HEAD-MEDIA INTERACTION IN MAGNETIC RECORDING

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HEAD-MEDIA INTERACTION IN MAGNETIC RECORDING

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Abstract. The head-tape interaction in magnetic recording is modelled by a coupled system of a second order differential equation for the pressure and a fourth order differential equation for the tape deflection. There is also the constraint that the spacing between the head and tape remains positive. In this paper, we study the stationary 1-d case: we establish the existence of a smooth solution and a boundary layer phenomenon observed both numerically and experimentally. The 2-d case is briefly discussed.

1. The model. The motion of magnetic media entrains air in between the head and media that forms a thin air film separating the head and the media. Figure 1 shows the one-dimensional head-media interaction system

![Diagram of head-media interaction](image)

Figure 1.

Here

\[ y = \text{length parameter}, \]
\[ \hat{u} = \text{deflection of the tape}, \]
\[ \hat{\delta} = \text{the profile of the head}, \]

and

\[ \hat{h} = \text{the spacing between head and tape}. \]

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This problem was studied numerically by several authors; see [1] [2] [6] and the references therein.

The mathematical model for the air pressure \( \hat{p} \) between the head and the tape is based on the modified Reynolds equation, which is the lubrication approximation to the Navier-Stokes equations (taking slip into account)

\[
12 \frac{\partial (\hat{p} \hat{h})}{\partial t} + 6V \frac{\partial (\hat{p} \hat{h})}{\partial y} - \frac{6\lambda p_a}{\mu} \frac{\partial}{\partial y} \left( \hat{h}^2 \frac{\partial \hat{p}}{\partial y} \right) - \frac{\partial}{\partial y} \left( \frac{\hat{h}^3 \hat{p}}{\mu} \frac{\partial \hat{p}}{\partial y} \right) = 0, \quad \hat{L}_1 < y < \hat{L}_2.
\]

Here

\[
\mu = \text{air viscosity},
\]

\[
\lambda = \text{mean free path length at ambient pressure},
\]

\[
p_a = \text{atmospheric pressure},
\]

\[
V = \text{tape velocity}.
\]

The tape deflection \( \hat{u} \) satisfies the elasticity equation

\[
\rho \frac{\partial^2 \hat{u}}{\partial t^2} + 2\rho V \frac{\partial^2 \hat{u}}{\partial y \partial t} - (T - \rho V^2) \frac{\partial^2 \hat{u}}{\partial y^2} + EI \frac{\partial^4 \hat{u}}{\partial y^4} = (\hat{p} - p_a) \chi_{[\hat{L}_1,\hat{L}_2]}(y), \quad 0 < y < \hat{L},
\]

and

\[
\hat{u}(y, t) = \hat{\delta}(y) + \hat{h}(y, t).
\]

Here

\[
\rho = \text{mass density of the tape},
\]

\[
T = \text{tape tension},
\]

\[
E = \text{Young's modulus of the tape},
\]

\[
I = \text{second moment of inertia of the tape section}.
\]

Typical magnitudes of the above quantities are:

\[
\mu = 2 \times 10^{-5} \text{Kg/m} \cdot \text{sec}, \quad \lambda = 6 \times 10^{-8} \text{m}, \quad p_a = 8 \times 10^4 \text{N/m}^2,
\]

\[
V = 2.5 \text{m/sec}, \quad \rho = 4 \times 10^{-2} \text{Kg/m}^2, \quad T = 3 \times 10^2 \text{N/m},
\]

\[
EI = 2 \times 10^{-5} \text{N} \cdot \text{m}, \quad \hat{L} = 10^{-1} \text{m}, \quad \hat{L}_2 - \hat{L}_1 = 10^{-2} \text{m},
\]

where \( N = \text{Newton} \). The spacing \( \hat{h} \) is of order of magnitude of \( 10^{-6} \text{m} \). For more details on the model see [1] and [4; Chap. 3].

We shall consider the steady state and nondimensionalize the system by introducing

\[
x = \frac{y}{10^{-2}}, \quad p(x) = \frac{\hat{p}(y)}{p_a}, \quad h(x) = \frac{\hat{h}(y)}{10^{-6}},
\]

\[
u(x) = \frac{\hat{u}(y)}{10^{-6}}, \quad \delta(x) = \frac{\hat{\delta}(y)}{10^{-6}}.
\]
We then obtain the following system:

\begin{align}
(1.1) \quad & \frac{\partial (ph)}{\partial x} - \varepsilon \frac{\partial}{\partial x} \left( \alpha h^2 \frac{\partial p}{\partial x} + \beta h^3 \frac{\partial p}{\partial x} \right) = 0, \quad L_1 < x < L_2, \\
(1.2) \quad & -\frac{\partial^2 u}{\partial x^2} + \eta \frac{\partial^4 u}{\partial x^4} = K(p-1)\chi_{[L_1,L_2]}, \quad 0 < x < L, \\
(1.3) \quad & u(x) = h(x) + \delta(x), \quad h(x) > 0 \quad \text{if } L_1 \leq x \leq L_2,
\end{align}

where

\[ \varepsilon = O(10^{-2}), \quad \beta = O(1), \quad \alpha = O(10^{-1}), \quad \eta = O(10^{-3}), \]

\[ L_2 - L_1 = O(1), \quad L = O(10), \quad K = O(10^4). \]

The system is supplemented by the boundary conditions:

\begin{align}
(1.4) \quad & p = 1 \quad \text{at } x = L_1 \text{ and } x = L_2, \\
(1.5) \quad & u = \frac{\partial u}{\partial x} = 0 \quad \text{at } x = 0 \text{ and } x = L. 
\end{align}

We assume throughout this paper that \( \delta(x) \) is in \( C^2[L_1,L_2] \) and

\[ \delta(x) > 0, \quad \delta''(x) < 0 \text{ if } L_1 \leq x \leq L_2, \]

\[ \frac{\delta(L_1)}{L_1} < \delta'(L_1). \]

**Theorem 1.1.** There exist positive constants \( \varepsilon^*, \eta^* \) such that if \( 0 < \varepsilon < \varepsilon^*, \)
\( 0 < \eta < \eta^* \), then the system (1.1)–(1.6) has a classical solution \( (p,h,u) \) with \( p, h \) in \( C^\infty[L_1,L_2] \), and \( p > 0, h > 0 \) in \( [L_1,L_2] \).

This result will be proved in Sections 2–5. In Section 6 we shall study the boundary layer behavior of the solution as \( \varepsilon \to 0, \eta \to 0 \). Finally, in Section 7 we discuss the two-dimensional case.

