating yields

\[
\frac{\partial z_1}{\partial h} = \frac{\partial \xi_1}{\partial h} + \frac{\partial \xi_1}{\partial z_0} \left( \frac{\partial \xi_0}{\partial h} + \frac{\partial \xi_0}{\partial z_1} \frac{\partial z_1}{\partial h} \right).
\]

Solving for \( \partial z_1 / \partial h \) gives

\[
\frac{\partial z_1}{\partial h} = \left\{ -[\beta^2(1 + J)(1 - p)(1 - z_0^2) + \beta](1 - z_1^2) \right\}
\]

\[
\cdot \left\{ [\beta^2(1 + J^2)p(1 - p) + \beta(2p - 1) - 1] + C_3(\beta)z_0^2 + C_4(\beta)z_1^2 + O(z_0^2 z_1^2) \right\}^{-1}.
\]

The numerator converges to a negative number as \( \beta \to \beta_c \). Notice that the first term in the denominator is the quadratic (4.31). In the case \( \beta < \beta_c \) we have \( z_0 = 0 \) so the denominator consists of only the first term, and (4.17) follows for \( x = 1 \). In case \( \beta > \beta_c \), use (4.40), (4.41), and some algebra to see that the denominator equals

\[
(\beta - \beta_c) \frac{C_1(\beta)}{C_2(\beta)} \left\{ \beta^2 J^2 p(1 - p) C_2(\beta) - \beta p(1 + \beta(1 - p)) - \beta^2 J^2 p(1 - p) \right.
\]

\[
+ \left( \frac{1 - \beta p}{\beta J(1 - p)} \right)^2 \left( \beta(1 - p)(1 - \beta p) - \beta^2 J^2 p(1 - p) \right) \Bigg\} + O(z_1^4)
\]

where the expression in \( \{ \} \)'s converges to

\[
-\frac{2}{3} \left( 1 + \beta_c(1 - p) + \frac{(1 - \beta_c p)^3}{\beta_c^2 J^2 (1 - p)} \right) < 0
\]

as \( \beta \to \beta_c \). (To get this, use repeatedly \( \beta_c^2 J^2 p(1 - p) - (1 + \beta_c(1 - p))(1 - \beta c p) = 0 \) and also \( 1 - \beta_c p > 0 \).) This proves (4.17) again for \( x = 1 \). To get (4.17) for \( x = 0 \) differentiate \( z_0 = \xi_0(h, z_1) \) to get

\[
\frac{\partial z_0}{\partial h} = \frac{1}{J(1 - p)} + C_5(\beta) \frac{\partial z_1}{\partial h},
\]
where $C_5(\beta) \to C_5(\beta_c) > 0$ as $\beta \nearrow \beta_c$.

(4.18) Let us first verify the remarks on specific energy stated before Theorem 4.14. Since $-\beta H_n^X = n F(\bar{L}_n)$, specific energy is given by

$$u(\beta, h) = -\beta^{-1} \lim_{n \to \infty} \gamma_n^X(F(\bar{L}_n)),$$

hence it is clear from the LDP’s (use Theorems 2.21 and 4.7) that

$$u(\beta, h) = -\beta^{-1} F(z_0(\beta, h), z_1(\beta, h)).$$

(In the case $\beta > \beta_c$, $h = 0$ the additional fact $F(z_0, z_1) = F(-z_0, -z_1)$ is needed, for the limit above is $-\beta^{-1}(F(z_0, z_1) + F(-z_0, -z_1))/2$.) The continuity of $u(\beta, 0)$ is clear by Lemma 4.13, and for $\beta \leq \beta_c$, $u(\beta, 0) = -\beta^{-1} F(0, 0) = 0$. For $\beta > \beta_c$ and $x = 0, 1$, $\partial z_x/\partial \beta$ exist by the implicit function theorem argument given above, and since $(z_0, z_1)$ minimizes $K - F$, $\partial F/\partial z_x(z_0, z_1) = \partial K/\partial z_x(z_0, z_1) > 0$, the inequality by noting that now $z_x > 0$. Consequently

$$\frac{\partial u}{\partial \beta} = \frac{\partial}{\partial \beta}(-\beta^{-1} F) = -\beta^{-1} \frac{\partial F}{\partial z_0} \frac{\partial z_0}{\partial \beta} - \beta^{-1} \frac{\partial F}{\partial z_1} \frac{\partial z_1}{\partial \beta}$$

$$= -\beta^{-1} \frac{\partial K}{\partial z_0} \frac{\partial z_0}{\partial \beta} - \beta^{-1} \frac{\partial K}{\partial z_1} \frac{\partial z_1}{\partial \beta} < 0,$$

where the last inequality follows from Lemma 4.13. This shows that $u(\beta, 0) < 0$ for $\beta > \beta_c$.