2. The special case \( \eta = 0, \varepsilon = 0 \). In the special case \( \eta = 0, \varepsilon = 0 \) the system
(1.1)–(1.3) reduces to

\begin{align}
(2.1) \quad & \frac{\partial (ph)}{\partial x} = 0, \quad h = u - \delta > 0 \text{ if } L_1 \leq x \leq L_2, \\
(2.2) \quad & -\frac{\partial^2 u}{\partial x^2} = K(p-1)\chi_{[L_1,L_2]} \text{ if } 0 < x < L.
\end{align}

Since we are dealing with a singular perturbation, only some of the boundary conditions in (1.4), (1.5) are preserved. As will be shown later on, the correct boundary conditions for (2.1), (2.2) are:

\begin{align}
(2.3) \quad & p = 1 \text{ at } x = L_1, \\
(2.4) \quad & u = 0 \text{ at } x = 0 \text{ and } x = L.
\end{align}
From (2.1), we deduce that $p \cdot h = \text{constant} = C$; since $p(L_1) = 1$,

$$C = h(L_1) = u(L_1) - \delta(L_1),$$

and (2.2) becomes

$$(2.5) \quad -u_{xx} = K \left( \frac{u(L_1) - \delta(L_1)}{u(x) - \delta(x)} - 1 \right) \chi_{[L_1, L_2]}.$$

To solve (2.5), (2.4) we use the shooting method: We solve (2.5) subject to the initial conditions

$$(2.6) \quad u(0) = 0, \quad u_x(0) = k \quad (> 0)$$

and vary the parameter $k$. Any solution of (2.5), (2.6) is continued as long as it remains larger than $\delta(x)$. We shall denote this solution (extended to its maximal existence interval) by $u(x, k)$. If $k \leq \delta(L_1)/L_1$, then the solution does not exist for $x > L_1$; hence we shall always take $k > \delta(L_1)/L_1$.

**Lemma 2.1.** If $k \geq \delta'(L_1)$, then $u_x(x, k) \geq 0$ for all $x \geq 0$.

**Proof.** It suffices to prove the lemma for $k > \delta'(L_1)$. Introduce the linear function $v(x) = kx$. Then $u - v = (u - v)_x = 0$ at $x = L_1$. Also $u_{xx}(L_1 + 0, k) = 0$ and

$$u_{xxx}(L_1 + 0, k) = \frac{K}{u(L_1, k) - \delta(L_1)}(k - \delta'(L_1)) > 0$$

so that $u_{xx}(x, k) > 0$ in some interval $L_1 < x < L_1 + \varepsilon_1$, and consequently $(u - v)_x > 0$, $(u - v) > 0$ if $L_1 < x < L_1 + \varepsilon_1$. Let $x_0$ denote the largest number $\leq L_2$ such that $(u - v)_x > 0$, $(u - v) > 0$ if $L_1 < x < x_0$. If $x_0 = L_2$ then the lemma follows. Suppose then that $x_0 < L_2$. Then $u_x(x_0, k) = v_x(x_0)$ and

$$u_{xx}(x_0, k) \leq v_{xx}(x_0) = 0.$$

However, since $u(x_0, k) > v(x_0)$ and $u(L_1, k) = v(L_1),$

$$u_{xx}(x_0, k) = -K \left( \frac{u(L_1, k) - \delta(L_1)}{u(x_0, k) - \delta(x_0)} - 1 \right) > -K \left( \frac{v(L_1) - \delta(L_1)}{v(x_0) - \delta(x_0)} - 1 \right)$$

whereas, by (1.6)

$$v(x_0) - \delta(x_0) > v(L_1) - \delta(L_1) > 0.$$

It follows that $u_{xx}(x_0, k) > 0$, a contradiction. \qed

**Lemma 2.2.** Let $\frac{\delta(L_1)}{L_1} < k < \delta'(L_1)$. Then

$$(2.7) \quad u_{xx}(x, k) < 0 \quad \text{if} \quad 0 < x - L_1 < \varepsilon_1$$
for some small \( \varepsilon_1 > 0 \), and

\[(2.8) \quad u_{xx}(x, k) \text{ will change sign at most once in the interval } (L_1, L_2).\]

**Proof.** Since \( u_{xx}(L_1, k) = 0 \) and

\[u_{xxx}(L_1 + 0, k) = \frac{K}{u(L_1, k) - \delta(L_1)(k - \delta'(L_1))} < 0,\]

(2.7) follows. To prove (2.8) suppose that \( u_{xx}(x, k) \) changes sign and let \( \bar{x} \) denote the first zero of \( u_{xx}(x, k) \) in the interval \( (L_1, L_2) \), i.e.,

\[u_{xx}(x, k) < 0 \quad \text{if } L_1 < x < \bar{x} < L_2,\]
\[u_{xx}(\bar{x}, k) = 0.\]

From the differential equation for \( u \) it follows that

\[u(x, k) - \delta(x) < u(L_1, k) - \delta(L_1) \quad \text{if } L_1 < x < \bar{x},\]
\[u(\bar{x}, k) - \delta(\bar{x}) = u(L_1, k) - \delta(L_1).\]

Hence

\[u_x(\bar{x}, k) \geq \delta'(\bar{x}).\]

Since also

\[u_{xx}(\bar{x}, k) = 0 > \delta''(\bar{x}),\]

it follows that

\[u_x(x, k) > \delta'(x) \quad \text{if } 0 < x - \bar{x} < \sigma\]

for some \( \sigma > 0 \), and therefore

\[u(x, k) - \delta(x) > u(\bar{x}, k) - \delta(\bar{x}) = u(L_1, k) - \delta(L_1) \quad \text{by (2.5) and } u_{xx}(\bar{x}, k) = 0,\]

so that, by the differential equation for \( u \),

\[u_{xx}(x, k) > 0 \quad \text{if } 0 < x - \bar{x} < \sigma.\]

Since \( \delta'' < 0 \), we can extend the solution beyond \( x = \bar{x} + \sigma \); furthermore, as long as \( u_{xx} \) remains positive, we have \( u_x(x, k) - \delta'(x) > 0 \) and

\[u(x, k) - \delta(x) > u(L_1, k) - \delta(L_1),\]
and so, by the differential equation, \( u_{xx} \) will remain uniformly positive. □

**Lemma 2.3.** Suppose

\[
\frac{\delta(L_1)}{L_1} < k_1 < k_2 < \delta'(L_1)
\]

and

\[
u_{xx}(x, k_1) \leq 0 \quad \text{for } 0 \leq x < \overline{x} \quad \text{and some } L_1 < \overline{x} \leq L_2.
\]

Then

\[
u(x, k_2) - u(L_1, k_2) > u(x, k_1) - u(L_1, k_1) \quad \text{for } L_1 < x < \overline{x},
\]

and

\[
u_x(x, k_2) > u_x(x, k_1) \quad \text{for } 0 < x < \overline{x}.
\]