We now know that $C_{h=0}(\beta) = 0$ for $\beta < \beta_c$. To get $C_{h=0}$ for $\beta > \beta_c$, start with (4.44) and use $z_0 = \xi_0(z_1)$ to get

$$\beta^{-1} C_{h=0} = \left( \frac{\partial K}{\partial z_1} + \frac{\partial K}{\partial \xi_0} \frac{\partial \xi_0}{\partial z_1} \right) \frac{\partial z_1}{\partial \beta} + \frac{\partial K}{\partial z_0} \frac{\partial z_0}{\partial \beta} = D(\beta) + \frac{\partial K}{\partial \xi_0} \frac{\partial \xi_0}{\partial z_0} \frac{\partial z_0}{\partial \beta}.$$

The last term on the right vanishes as $\beta \searrow \beta_c$ so we ignore it. Replace $h$ by $\beta$ in (4.42), solve for $\partial z_1/\partial \beta$, and insert this in $D(\beta)$. Do the other derivatives in $D(\beta)$ directly from (4.2) and (4.27). Then use (4.38) and (4.40) to get

$$D(\beta) = \frac{-A_1(\beta) z_1^2 + O(z_1^2)}{C_1(\beta)(\beta - \beta_c) - A_2(\beta) z_1^2 + O(z_1^2)},$$

with

$$A_1(\beta) = \left( p + \frac{(1 - \beta p)^2}{\beta^2 J^2(1 - p)} \right) \cdot \frac{(1 - \beta p) + (1 + \beta(1 - p))}{\beta^3 J^2 p(1 - p)}$$

and

$$A_2(\beta) = \frac{(1 - \beta p)^2}{\beta^2 J^2(1 - p)^2} \cdot \left( 1 - \frac{1 - \beta p}{\beta J^2 p} \right) + 1 + \frac{1 + \beta(1 - p)}{\beta J^2(1 - p)}.$$

$A_1(\beta) \to A_1(\beta_c) > 0$ and $A_2(\beta) \to 3C_2(\beta_c)$ as $\beta \searrow \beta_c$. By (4.41) $D(\beta) \to A_1(\beta_c)/2C_2(\beta_c) > 0$ as $\beta \searrow \beta_c$. □
5 Proofs of the general theorems

We start by proving the large deviation results presented at the end of Section 2. Let \( \mathcal{E}_n \) denote the \( \sigma \)-field on \( \Omega \times \Sigma \) generated by \( B_0^\Omega \) and \( B^\Sigma \). Define \( \Phi \in \mathcal{M}_\Theta(\Omega \times \Sigma) \) by \( \Phi(d\sigma, dy) = P_y(d\sigma)\pi(dy) \), and let \( \Gamma \in \mathcal{M}_\Theta(\Omega \times \Sigma) \) have \( \Sigma \)-marginal \( \pi \), or \( \Gamma_\Sigma = \pi \) for short. Then by Theorem 2.13 in [Sel], the specific entropy

\[
h^\mathcal{E}(\Gamma | \Phi) = \lim_{n \to \infty} \frac{1}{|V_n|} H_{\mathcal{E}_n}(\Gamma | \Phi)
\]

exists, and \( k(Q) = \inf \Gamma h^\mathcal{E}(\Gamma | \Phi) \) where the infimum is over \( \Gamma \) with marginals \( Q \) and \( \pi \). Recall the definitions of \( \Psi_\nu \) and \( j_\infty(\nu) \) given in (2.16) and in the paragraph preceding it. Write \( \Psi_\nu = \pi \otimes \Psi_\nu \) for short.

5.1. Lemma. \( K(\nu) = k(j_\infty(\nu)) = h^\mathcal{E}(\Psi_\nu | \Phi) \) for \( \nu \in \mathcal{M}_1(S) \) such that \( K(\nu) < \infty \).

Proof. Since entropies of product measures are sums of the entropies of the factors, we get

\[
H_{\mathcal{E}_n}(\Psi_\nu | \Phi) = \int H_{\mathcal{E}_n}(\Psi_\nu | P^\nu \pi(dy)
= |V_n| \int H(\psi_\nu | p^\nu \pi(dy)
= |V_n| H(\psi_\nu | \varphi).
\]

The first and last equalities follow from the conditional entropy formula ((4.4.8) in [DS] or (10.2) in [V]). Hence

\[
k(j_\infty(\nu)) \leq h^\mathcal{E}(\Psi_\nu | \Phi) = H(\psi_\nu | \varphi) = K(\nu) \leq k(j_\infty(\nu)),
\]

where the last inequality follows from the contraction principle \( K(\nu) = \inf_{Q_0=\nu} k(Q) \). \( \square \)

Order \( \mathbb{Z}^d \) lexicographically, and let \( W^- = \{ i \in \mathbb{Z}^d : i < 0 \} \) be the past of the origin. For \( \Gamma \in \mathcal{M}_\Theta(\Omega \times \Sigma) \), let \( \Gamma_0^{\sigma_w \cdot y} \) be a conditional distribution of the spin \( \sigma_0 \) at the origin under \( \Gamma \), given the past \( \sigma_{W^-} = \{ \sigma_i : i \in W^- \} \) and \( y \). Analogously to (3.13) in [Fö], we have the representation

\[
h^\mathcal{E}(\Gamma | \Phi) = \int H(\Gamma_0^{\sigma_w \cdot y} | p^y) \Gamma(d\sigma_{W^-}, dy)
\]

whenever \( \Gamma_\Sigma = \pi \).

5.2. Lemma. Suppose \( k(Q) < \infty \). Then \( k(Q) > k(j_\infty(Q_0)) \) unless \( Q = j_\infty(Q_0) \).

Proof. Pick \( \Gamma \in \mathcal{M}_\Theta(\Omega \times \Sigma) \) with marginals \( Q \) and \( \pi \) so that \( k(Q) = h^\mathcal{E}(\Gamma | \Phi) \). Let \( \gamma \in \mathcal{M}_1(S \times \Sigma) \) be the \( \mathcal{E}_0 \)-marginal of \( \Gamma \). Let \( \psi_{Q_0} \) be the minimizer for \( Q_0 \) in (2.15).

\[
k(Q) = h^\mathcal{E}(\Gamma | \Phi) = \int H(\Gamma_0^{\sigma_w \cdot y} | p^y) \Gamma(d\sigma_{W^-}, dy)
\geq \int H(\Gamma_0^{\sigma_w \cdot y} \Gamma^y(d\sigma_{W^-}) | p^y) \pi(dy)
= \int H(\gamma | p^y) \pi(dy) = H(\gamma | \varphi)
\geq K(Q_0) = k(j_\infty(Q_0)).
\]
The first inequality above follows from convexity, the second from (2.15). Forcing the second inequality to be an equality gives $\gamma = \psi_{Q_0}$. Equality in the first inequality together with the strict convexity of relative entropy implies that $I_0^0 \sigma_0^w = \gamma \sigma^w \sigma_0$ a.s. In other words, $\sigma_0$ is independent of the past $\sigma_0^w$, given $\gamma$, and by shift-invariance this independence holds for all spins. We get $I^\gamma(d\sigma) = \otimes_1 \psi_0^{Q_0}(d\sigma_1)$, from which $Q = \int I^\gamma \pi(dy) = j_\infty(Q_0)$. ☐

**Proof of part of Lemma 2.20.**

\[ r_\infty = \inf_Q \{ k(Q) - F^u(Q_0) \} \]
\[ = \inf_Q \{ j_\infty(Q_0) - F^u(Q_0) \} \]
\[ = \inf_Q \{ K(Q_0) - F^u(Q_0) \} = r_1. \]

$K_\infty = j_\infty(K_1)$ is now immediate from Lemma 5.2. $I_\infty$ is lower semicontinuous by its definition, so from $\{ I_\infty \leq b \} \subset \{ k \leq r_1 + b + c \}$ we see that $\{ I_\infty \leq b \}$ is a closed subset of a compact set, hence itself compact. Taking $b = 0$ shows that $K_\infty$ is compact. Since $j_\infty(\nu) = \nu$, the projection $Q \mapsto Q_0$ is a continuous one-to-one map of the compact set $K_\infty$ onto $K_1$, hence its inverse $j_\infty$ is a homeomorphism. The remaining statement of Lemma 2.20 about $K_\infty^r$ will be proved in Lemma 5.12 below where it will follow naturally from considerations in the skew model setting. ☐

Theorem 2.21 follows immediately from the next lemma. For the lower bound, define

\[ I(Q) = k(Q) - F_\ell(Q_0) - r_1 \]

for $Q \in M_\Theta(\Omega)$ and $I(Q) = \infty$ for noninvariant $Q$.

**5.3. Lemma.** Under Assumption A, we have for any Borel set $E \subset M_1(\Omega)$,

\[
- \inf_{Q \in \mathcal{E}_0} I(Q) \leq \liminf_{n \to \infty} \frac{1}{|V_n|} \log \gamma_n^x(R_n \in E)
\leq \limsup_{n \to \infty} \frac{1}{|V_n|} \log \gamma_n^x(R_n \in E)
\leq - \inf_{Q \in \mathcal{E}} I_\infty(Q). \tag{5.4}
\]

**Proof.** The inequalities follow from the LDP of Theorem 2.1 in [Se1]: Apply Lemmas 2.1.7 and 2.1.8 of [DS] to the upper semicontinuous functions $\log 1_E(Q) + F^u(Q_0)$ and $F^u(Q_0)$ and to the lower semicontinuous functions $\log 1_E(Q) + F_\ell(Q_0)$ and $F_\ell(Q_0)$, defined for $Q \in M_1(\Omega)$. Use (2.5). ☐

**Proof of Theorem 2.7.** Since $K_\infty = \{ I_\infty = 0 \}$ is compact, the upper bound of (5.4) implies that the laws $\gamma_n^x(R_n \in \cdot)$ are tight. Consequently so are the measures $\xi_n = \gamma_n^x(R_n(\cdot))$. 
Suppose $\zeta_{n_j}^x \to Q$ as $j \to \infty$. By passing to a subsequence, we may assume that $\gamma_{n_j}^x(R_{n_j} \in \cdot) \to \Pi$. By the upper bound of (5.4) $\Pi(K_{\infty}) = 1$, and so

\begin{equation}
Q = \int_{K_{\infty}} M \Pi(dM).
\end{equation}

Let $\Lambda(d\nu) = \Pi(M_0 \in d\nu)$ be the image of $\Pi$ under the projection $M \mapsto M_0$. Since $M = j_{\infty}(M_0)$ for $M \in K_{\infty}$, (5.5) implies (2.8). Since $L_n$ is the projection of $R_n$, $\gamma_{n_j}^x(L_{n_j} \in \cdot) \to \Lambda$. $\square$

5.6. Lemma. Suppose $\pi$ is ergodic. Then $\Psi = \Psi_\nu$ is ergodic, whenever it is defined. In particular, all elements of $K_{\infty}$ are ergodic.