**Proof.** Since (2.10) is a consequence of (2.11) whereas (2.11) is already valid for \( 0 \leq x \leq L_1 \), it remains to prove that (2.11) holds for all \( x \) in \((L_1, \overline{x})\). If this is not the case then there exists a point \( x^* \) in \((L_1, \overline{x})\) such that

\[
u(x, k_2) - u(L_1, k_2) > u(x, k_1) - u(L_1, k_1), \quad L_1 < x < x^*,
\]

\[
u_x(x, k_2) > u_x(x, k_1), \quad 0 \leq x < x^*
\]

and

\[
u_x(x^*, k_2) = u_x(x^*, k_1).
\]

It follows that \( u_{xx}(x^*, k_2) \leq u_{xx}(x^*, k_1) \leq 0 \), or, from the differential equation,

\[
u(L_1, k_2) - \delta(L_1)

u(x^*, k_2) - \delta(x^*) \geq u(x^*, k_1) - \delta(x^*)
\]

Combining this with the first inequality in (2.12) for \( x = x^* \), we find that

\[
u(x^*, k_1) - \delta(x^*) \frac{u(L_1, k_1) - \delta(L_1)}{u(L_1, k_1) - \delta(L_1)} > 1.
\]

This implies, by (2.5), that \( u_{xx}(x^*, k_1) > 0 \), which is a contradiction. □

**Lemma 2.4.** Let \( \frac{\delta(L_1)}{L_1} < k < \delta'(L_1) \). If

\[
u_x(x, k) \geq -C \quad \text{for } L_1 < x < \overline{x} \quad \text{and some } \overline{x} \in (L_1, L_2),
\]

then \( u(\overline{x}, k) > \delta(\overline{x}) \) and hence the solution \( u(x, k) \) can be extended beyond \( x = \overline{x} \).
Proof. Since \(-u_{xx} \geq -K\) in \((L_1, L_2)\), \(u_{xx} \leq K\) and \(u_x\) is bounded from above. Under the assumption (2.13) it then follows that \(u(\bar{x} - 0, k)\) exists; further, by (2.13),
\[
u(x, k) \leq u(\bar{x}, k) + C(\bar{x} - x) \quad \text{if } L_1 < x < \bar{x}.
\]
This implies that
\[
u_{xx}(x, k) \geq K \left( \frac{u(L_1, k) - \delta(L_1)}{C(\bar{x} - x) + u(\bar{x}, k) - \delta(x)} - 1 \right) \quad \text{for } L_1 < x < \bar{x}.
\]
We need to show that \(u(\bar{x}, k) > \delta(\bar{x})\). Suppose \(u(\bar{x}, k) = \delta(\bar{x})\). Then, since \(C\) may be assumed to be larger than \(\sup |\delta'|\),
\[
u_x(\bar{x} - 0, k) - \nu_x(x, k) = \int_x^{\bar{x} - 0} \nu_{xx}(\xi, k) d\xi \\ \leq -K \int_x^{\bar{x} - 0} \left( \frac{u(L_1, k) - \delta(L_1)}{C(\bar{x} - \xi) + \delta(\bar{x}) - \delta(\xi)} - 1 \right) d\xi = -\infty
\]
for \(L_1 < x < \bar{x}\), which is a contradiction to (2.13). \(\square\)

We define
\[
\bar{k} \equiv \inf \left\{ k > 0 ; \inf_{L_1 \leq x \leq L_2} [u(x, \bar{k}) - \delta(x)] > 0 \quad \forall \ k \leq \bar{k} < \delta'(L_1) \right\}.
\]
Then
\[
\frac{\delta(L_1)}{L_1} < \bar{k} < \delta'(L_1),
\]
where the second inequality follows from Lemma 2.1.

**Lemma 2.5.** There holds:

\begin{align*}
(2.14) & \quad u(x, \bar{k}) > \delta(x) \quad \text{if } L_1 \leq x < L_2, \\
(2.15) & \quad u(L_2, \bar{k}) = \delta(L_2),
\end{align*}

i.e., the maximal existence interval for \(u(x, \bar{k})\) is \(0 \leq x < L_2\).

**Remark 2.1.** Since \(u_{xx}\) has a fixed sign near \(x = L_2\), Lemma 2.4 and (2.15) imply that
\[
u_x(L_2 - 0, \bar{k}) = -\infty.
\]

**Proof.** Suppose (2.14) is not true. Then there exists an \(x^* \in (L_1, L_2)\) such that
\[
u(x, \bar{k}) > \delta(x) \quad \text{if } L_1 \leq x < x^*, \\
u(x^*, \bar{k}) = \delta(x^*).
\]

As in Remark 2.1
\[
u_x(x^* - 0, \bar{k}) = -\infty
\]
and so for any large positive number $N$ there is an $x^{**} < x^*$ ($x^* - x^{**}$ small) such that $u_x(x^{**}, k) < -3N$; then also

$$u_x(x^{**}, k + \sigma) < -2N$$

if $\sigma$ is positive and small enough. Since $u_{xx} \leq K$, we deduce that

$$u_x(x, k + \sigma) < -N$$

for all $x^{**} < x < L_2$. Taking $N$ large enough we find that the curve $y = u(x, k + \sigma)$ intersects the curve $y = \delta(x)$ at some point between $x^{**}$ and $L_2$ (in fact, near $x^*$), and this contradicts the definition of $k$.

Next, if (2.15) is not true, i.e., if $u(L_2, k) > \delta(L_2)$ then by continuity we have that $u(x, k - \sigma)$ exists for all $0 < x \leq L_2$, for any $\sigma$ positive and small enough, and

$$\inf_{L_1 \leq x \leq L_2} [u(x, k - \sigma) - \delta(x)] > 0;$$

this however contradicts the definition of $k$. \qed

From (2.16), we have, for any $N \gg 1$,

$$u_x(L_2 - \bar{\delta}, k) < -3N$$

if $\bar{\delta}$ is small enough and, by the continuous dependence of the solution on $k$,

$$u_x(L_2 - \bar{\delta}, k + \sigma) < -2N$$

if $\sigma > 0$ and small. Since $u_{xx} \leq K$ it follows that $u_x(x, k + \sigma) < -N$ if $x > L_2 - \bar{\delta}$, and therefore

(2.17) \quad $u(L, k + \sigma) < 0$.

The inequality

$$u(L_2, k + \sigma) - \delta(L_2) < (k + \sigma)L_1 - \delta(L_1)$$

is true for $\sigma = 0$ and by continuity also for $\sigma > 0$ and sufficiently small. It implies (by the differential equation for $u$) that

(2.18) \quad $u_{xx}(L_2, k + \sigma) < 0$

and therefore, by Lemma 2.2,

(2.19) \quad $u_{xx}(x, k + \sigma) \leq 0$ \quad if $0 \leq x < L_2$. 