Proof. Let $A$ be an invariant Borel subset of $\Omega \times \Sigma$. Let $\epsilon > 0$ and find $m$ and $B \in \mathcal{E}_m$ such that $\Psi(A \triangle B) < \epsilon$. Let $n > m$, to eventually let $n \to \infty$. Since $A$ is invariant, $A = A \cap \theta_{-j}A$, and consequently

$$\left| \Psi(A) - \frac{1}{|V_n|} \sum_{j \in V_n} \Psi(B \cap \theta_{-j}B) \right| < 2\epsilon.$$ 

Let $H \subset \mathbb{Z}^d$ be such that $0 \in H$ and $\{i + V_m : i \in H\}$ is a disjoint cover of $\mathbb{Z}^d$. Now remove the $3^d$ central sites from $H$ so that $k + i + V_m \cap V_m = \emptyset$ for all $k \in V_m$ and $i \in H$. Then $\mathcal{E}_m$ and $\mathcal{E}_{i+k+V_m}$ are independent under $\Psi^y$ for $k \in V_m$ and $i \in H$, and since $(\theta_{-j}B)^y = \theta_{-j}(B^{\theta j}y)$ holds for $y$-sections, we get

$$\Psi(B \cap \theta_{-k-i}B) = \int \Psi^y(B^y) \Psi^{\theta i+k}(B^{\theta i+k}y) \pi(dy).$$

Let $H_n = \{i \in H : i + V_m \subset V_n\}$. Averaging over $k \in V_m$ and $i \in H_n$ gives

$$\frac{1}{|V_n|} \sum_{j \in V_n} \psi(B \cap \theta_{-j}B) = \int \Psi^y(B^y) \frac{1}{|V_n|} \sum_{j \in V_n} \psi^{\theta j y}(B^{\theta j}y) \pi(dy) + o(1),$$

which tends to $\Psi(B)^2$ as $n \to \infty$ by $\pi$'s ergodicity. We deduce that $\Psi(A) - \Psi(A)^2 \leq 4\epsilon$, and since $\epsilon$ was arbitrary, $\Psi(A) = 0$ or 1. $\square$

Proof of Theorem 2.9. Since the three sequences in question are tight, it suffices to show that there is a 1-1 correspondence between their limit points: $Q \leftrightarrow \Pi \leftrightarrow \Lambda$. $\Pi \leftrightarrow \Lambda$ follows from $\Pi(K_{\infty}) = 1$, $\Lambda(K_1) = 1$, and the homeomorphism $K_{\infty} \simeq K_1$. Since $K_{\infty}$ is now a set of ergodic measures, (5.5) is the ergodic decomposition of $Q$, and the 1-1 correspondence $Q \leftrightarrow \Pi$ holds by the uniqueness of the ergodic decomposition. That $\Pi = \Lambda(j_{\infty} \in \cdot)$ is clear since $\Lambda$ is the image of $\Pi$ under the inverse of $j_{\infty}$, namely the projection. $\square$

For the remainder of this section we work under Assumption B. Theorem 2.10 follows immediately and needs no proof. The first thing to check is that permuting finitely many coordinates of $x$ does not affect the limiting behavior of $\zeta_n^x$. So suppose $\tau$ is a permutation of $V_k$ for some $k$, and set $(\tau x)_i = x_{\tau(i)}$ for $i \in V_k$, $(\tau x)_i = x_i$ for $i \in V_k^c$. Let $f \in bB_m$ for some $m$. 


5.7. Lemma. \( \lim_{n \to \infty} |\zeta_n^x(f) - \zeta_n^{\tau x}(f)| = 0. \)

Proof. Letting \( \tau \) act on \( \Omega \) the same way it acts on \( \Sigma \), it is clear that \( P^{\tau x} = P^x \circ \tau^{-1} \). If \( n > k \), \( F(L_n) \) is invariant under \( \tau \), and consequently

\[
\zeta_n^{\tau x}(f) = P^{\tau x}(R_n(f) e^{|V_n| F(L_n)})/Z_n^{\tau x} = P^x(R_n(f) \circ \tau \cdot e^{|V_n| F(L_n)})/Z_n^x.
\]

Now notice that \( f(\theta_i \tau(\sigma)) = f(\theta_i \sigma) \) whenever \( i + V_m \cap V_k = \emptyset \). The fraction of sites in \( V_n \) for which this condition fails vanishes as \( n \to \infty \), hence

\[
\lim_{n \to \infty} \|R_n(f) \circ \tau - R_n(f)\| = 0. \quad \square
\]

Proof of Theorem 2.11. Without loss of generality we prove the theorem for the original sequence \( \{n\} \). To construct \( g \), let \( \{B_j\} \) be a countable basis for the topology of \( M_1(\Omega) \) with the property that only finitely many \( B_j \) have diameter \( \geq \varepsilon \) for any fixed \( \varepsilon > 0 \). Set

\[
g(x, Q) = \limsup_{j \to \infty}[1_{B_j}(Q) \cdot \limsup_{n \to \infty} 1_{B_j}(\zeta_n^x)].
\]

Clearly \( g(x, Q) \) is either 0 or 1, and it is 1 precisely when there are infinitely many \( B_j \) that both contain \( Q \) and are visited infinitely often by \( \zeta_n^x \). Since \( \text{diam} \ B_j \to 0 \), this is equivalent to saying that \( Q \) is a limit point of \( \{\zeta_n^x\}_{n=1}^\infty \). The previous lemma then implies that \( g(x, Q) = g(\tau x, Q) \) for all finite permutations \( \tau \).

To construct \( \Sigma_0 \), employ the functions

\[
h_j(x) = \lim_{n \to \infty} \left[ \max_{k \geq n} 1_{B_j^{(1/n)}}(\zeta_n^x) \right].
\]

Here \( B_j^{(1/n)} \) is a \( 1/n \)-fattening of \( B_j \). So for \( x \) and \( y \) such that \( \{\zeta_n^x\}_{n=1}^\infty \) and \( \{\zeta_n^y\}_{n=1}^\infty \) are tight, \( h_j(x) = 1 \) iff \( \{\zeta_n^x\}_{n=1}^\infty \) has a limit point in \( \overline{B_j} \), and \( \{\zeta_n^x\}_{n=1}^\infty \) and \( \{\zeta_n^y\}_{n=1}^\infty \) have the same limit points iff \( h_j(x) = h_j(y) \) for all \( j \). From the above lemma again we know that \( h_j \) is invariant under finite permutations. Set

\[
\Sigma_0 = \left\{ x : h_j(x) = \int h_j \, d\pi \text{ for all } j \right\}.
\]

An application of the Hewitt-Savage 0-1 law concludes the proof of the theorem. \( \square \)

Our next goal is Theorem 2.12 about the convergence of the maps \( \gamma_n \). First a simple general lemma about preserving tightness under averaging: Let \( Z \) and \( W \) be Polish spaces, \( \kappa \in M_1(W) \), \( I \) an index set, and \( w \mapsto \rho_\alpha^w, \alpha \in I \), a collection of measurable maps from \( W \) into \( M_1(Z) \). For \( \alpha \in I \), define \( \mu_\alpha \in M_1(Z) \) by \( \mu_\alpha = \int \rho_\alpha^w \pi(dw) \).
5.8. Lemma. If \( \{\rho^w_\alpha : \alpha \in I\} \) is tight for \( \kappa \)-a.e. \( w \), then the collection \( \{\mu_\alpha : \alpha \in I\} \) is also tight.

Proof. Let \( \{z_j\} \) be a countable dense subset of \( Z \), and let

\[
A_{k,n} = \bigcup_{j=1}^{n} \overline{B}_{1/k}(z_j),
\]

where \( \overline{B}_{1/k}(z_j) \) is the closed \( 1/k \)-radius ball around \( z_j \). Let \( \varepsilon > 0 \) and find \( \varepsilon_k > 0 \) such that \( 2 \sum_k \varepsilon_k < \varepsilon \). Let \( W_0 \) be the set of \( w \in W \) for which \( \{\rho^w_\alpha : \alpha \in I\} \) is tight. For each \( w \in W_0 \) and \( k \) find \( n(w,k) \) such that \( \inf_{\alpha} \rho^w_\alpha(A_{k,n(w,k)}) > 1 - \varepsilon_k \). Let

\[
W_{k,n} = \{w \in W_0 : n(w,k) = n\}.
\]

For each \( k \) find \( m(k) \) such that \( \kappa(\cup_{n=1}^{m(k)} W_{k,n}) > 1 - \varepsilon_k \). Then for all \( k \) and \( \alpha \),

\[
\mu_\alpha(A_{k,m(k)}) = \sum_{n=1}^{\infty} \int_{W_{k,n}} \rho^w_\alpha(A_{k,m(k)}) \kappa(dw)
\]
\[
\geq \sum_{n=1}^{m(k)} \int_{W_{k,n}} (1 - \varepsilon_k) \kappa(dw)
\]
\[
\geq (1 - \varepsilon_k)^2 \geq 1 - 2\varepsilon_k.
\]

Now set \( A = \cap_k A_{k,m(k)} \), and deduce \( \mu_\alpha(A) > 1 - \varepsilon \) for all \( \alpha \). \( A \subset \cup_{j=1}^{m(k)} \overline{B}_{1/k}(z_j) \) for all \( k \), so \( A \) is compact by Lemma 3.1 on p. 29 of [Pa]. \( \square \)

A further useful technical twist is periodization: Given \( \sigma \in \Omega \), define \( \sigma^{(n)} \in \Omega \) by \( \sigma^{(n)}_i = \sigma_i \) for \( i \in V_n \) and by requiring \( \sigma^{(n)}_{i+(2n+1)j} = \sigma^{(n)}_i \) for all \( i \) and \( j \). Using this we form the stationary version of the empirical process:

\[
R^{s}_n = \frac{1}{|V_n|} \sum_{i \in V_n} \delta_{\theta_i \sigma^{(n)}}.
\]

Under Assumption B we can perform the same operation also on quenched variables.