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We now define
\[ k^* = \sup \{ k; \; k \in (\bar{k}, \delta'(L_1)) \text{ and } u(L, k) \leq 0 \}, \]
\[ k_* = \inf \{ k; \; k \in (\bar{k}, \delta'(L_1)) \text{ and } u(L, k) \geq 0 \}. \]

Lemma 2.1 and (2.17) show that \( k^* \) and \( k_* \) are well defined, and then, clearly, \( u(x, k^*) \), \( u(x, k_*) \) are solutions of (2.5), (2.4).

**Lemma 2.6.** For any solution \( u(x) \) of (2.5), (2.4),

\[ u(x, k_*) \leq u(x) \leq u(x, k^*), \quad 0 < x < L. \]  

**Proof.** We shall show that any two non-identical solutions \( u_1(x) \) and \( u_2(x) \) cannot intersect; this clearly implies the assertion (2.20), since \( k_* \leq k \leq k^* \) if \( u(x) = u(x, k) \).

If \( u_1(L_1) = u_2(L_1) \) then obviously \( u_1 \equiv u_2 \). Hence we may suppose that \( u_1(L_1) \neq u_2(L_1) \). Consider for definiteness the case \( u_1(L_1) > u_2(L_1) \). Then

\[ -(u_1)_{xx} = K \left( \frac{u_1(L_1) - \delta(L_1)}{u_1(x) - \delta(x)} - 1 \right) \chi_{[L_1, L_2]} \]

\[ \geq K \left( \frac{u_2(L_1) - \delta(L_1)}{u_1(x) - \delta(x)} - 1 \right) \chi_{[L_1, L_2]}, \]

or

\[ -(u_1)_{xx} + F(u_1, x) \geq 0 \]

where

\[ F(s, x) = -K \left( \frac{u_2(L_1) - \delta(L_1)}{s - \delta(x)} - 1 \right) \chi_{[L_1, L_2]} \]

Since \( -(u_2)_{xx} + F(u_2, x) = 0 \) and

\[ \frac{\partial F(s, x)}{\partial s} > 0, \]

the maximum principle shows that \( u_1(x) > u_2(x) \) for \( 0 < x < L \). \( \square \)

We summarize the main results of this section:

**Theorem 2.7.** (i) There exists at least one classical solution to the system (2.1)–(2.4); (ii) No two distinct solutions of (2.5), (2.4) can intersect, and every solution \( u(x) \) satisfies (2.20); (iii) For every solution \( u(x) \), \( u_{xx}(L_1 + \xi) < 0 \) for \( \xi \) positive and small, and \( u_{xx} \) can change sign at most once in the interval \( L_1 < x < L_2 \).

**Remark 2.2.** If

\[ u_{xx}(x, k_*) \leq 0 \quad \text{for } L_1 < x < L_2, \]
then the solution $u(x)$ is unique. Indeed, if $k > k_*$, then $u(L_1, k) > u(L_1, k_*)$ and, by (2.10) of Lemma 2.3, $u(L_2, k) > u(L_2, k_*)$. Since, by (2.11) of Lemma 2.3, also $u_L(L_2, k) \geq u_L(L_2, k_*)$, it follows that $u(L, k) > 0$.

**Remark 2.3.** If

$$\delta'(L_2) \geq -\frac{\delta(L_2)}{L - L_2} \tag{2.22}$$

then (2.21) is valid, and so uniqueness holds. Indeed, if (2.21) is not valid, then

$$u_{xx}(x, k_*) \leq 0 \quad \text{if} \quad L_1 < x < \hat{x},$$

$$u_{xx}(x, k_*) \geq 0 \quad \text{if} \quad \hat{x} < x < L_2$$

for some $\hat{x} \in (L_1, L_2)$. Using the differential equation we see that the values of $u(x, k_*) - \delta(x)$ at $x < \hat{x}$ are smaller than the values at $x > \hat{x}$, so that

$$u_x(\hat{x}, k_*) \geq \delta'(\hat{x})$$

and hence

$$u_x(L_2, k_*) \geq u_x(\hat{x}, k_*) \geq \delta'(\hat{x}) > \delta'(L_2) \geq -\frac{\delta(L_2)}{L - L_2}.$$ 

But this implies that $u(L, k_*) > 0$, which is a contradiction.

**Remark 2.4.** For any solution of (2.5), (2.4), we have the bound

$$u(L_1) - \delta(L_1) < L_1 \delta'(L_1) - \delta(L_1). \tag{2.23}$$

Indeed, if (2.23) is not true, then writing $u(x) = u(x, k)$ we have $k = u(L_1)/L_1 \geq \delta'(L_1)$, which contradicts Lemma 2.1.

We end this section by proving a uniqueness theorem for symmetric heads.

**Theorem 2.8.** Suppose, in addition to (1.6), that

$$\delta(x) = \delta(2L_m - x) \quad \text{for} \quad L_m < x \leq L_2 \tag{2.24}$$

for some $L_m$ such that

$$\frac{L_1 + L_2}{2} \leq L_m < L_2, \quad L - L_m \leq L_m. \tag{2.25}$$

Then the solution to the system (2.1)–(2.4) is unique.

**Proof.** According to Remark 2.2, it suffices to prove that $u_{xx}(x, k_*) \leq 0$ for $L_1 < x < L_2$. We first claim that

$$u_{xx}(x, k_*) \leq 0 \quad \text{for} \quad L_1 < x \leq L_m. \tag{2.26}$$
In fact, if this is not true, then by Lemma 2.2, there exists a point \( \hat{x} \in (L_1, L_m] \) such that

\[
(2.27) \quad u_{xx}(x, k_*) < 0 \quad \text{for} \quad L_1 < x < \hat{x},
\]
\[
(2.28) \quad u_{xx}(x, k_*) > 0 \quad \text{for} \quad \hat{x} < x < L_2.
\]

Notice that the assumption (2.24) implies that \( \delta'(L_m) = 0 \). Using the differential equation, we obtain

\[
u_x(\hat{x}, k_*) \geq \delta'(\hat{x}) \geq \delta'(L_m) = 0.
\]

By (2.28) we then have \( u_x(x, k_*) > 0 \) for all \( x > \hat{x} \), which is a contradiction.