5.9. Lemma. Let \( f \in bB^\Omega \), \( g \in bB^\Sigma \), and \( i \in \mathbb{Z}^d \).

1. \( \int f(\theta_i \sigma^{(n)}) P^{y^{(n)}}(d\sigma) = \int f(\sigma^{(n)}) P^{\theta_i y^{(n)}}(d\sigma) \).

2. Suppose \( \pi \) is exchangeable. Then \( \int g(\theta_i y^{(n)}) \pi(dy) = \int g(y^{(n)}) \pi(dy) \).

3. If \( f \in bB^\Omega_n \), then \( \frac{1}{|V_n|} \sum_{i \in V_n} \gamma^{\theta_i y^{(n)}}_n (f) = \gamma^{y^{(n)}}_n (R^{s}_n (f)) \).
Proof. (1) It suffices to consider functions of the form \( f(\sigma) = \prod_{j \in W} f_j(\sigma_j) \) for a finite set \( W \). Write \( i \equiv j \) if \( i - j \in (2n + 1)\mathbb{Z}^d \). Then
\[
\int f(\theta_1 \sigma^{(n)}) P^{y^{(n)}}(d\sigma) = \int \prod_{j} f_j(\sigma_{i+j}) P^{y^{(n)}}(d\sigma) = \prod_{k \in V_n} \int_{j : j \equiv k} f_j d\gamma^{(n)}
\]
\[
= \prod_{l \in V_n} \int \prod_{j : j \equiv l} f_j d\gamma^{(n)} = \prod_{l \in V_n} \int \prod_{j : j \equiv l} f_j d\gamma^{(n)}
\]
\[
= \int f(\sigma^{(n)}) P^{\theta_1 y^{(n)}}(d\sigma).
\]

(2) It suffices to consider only IID \( \pi \), for the statement of (2) is linear in \( \pi \) and a general exchangeable measure is a mixture of IID’s. But for an IID \( \pi \) (2) is a special case of (1).

For (3), note that \( L_n(\sigma) = L_n(\sigma^{(n)}) = L_n(\theta_1 \sigma^{(n)}) \) and that \( f(\sigma) = f(\sigma^{(n)}) \) for \( f \in bB_{\Omega} \). If \( \gamma^{(n)} = P^{y^{(n)}} \), then also \( \gamma_{\theta_1 y^{(n)}} = P^{\theta_1 y^{(n)}} \), and (3) follows from (1). Otherwise use (1) to get
\[
\gamma_{\theta_1 y^{(n)}} (f) = P^{\theta_1 y^{(n)}} (f e^{V_n | F(L_n)}) / Z_{\theta_1 y^{(n)}}
\]
\[
= \int f(\theta_1 \sigma^{(n)}) e^{V_n | F(L_n(\sigma))} P^{y^{(n)}}(d\sigma) / Z_{\theta_1 y^{(n)}} = \int f(\theta_1 \sigma^{(n)}) \gamma^{y^{(n)}}(d\sigma).
\]

Now average over \( i \in V_n \) to conclude. \( \square \)

We now introduce the skew model associated with the setting of Assumption B: The quenched variable is adjoined to the process as a deterministic component, by defining \( \tilde{S} = S \times \mathcal{X}, \tilde{\Omega} = \Omega \times \Sigma, \tilde{p}^x = p^{x_0} \otimes \delta_{x_0}, \) and \( \tilde{P}^x = P^x \otimes \delta_x \). This defines a new setting of precisely the same type as before. By Theorem 3.3 in [Se1], the entropy of \( \tilde{\nu} \in M_{1}(\tilde{S}) \) is given by

\[
(5.10) \quad \tilde{K}(\tilde{\nu}) = \begin{cases} 
H(\tilde{\nu} | \varphi_0) \quad \text{if } \tilde{\nu} \text{ has marginal } \pi, \\
\infty \quad \text{otherwise}.
\end{cases}
\]

(In the setting of Section 3 in [Se1], \( U = \{0\} \) because \( x \mapsto \tilde{p}^x \) is \( B_{\tilde{S}} \)-measurable by Assumption B.) Write \( \tilde{\nu}_S \) for the \( S \)-marginal of a probability measure \( \tilde{\nu} \) on \( \tilde{S} \). We get the LDP of the original model by a contraction to the marginal \( [V, \text{Remark 1, p. 5}] \), hence by the uniqueness of the rate function

\[
(5.11) \quad K(\nu) = \inf_{\tilde{\nu} ; \tilde{\nu}_S = \nu} \tilde{K}(\tilde{\nu}),
\]

which together with (5.10) justifies (2.17).

Define \( \tilde{F} : M_{1}(\tilde{S}) \to [-\infty, c] \) by \( \tilde{F}(\tilde{\nu}) = F(\tilde{\nu}_S) \). Write \( \tilde{L}_n \) and \( \tilde{R}_n \) for the skew empirical distribution and field, so for example
\[
\tilde{R}_n = \frac{1}{|V_n|} \sum_{i \in V_n} \delta_{(\theta, \sigma, \theta_i x)}.\]
Let $\tilde{\gamma}_n^x$ be defined as in (2.1), and $\zeta_n^x(\cdot) = \tilde{\gamma}_n^x(\mathbf{R}_n(\cdot))$. Since $F(\mathbf{L}_n)$ does not depend on the $\Sigma$-valued coordinate, we have $\tilde{\gamma}_n^x = \gamma_n^x \otimes \delta_x$. The set

$$\overline{K}_1 = \{ \tilde{\nu} \in \mathcal{M}_1(\overline{S}) : \tilde{K}(\tilde{\nu}) - F^u(\tilde{\nu}) = \tilde{r}_1 \}$$

and its infinite-volume counterpart $\overline{K}_\infty$ are defined as before, and they are related by the homeomorphism $\tilde{j}_\infty : \overline{K}_1 \to \overline{K}_\infty$.

5.12. Lemma.

(1) $\overline{K}_\infty = \{ \Psi_\nu : \nu \in K_1 \}$ and the map $\nu \mapsto \Psi_\nu$ is a homeomorphism from $K_1$ to $\overline{K}_\infty$.

(2) $K_\infty = \{ \Psi_\nu : \nu \in K_1 \}$ and the map $\nu \mapsto \Psi_\nu$ is a homeomorphism from $K_1$ to $K_\infty$.

(3) The map $(\nu, y) \mapsto \Psi_\nu^y$ is jointly measurable on $K_1 \times \Sigma$.

Proof. (1) Since

$$\tilde{r}_1 = \inf_{\nu \in \mathcal{M}_1(\overline{S})} \{ \inf_{\tilde{\nu} \in \nu} \tilde{K}(\tilde{\nu}) - F^u(\nu) \},$$

it is clear from (2.17), (5.10), and the uniqueness of $\psi_\nu$ that $\tilde{r}_1 = r_1$ and $\overline{K}_1 = \{ \psi_\nu : \nu \in K_1 \}$. Since $K_1 \ni \nu \mapsto \psi_\nu \in \overline{K}_1$ is the inverse of a one-to-one continuous map (the projection) on a compact set, it is a homeomorphism. For (1) we need to show that $\tilde{j}_\infty(\psi_\nu) = \Psi_\nu$, by applying to the skew model the part of Lemma 2.20 already proved. But this is really only a matter of seeing through the formalities: Suppose $\tilde{\nu}_x = \pi_0$. Strictly speaking the measure $\tilde{\psi}_\nu$ associated to $\tilde{\nu}$ by (2.17) lives on $\overline{S} \times \mathcal{X} = S \times \mathcal{X} \times \mathcal{X}$, but by (5.10) we can neglect the unnecessary extra $\mathcal{X}$-factor and identify $\tilde{\psi}_\nu$ with $\tilde{\nu}$ itself. Under this identification the kernel $\tilde{\psi}_\nu$ associated to $\tilde{\nu}$ becomes just the conditional distribution $\tilde{\nu}_x$ of $\tilde{\nu}$ on $\overline{S}$, given $x \in \mathcal{X}$, and then $\tilde{j}_\infty(\tilde{\nu}) = \bigotimes_i \tilde{\nu}_x^i (d\sigma_i) \pi(dy)$. Taking $\tilde{\nu} = \psi_\nu$ then gives $\tilde{j}_\infty(\psi_\nu) = \Psi_\nu$.

There is also an indirect argument via entropy: By Lemma 5.1, (5.10), and (3.6) in [Se1],

$$\bar{k}(\tilde{j}_\infty(\psi_\nu)) = \bar{K}(\psi_\nu) = H(\psi_\nu | \varphi_o) = K(\nu) = h^x(\psi_\nu | \Phi) = \bar{k}(\psi_\nu).$$

Hence by Lemma 5.2 $\tilde{j}_\infty(\psi_\nu) = \Psi_\nu$.

(2) By Lemma 5.1 and (3.6) in [Se1], $k(\pi \psi_\nu^x) = h^x(\pi \otimes \psi_\nu^x | \Phi) = \bar{k}(\pi \otimes \psi_\nu^x)$, hence it is clear that the homeomorphism $\pi^x \mapsto \psi_\nu^x$ from $\mathcal{M}_x$ into $\mathcal{M}_1(\Omega \times \Sigma)$ restricts to a homeomorphism from $K_\infty$ onto $\overline{K}_\infty$. (2) now follows from (1).

(3) Let $(\tilde{\nu}, x) \mapsto q(\tilde{\nu}, x)$ be a jointly measurable conditional distribution map on $\overline{S}$, that is, $x \mapsto q(\tilde{\nu}, x)$ is a version of $\tilde{\nu}_x$. (Since the conditioning $\sigma$-field $B_\mathcal{X}$ is countably generated, $q(\tilde{\nu}, x)$ can be defined by the martingale convergence theorem.) Write $\Psi_\nu^y = \bigotimes_i q(\psi_\nu^y, y_i)$ and recall that $\nu \mapsto \psi_\nu$ is continuous.