Note that under the assumption (2.25), \( 2L_m - L \geq 0 \) and \( 2L_m - L_2 \geq L_1 \). We next claim that

\[
u(x, k_*) \geq u(2L_m - x, k_*) \quad \text{for} \quad 2L_m - L < x < L_m.
\]

To prove this, we let

\[
v(x) = u(x, k_*) - u(2L_m - x, k_*) \quad \text{for} \quad 2L_m - L < x < L_m.
\]

Using (2.24) and (2.26), we find that \( v(x) \) satisfies

\[-v_{xx} + c(x)v(x) \geq 0 \quad \text{for} \quad 2L_m - L < x < L_m,
\]
\[v(2L_m - L) \geq 0, \quad v(L_m) = 0,
\]

where

\[
c(x) = \frac{K[u(L_1, k_*) - \delta(L_1)]}{[u(x, k_*) - \delta(x)][u(2L_m - x, k_*) - \delta(x)]} \geq 0.
\]

Therefore the maximum principle implies that \( v(x) \geq 0 \), which is the assertion (2.29). Using (2.29) in (2.5) we get \( u_{xx}(x, k_*) \geq u_{xx}(2L_m - x, k_*) \) for \( 2L_m - L < x < L_m \), and then, by (2.26), \( u_{xx}(2L_m - x, k_*) \leq 0 \) for \( L_1 < x < L_m \). Consequently, altogether, \( u_{xx}(x, k_*) \leq 0 \) for \( L_1 < x < L_2 \). \( \square \)

3. A variational inequality for \( \varepsilon = 0, \eta = 0 \). We shall need to recast the solution of (2.1)–(2.4) (or rather ((2.5), (2.4))) as a solution of a variational inequality. We first note that from the results of §2 ((2.17)–(2.19)) it follows that if \( 0 < k - \bar{k} \ll 1 \), then

\[
u_{xx}(x, k) < 0 \quad \text{for} \quad L_1 < x \leq L_2 - 0
\]

and \( u(L, k) < 0 \). It follows that if \( \bar{\alpha} \) is positive and sufficiently small, then there exists a solution of

\[
g_{xx} = -K\left(\frac{g(L_1) - \delta(L_1)}{g(x) - \delta(x)} - 1\right) \chi_{[L_1, L_2]}, \quad 0 < x < L,
\]
\[
g(0) = -\bar{\alpha}, \quad g(L) < 0
\]
and

\[ u(x, k_\ast) - g(x) \geq m > 0 \quad \text{for } 0 < x < L, \]

where \( m \) is a constant; in fact, \( g'(0) = k \) where \( 0 < k - \bar{k} \ll 1 \). We introduce a truncation of the linear function \( s \),

\[
f(s) = \begin{cases} 
  s & \text{if } s \leq 1 + L_1\delta'(L_1) - \delta(L_1) \\
  1 + L_1\delta'(L_1) - \delta(L_1) & \text{if } s > 1 + L_1\delta'(L_1) - \delta(L_1),
\end{cases}
\]

and the variational inequality

\[
-u_{xx} \geq K \left( \frac{f(u(L_1) - \delta(L_1))}{u(x) - \delta(x)} - 1 \right) \chi_{[L_1, L_2]},
\]

\[ u(x) \geq g(x), \]

\[ (u - g) \left\{ -u_{xx} - K \left( \frac{f(u(L_1) - \delta(L_1))}{u(x) - \delta(x)} - 1 \right) \chi_{[L_1, L_2]} \right\} = 0 \]

for \( 0 < x < L \), with the boundary conditions

\[ u(0) = u(L) = 0. \]

**Lemma 3.1.** Any solution \( u(x) \) of (3.4), (3.5) satisfies

\[ u(x) - g(x) \geq m > 0 \quad \text{if } L_1 \leq x \leq L_2, \]

\[ u(L_1) - \delta(L_1) \leq L_1\delta'(L_1) - \delta(L_1); \]

and, consequently, it is also a solution of (2.5), (2.4).

**Proof.** Since \( u(L_1) \geq g(L_1) \) and \( f(s) \) is monotone in \( s \), we have \( f(u(L_1) - \delta(L_1)) \geq f(g(L_1) - \delta(L_1)) = g(L_1) - \delta(L_1) \); the last equation follows from (3.3) and Remark 2.4. It follows that

\[
-u_{xx} \geq K \left( \frac{g(L_1) - \delta(L_1)}{u(x) - \delta(x)} - 1 \right) \chi_{[L_1, L_2]}.
\]

Comparing with (3.1) and using the strong maximum principle, we conclude that \( u(x) \geq g(x) \) for \( 0 \leq x \leq L \) and therefore \( u \) satisfies the equation

\[
-u_{xx} = K \left( \frac{f(u(L_1) - \delta(L_1))}{u(x) - \delta(x)} - 1 \right) \chi_{[L_1, L_2]}.
\]

Using the proof for Lemma 2.1, we conclude that \( u(L_1) \leq L_1\delta'(L_1) \) and so the second inequality of (3.6) is satisfied. Hence \( u \) is a solution of (2.5), (2.4) and then of course

\[ u(x) \geq u(x, k_\ast) \geq g(x) + m \quad \text{by (3.3)}. \]

In Remark 4.1 we shall explain why the truncation was needed.
4. A variational system for small $\varepsilon$ and $\eta$. To avoid the difficulty of possible degeneracy when $h$ is near zero, we first impose in (1.1), (1.2) the restriction that $u(x) \geq g(x)$ (recall that $g(x) > \delta(x)$ for $L_1 \leq x \leq L_2$). The boundary condition for $p$ will also be modified for technical reasons. Thus we consider the system

\begin{equation}
\frac{\partial (ph)}{\partial x} - \varepsilon \frac{\partial}{\partial x} \left( \alpha h^2 \frac{\partial p}{\partial x} + \beta h^3 \frac{\partial^2 p}{\partial x^2} \right) = 0, \quad L_1 < x < L_2,
\end{equation}

\begin{equation}
\frac{\partial^2 u}{\partial x^2} + \eta \frac{\partial^4 u}{\partial x^4} \geq K(p - 1)\chi_{[L_1,L_2]}, \quad 0 < x < L,
\end{equation}

(4.2)

\begin{equation}
|u(x) - g(x)| \left\{ - \frac{\partial^2 u}{\partial x^2} + \eta \frac{\partial^4 u}{\partial x^4} - K(p - 1)\chi_{[L_1,L_2]} \right\} = 0, \quad 0 < x < L,
\end{equation}

and

\begin{equation}
h(x) = u(x) - \delta(x), \quad L_1 \leq x \leq L_2
\end{equation}

with the boundary conditions

\begin{equation}
p(L_1) = p(L_2) = \frac{f(h(L_1))}{h(L_1)},
\end{equation}

(4.4)

\begin{equation}
u = u_x = 0 \quad \text{at} \quad x = 0, \quad x = L.
\end{equation}

(4.5)

Let

\begin{equation}
G = \left\{ p \in C[L_1,L_2]; \quad 0 \leq p \leq 1 + \frac{1 + L_1\delta'(L_1) - \delta(L_1)}{l} \right\}
\end{equation}

(4.6)

where

\begin{equation}
l = \min_{L_1 \leq x \leq L_2} [g(x) - \delta(x)] > 0.
\end{equation}

For each $p \in G$ we solve the variational inequality (4.2) with boundary conditions (4.5). By general theory, $u$ is uniquely determined as the minimizer of

\begin{equation}
\int_0^L \left\{ u_x^2 + \eta u_{xx}^2 - K(p - 1)\chi_{[L_1,L_2]} \right\} dx
\end{equation}

subject to the boundary condition (4.5), and consequently

\begin{equation}
\int_0^L u_x^2 dx \leq C, \quad C \text{ independent of } p \text{ and } \eta.
\end{equation}

(4.7)

We shall need the following stronger estimate:

**Lemma 4.1.** The following estimate holds:

\begin{equation}
|u_x(x)| \leq C, \quad 0 \leq x \leq L
\end{equation}

(4.8)
where $C$ is a constant independent of $p$ and $\eta$.

Proof. The proof relies on the fact that $g \in W^{2,\infty}(0,L)$. We also note that $u \in H^2(0,L)$ so that $u \in C^1[0,L]$. At any point $x_0$ where $u(x_0) = g(x_0)$, we have

$$|u_x(x_0)| = |g_x(x_0)| \leq C.$$

Thus it remains to estimate $u_x$ in the open set $\{u > g\}$, which consists of intervals $\{x_1 < x < x_2\}$. If at least one end point is 0 or $L$, then $x_2 - x_1 \geq c > 0$, $c$ independent of $\eta$, and so

$$\frac{x_2 - x_1}{\sqrt{\eta}} \geq 3$$

if $\eta$ is small enough.

We shall first consider the case where $0 < x_1 < x_2 < L$. Then

$$u(x_i) = g(x_i), \quad u_x(x_i) = g_x(x_i) \quad \text{for } i = 1, 2,$$

and

$$-\frac{\partial^2 u}{\partial x^2} + \eta \frac{\partial^4 u}{\partial x^4} = q(x) \quad \text{if } x_1 < x < x_2,$$

where $q(x) = K(p-1)\chi_{[L_1,L_2]}$. Multiplying (4.11) by $u - g$ and integrating over $(x_1, x_2)$, we obtain

$$\int_{x_1}^{x_2} u_x^2 dx + \eta \int_{x_1}^{x_2} u_{xx}^2 dx = \int_{x_1}^{x_2} u_x g_x dx + \eta \int_{x_1}^{x_2} u_{xx} g_{xx} dx + \int_{x_1}^{x_2} (u - g) q(x) dx$$

$$\leq \frac{1}{2} \left( \int_{x_1}^{x_2} u_x^2 dx + \int_{x_1}^{x_2} g_x^2 dx + \eta \int_{x_1}^{x_2} u_{xx}^2 dx + \eta \int_{x_1}^{x_2} g_{xx}^2 dx \right)$$

$$+ C(x_2 - x_1) \int_{x_1}^{x_2} |u_x - g_x| dx,$$

so that

$$\eta \int_{x_1}^{x_2} u_{xx}^2 dx \leq C(x_2 - x_1).$$

Hence

$$|u_x(x)| \leq |g_x(x_1)| + \int_{x_1}^{x_2} |u_{xx} - g_{xx}| dx$$

$$\leq C + \int_{x_1}^{x_2} |u_{xx}| dx$$

$$\leq C + (x_2 - x_1)^{1/2} \left( \int_{x_1}^{x_2} u_{xx}^2 dx \right)^{1/2}$$

$$\leq C + \frac{C(x_2 - x_1)}{\sqrt{\eta}}.$$
and (4.8) follows provided \((x_2 - x_1)/\sqrt{\eta} < 3\).

It remains to consider the case (4.9). Let

\[
(4.12) \quad s(x) = v(x) - \int_{x_1}^{x} \int_{x_1}^{\tau} q(\xi)d\xi d\tau
\]

where \(v(x)\) satisfies

\[
(4.13) \quad -v + \eta v_{xx} = \eta q, \quad x_1 < x < x_2, \quad v(x_1) = v(x_2) = 0.
\]

Thus \(-s_{xx} + \eta s_{xxx} = q(x)\) so that the solution \(u\) has the form

\[
(4.14) \quad u(x) = C_1 + C_2x + C_3 \exp\left(-\frac{x - x_1}{\sqrt{\eta}}\right) + C_4 \exp\left(-\frac{x_2 - x}{\sqrt{\eta}}\right) + s(x),
\]

\[x_1 < x < x_2.\]

By the maximum principle

\[
\|v\|_{L^\infty(x_1, x_2)} \leq \eta \|q\|_{L^\infty(x_1, x_2)}
\]

and, therefore, from the differential equation in (4.13),

\[
\|v_{xx}\|_{L^\infty(x_1, x_2)} \leq C \|q\|_{L^\infty(x_1, x_2)}.
\]

Since \(v(x_1) = v(x_2) = 0\), there is a point \(\xi \in (x_1, x_2)\) such that \(v_{x}(\xi) = 0\), and consequently

\[
(4.15) \quad \|v_{x}\|_{L^\infty(x_1, x_2)} \leq C(x_2 - x_1) \|q\|_{L^\infty(x_1, x_2)}.
\]

Using the last two estimates on \(v\), we deduce from (4.12) that

\[
|s(x_i)| + |s_{x}(x_i)| \leq C,
\]

\[
|s(x_2) - s(x_1)| + |s_{x}(x_2) - s_{x}(x_1)| \leq C(x_2 - x_1).
\]

From (4.10) and the fact that \(g \in W^{2,\infty}\) it follows that \(u\) satisfies the same estimates, and thus the function \(w(x) = u(x) - s(x)\) also satisfies

\[
(4.16) \quad |w(x_i)| + |w_{x}(x_i)| \leq C,
\]

\[
|w(x_2) - w(x_1)| + |w_{x}(x_2) - w_{x}(x_1)| \leq C(x_2 - x_1).
\]

We need to choose the constants \(C_i\) in (4.14) such that

\[
C_1 + C_2 x_1 + C_3 + C_4 \exp\left(-\frac{x_2 - x_1}{\sqrt{\eta}}\right) = w(x_1),
\]

\[
C_2 + \frac{C_3}{\sqrt{\eta}} + \frac{C_4}{\sqrt{\eta}} \exp\left(-\frac{x_2 - x_1}{\sqrt{\eta}}\right) = w_{x}(x_1),
\]

\[
C_1 + C_2 x_2 + C_3 \exp\left(-\frac{x_2 - x_1}{\sqrt{\eta}}\right) + C_4 = w(x_2),
\]

\[
C_2 - \frac{C_3}{\sqrt{\eta}} \exp\left(-\frac{x_2 - x_1}{\sqrt{\eta}}\right) + \frac{C_4}{\sqrt{\eta}} = w_{x}(x_2).
\]
Eliminating $C_1$ and $C_2$, we obtain

\[
\begin{align*}
\frac{C_3 + C_4}{\sqrt{\eta}} &= \frac{w(x_1) - w(x_2)}{1 - \gamma}, \\
\frac{C_3 - C_4}{\sqrt{\eta}} &= \frac{1}{\frac{1}{2}(1 + \gamma) - \frac{1}{\beta}(1 - \gamma)} \left\{ \frac{w(x_1) - w(x_2)}{x_1 - x_2} - \frac{1}{2} [w(x_1) + w(x_2)] \right\},
\end{align*}
\]

where $\gamma = e^{-\beta}$, $\beta = (x_2 - x_1)/\sqrt{\eta} \geq 3$. Since the right-hand sides are bounded independently of $\eta$ (by (4.16)), $C_3/\sqrt{\eta}$ and $C_4/\sqrt{\eta}$ are also bounded independently of $\eta$. We can now easily check that also $C_1$ and $C_2$ are bounded independently of $\eta$, and then the bound (4.8) for $x_1 < x < x_2$ follows by (4.14) upon recalling (4.12) and the estimate (4.15).