Proof of Theorem 2.7. By Theorem 2.7 applied to the skew model and Lemma 5.8, the measures $\pi_\nu^x = \int \tilde{\gamma}_n^x \pi(dy)$ are tight. Thus the relative compactness of $\{ \gamma_n \}_{n=1}^\infty$ will follow from proving

$$\lim_{n \to \infty} \left| \int \gamma_n^x(f) \pi(dy) - \int \tilde{\gamma}_n^x(f) \pi(dy) \right| = 0$$

for all $f \in L^1(\mathcal{X})$. Since $\nu \mapsto \psi_\nu$ is a homeomorphism from $K_1$ to $\overline{K}_1$, it suffices by Lemma 5.12 to prove this for all $f \in L^1(\mathcal{X})$.
for any \( f \in bB_m^{\Sigma} \), where the \( \sigma \)-field \( B_m^{\Sigma} \) on \( \overline{\Omega} \) is generated by \( (\sigma_i, x_i : i \in V_m) \). So fix such an \( f \) and let \( n > m \). In the following calculation, use \( \gamma_n^s = \gamma_n^s \otimes \delta_y \), the fact that \( \gamma_n^{s}(f) = \gamma_n^{s}(n)(f) \) by \( B_n^{s} \)-measurability, Lemma 5.9(2) and (3), again \( B_n^{s} \)-measurability, then the fact that
\[
\lim_{n \to \infty} \| \overline{R}_n(f) - \overline{R}_n^s(f) \| = 0,
\]
and finally the definition of \( \zeta_n^s \).

\[
\int \gamma_n^s(fy) \pi(dy) = \int \overline{\gamma}_n^s(f) \pi(dy) = \int \overline{\gamma}_n^{s}(n)(f) \pi(dy)
\]
\[
= \int \frac{1}{|V_n|} \sum_{i \in V_n} \gamma_i^{s}(n)(f) \pi(dy) = \int \overline{\gamma}_n^{s}(\overline{R}_n(f)) \pi(dy) = \int \overline{\gamma}_n^{s}(\overline{R}_n(f))^s \pi(dy)
\]
\[
= \int \overline{\gamma}_n^s(\overline{R}_n(f)) \pi(dy) + o(1) = \int \zeta_n^s(f) \pi(dy) + o(1).
\]

This proves the relative compactness of \( \{\gamma_n^s\}_n \).

Suppose \( \gamma_{n_j}^s \to \gamma^s \) as \( j \to \infty \). Applying (2.22) to the skew model shows the laws \( \overline{\gamma}_n^s(\overline{R}_n \in \cdot) \) tight for \( \pi \)-a.e. \( x \), so an application of Lemma 5.8 and a passage to a subsequence if necessary imply that \( \pi \gamma_{n_j}(\overline{R}_{n_j} \in \cdot) \to \overline{\mu} \) as \( j \to \infty \), for some probability measure \( \overline{\mu} \) on \( M_{\Theta}(\Omega \times \Sigma) \). The equality of the first and penultimate term in (5.13) gives, upon passing to the limit along \( \{n_j\} \),

\[
\int \gamma_n^s(fy) \pi(dy) = \int \mathcal{M}_{\Theta}(\Omega \times \Sigma) \mu(f) \overline{\mu}(d\mu).
\]

(2.22) also implies that \( \overline{\gamma}_n^s(\overline{R}_n \in U^c) \to 0 \) for any open neighborhood \( U \) of \( \overline{\mathcal{K}}_\infty \), hence by dominated convergence \( \overline{\mu}(\overline{\mathcal{K}}_\infty) = 1 \). Letting \( \Lambda = \lim_{j \to \infty} \pi \gamma_{n_j}(L_{n_j} \in \cdot) \), Lemma 5.12(1) implies \( \overline{\mu} = \Lambda(\nu : \Psi_\nu \in \cdot) \). We can rewrite (5.14) as

\[
\int \gamma_n^s(f) \pi(dy) = \int \Psi_\nu(f) \Lambda(d\nu) = \int \int \Psi_\nu^s(f \pi) \Lambda(d\nu) \pi(dy).
\]
This gives (2.13), and the invariance of \( \gamma \) follows from (2.13). \( \square \)

**Proof of Theorem 2.14.** The a.s. convergence \( \zeta_n^s \to Q \) follows from Theorem 2.11. Let \( \Lambda \) be the measure appearing in the decomposition (2.8) of \( Q \). Let \( \gamma^s \) be a limit point of \( \{\gamma_{n_j}^s\} \) realized along a further subsequence \( \{n'_j\} \), with mixing measure \( \Lambda' \) in its decomposition (2.13). Take \( f \in bB_m^\Omega \) in (5.13) and (5.15) to see that

\[
\int j_{\infty}(\nu) \Lambda(d\nu) = Q = \lim_{j \to \infty} \pi \zeta_{n'_j} = \lim_{j \to \infty} \pi \gamma_{n'_j} = \pi \gamma^s
\]
\[
= \int \int \Psi_\nu^s \Lambda'(d\nu) \pi(dy) = \int \int \Psi_\nu^s \Lambda'(d\nu) \pi(dy).
\]

The above decomposition of \( Q \) is unique because the \( j_{\infty}(\nu) \)'s are IID, so \( \Lambda' = \Lambda \). The relatively compact sequence \( \{\gamma_{n'_j}\} \) must converge since it has a unique limit point. \( \square \)
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