The above proof is valid also if $x_1 = 0$ or $x_2 = L$, and the proof of the lemma is thus complete. \[ \square \]

Having solved the variational inequality (4.2), (4.5) for $u$, given $p \in G$, we set $h = u - \delta$, and recall that

\begin{equation}
(4.17) \quad h(x) \geq g(x) - \delta(x) \geq l > 0, \quad L_1 \leq x \leq L_2.
\end{equation}

**Lemma 4.2.** Given $h$ as above, there exists a unique positive solution $\hat{p}$ of (4.1), (4.4).

**Proof.** By integration, $\hat{p}$ must satisfy

\begin{equation}
(4.18) \quad \begin{align*}
\hat{p}h - \varepsilon(\alpha h^2 + \beta h^3 \hat{p})\hat{p}_x &= \lambda, \quad L_1 \leq x \leq L_2, \\
\hat{p}(L_1) &= \frac{f(h(L_1))}{h(L_1)} \quad (\leq 1),
\end{align*}
\end{equation}

where $\lambda$ is a constant that clearly determines $\hat{p}$ uniquely. Thus it remains to show that $\lambda$ can be uniquely determined such that

\begin{equation}
(4.19) \quad \hat{p}(L_2) = \frac{f(h(L_1))}{h(L_1)}
\end{equation}

If $\lambda \leq 0$, then $\hat{p}_x > 0$ and so $\hat{p}(L_2) > \hat{p}(L_1)$ and (4.19) cannot be satisfied. If $\lambda$ is positive and large then $\hat{p}_x$ is negative and large and so $\hat{p}$ cannot remain positive throughout the interval $[L_1, L_2]$. Denoting the solution of (4.18) by $\hat{p}(x, \lambda)$, we let

\[
\lambda_0 = \inf \{ \lambda; \quad \lambda > 0, \quad \hat{p}(x, \lambda) \geq 0 \quad \text{for} \quad L_1 \leq x \leq L_2 \}.
\]

Then $\hat{p}(x, \lambda_0) \geq 0$ and $\hat{p}(x, \lambda_0)$ must vanish somewhere in the interval $\{L_1 < x \leq L_2\}$. It cannot vanish at an interior point $x_0$, for otherwise also $\hat{p}_x(x, \lambda_0) = 0$ so that (by the differential equation in (4.18)) $\lambda_0 = 0$, a contradiction. Thus

\[
\hat{p}(x, \lambda_0) > 0 \quad \text{if} \quad L_1 \leq x < L_2, \quad \hat{p}(L_2, \lambda_0) = 0.
\]
We claim:

\[(4.20) \quad \text{if } 0 < \lambda_1 < \lambda_2 \leq \lambda_0 \quad \text{then } \tilde{p}(x, \lambda_1) > \tilde{p}(x, \lambda_2) \quad \text{for } L_1 < x \leq L_2.\]

Indeed, since \(\tilde{p}_x(L_1, \lambda_1) > \tilde{p}_x(L_1, \lambda_2)\), the asserted inequality is valid for all \(x\) near \(x = L_1\). If the assertion is not true then there is a smallest number \(x_0\) such that

\[\tilde{p}(x, \lambda_1) > \tilde{p}(x, \lambda_2) \quad \text{if } L_1 \leq x < x_0, \quad \text{and } \tilde{p}(x, \lambda_1) = \tilde{p}(x_0, \lambda_2).\]

Hence \(\tilde{p}_x(x_0, \lambda_1) \leq \tilde{p}_x(x_0, \lambda_2)\) and, by (4.18), \(\lambda_1 \geq \lambda_2\), a contradiction.

Setting

\[\hat{\lambda} = \inf \{\lambda; \ 0 < \lambda < \lambda_0, \ \tilde{p}(L_2, \lambda) < \frac{f(h(L_1))}{h(L_1)}\},\]

it follows from (4.20) that \(\tilde{p}(x, \hat{\lambda})\) is the unique positive solution (4.1), (4.4). □

We define the mapping \(T\) on \(G\) by

\[(4.21) \quad (Tp)(x) = \tilde{p}(x, \hat{\lambda})\]

and wish to prove that \(T\) has a fixed point, which is then the solution of the variational system (4.1)–(4.5).

Let \(C^*\) be any positive constant such that

\[(4.22) \quad C^* \geq \sup_{L_1 \leq x \leq L_2} (|h(x)| + |h_x(x)|) \quad \text{and} \quad \frac{l}{C^* + 1} < \frac{1}{L_2 - L_1},\]

where \(l\) is the positive constant in (4.17); by Lemma 4.1, \(C^*\) can be chosen independently of \(p \in G\) and \(\eta\).

**Lemma 4.3.** If \(\varepsilon\) is positive and small enough, then \(T\) maps \(G\) into itself, and

\[(4.23) \quad |\hat{\lambda} - f(h(L_1))| \leq C\varepsilon,\]

where \(C\) is a constant independent of \(\varepsilon\) and \(\eta\).

**Proof.** We shall first establish the bound

\[(4.24) \quad \tilde{p}_x(L_1, \hat{\lambda}) \geq -\frac{C^* + 1}{l}.\]

Introduce the differential operator

\[(4.25) \quad \mathcal{L}(q) = \frac{\partial}{\partial x}(qh) - \varepsilon \frac{\partial}{\partial x}[(\alpha h^2 + \beta h^2 q) \frac{\partial q}{\partial x}]\]

and the function

\[v(x) = \frac{f(h(L_1))}{h(L_1)} - \frac{C^* + 1}{l}(x - L_1) - \sigma, \quad L_1 \leq x < \hat{x}\]

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for any small $\sigma > 0$, where $\hat{x}$ is such that $v(\hat{x}) = 0$. By (4.22), $L_1 < \hat{x} < L_2$. By direct computation,

$$L(v) = -\frac{C^* + 1}{l} h(x) + v(x) h_x(x) + \frac{C^* + 1}{l} \frac{\partial}{\partial x} (\alpha h^2 + \beta h^3 v)$$

$$\leq -1 + \frac{C^* + 1}{l} \varepsilon \frac{\partial}{\partial x} (\alpha h^2 + \beta h^3 v)$$

$$< 0, \quad L_1 \leq x \leq \hat{x}$$

if $\varepsilon$ is small enough depending only on $C^*$, $l$. We claim that for any $0 \leq \lambda \leq \hat{\lambda}$,

$$(4.26) \quad \hat{p}(x, \lambda) > v(x) \quad \text{if } L_1 < x \leq \hat{x}.$$ 

Indeed, this is clearly true for $\lambda = 0$. If (4.25) does not hold for some $\lambda^* \in (0, \hat{\lambda}]$, we take the smallest such $\lambda^*$ and then

$$\hat{p}(x^*, \lambda^*) = v(x^*)$$

must hold for some $x^* \in [L_1, \hat{x}]$, whereas

$$\hat{p}(x, \lambda^*) \geq v(x) \quad \text{elsewhere}.$$ 

Since $\hat{p}(x, \lambda^*) > v(x)$ at both $x = L_1$ and $x = \hat{x}$, $x^*$ lies in the open interval $(L_1, \hat{x})$, so that

$$\hat{p}_x(x^*, \lambda^*) = v_x(x^*), \quad \hat{p}_{xx}(x^*, \lambda^*) \geq v_{xx}(x^*).$$

Consequently

$$(L\hat{p})(x^*, \lambda^*) \leq (Lv)(x^*) < 0,$$

a contradiction.

Taking $\sigma \downarrow 0$ in (4.26) we deduce that

$$\hat{p}(x, \hat{\lambda}) \geq \frac{f(h(L_1))}{h(L_1)} - \frac{C^* + 1}{l} (x - L_1)$$

and, since $\hat{p}(L_1, \hat{\lambda}) = f(h(L_1))/h(L_1)$,

$$(4.27) \quad \hat{p}_x(L_1, \hat{\lambda}) \geq -\frac{C^* + 1}{l}.$$ 

From (4.18) we then obtain

$$(4.28) \quad \hat{\lambda} - f(h(L_1)) \leq -\varepsilon h^2(L_1)(\alpha + \beta h(L_1)) \frac{C^* + 1}{l} \leq C\varepsilon.$$ 

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We can now estimate \( \hat{\rho}(x, \hat{\lambda}) \) from above. If the maximum is attained at the boundary point, then

\[
\max_{L_1 \leq x \leq L_2} \hat{\rho}(x, \hat{\lambda}) = \frac{f(h(L_1))}{h(L_1)} \leq 1
\]

whereas if it occurs at an interior point \( \tilde{x} \), then \( \hat{p}_x(\tilde{x}, \hat{\lambda}) = 0 \) and, by (4.18), \( \hat{p}(\tilde{x}, \hat{\lambda})h(\tilde{x}) = \hat{\lambda} \). Using the upper bound on \( \hat{\lambda} \) obtained in (4.28), we conclude that

\[
\max_{L_1 \leq x \leq L_2} \hat{\rho}(x, \hat{\lambda}) \leq \frac{\hat{\lambda}}{l} \leq \frac{1 + L_1 \delta(L_1) - \delta(L_1)}{l} + \frac{C\varepsilon}{l}.
\]

Thus if \( \varepsilon \) is chosen small enough so that \( C\varepsilon < 1 \), then \( T \) maps \( G \) into itself.

To complete the proof of the lemma it remains to prove that

\[
(4.29) \quad \hat{\lambda} - f(h(L_1)) \geq -C\varepsilon.
\]

We shall compare \( \hat{\rho}(x, \hat{\lambda}) \) with the function

\[
w(x) = \frac{f(h(L_1))}{h(L_1)} + \frac{C^* + 1}{l}(x - L_1) + \sigma, \quad L_1 \leq x \leq L_2
\]

where \( \sigma \) is any small positive constant. We have \( \mathcal{L}(w) > 0 \) in \( L_1 < x < L_2 \) and \( w > 1 \) at \( x = L_1, x = L_2 \). If \( \lambda \) is large, then \( w(x) > \hat{p}(x, \lambda) \) in the largest interval \( L_1 \leq x < x_\lambda \) where \( \hat{p}(x, \lambda) \) is positive. By decreasing \( \lambda \) and arguing as before we deduce that

\[
\hat{p}_x(L_1, \hat{\lambda}) \leq \frac{C^* + 1}{l}
\]

which yields (by (4.18)) the bound (4.29).

For each \( \varepsilon > 0 \), the \( \hat{p}(x, \hat{\lambda}) \) form a bounded set in \( C^{2+\alpha}[L_1, L_2] \) and thus \( T \) maps \( G \) (endowed with the \( C[L_1, L_2] \) norm) into a compact subset of \( G \). It is also easily seen that \( T \) is a continuous map. Invoking the Schauder fixed point theorem we conclude that \( T \) has a fixed point. Thus we have proved:

**Theorem 4.4.** If \( \varepsilon \) is sufficiently small and, say, \( 0 < \eta < 1 \), then there exists a classical solution of (4.1)–(4.5), such that

\[
P(x) > 0, \quad h(x) = u(x) - \delta(x) \geq l > 0 \quad \text{if} \quad L_1 \leq x \leq L_2,
\]

\[
|u_x(x)| \leq C \quad \text{if} \quad 0 \leq x \leq L,
\]

and

\[
|\lambda - f(h(L_1))| \leq C\varepsilon
\]

where \( \lambda \) is defined as in (4.18) with \( \hat{\rho} = \rho \) and \( C \) is a constant independent of \( \varepsilon \) and \( \eta \).

Remark 4.1. From the proof of Lemma 4.3, it is clear that the truncation on the boundary condition for \( p \) in (4.4) is necessary. Without the truncation, (4.28) becomes

\[
\hat{\lambda} \leq h(L_1) + C\varepsilon;
\]

and then the maximum norm of \( p \) can only be estimated in terms of \( Ch(L_1) \); this does not allow us to conclude that \( T \) maps \( G \) into itself, even if \( \varepsilon \) is small.
5. Existence for small $\varepsilon$ and $\eta$. In this section we prove Theorem 1.1 by establishing:

**Lemma 5.1.** There exist $\varepsilon^* > 0$, $\eta^* > 0$ such that if

\begin{equation}
0 < \varepsilon < \varepsilon^*, \quad 0 < \eta < \eta^*,
\end{equation}

then the solution $(p, h, u)$ of (4.1)-(4.5) established in §4 is a solution asserted in Theorem 1.1.

**Proof.** It suffices to show that

\begin{equation}
0 < x < L
\end{equation}

and

\begin{equation}
h(L) \leq 1 + L_1 \delta'(L_1) - \delta(L_1)
\end{equation}

for $\varepsilon$ and $\eta$ positive and sufficiently small.

Suppose either (5.2) or (5.3) is not satisfied for a sequence $(\varepsilon_j, \eta_j) \to (0, 0)$; we shall derive a contradiction. We denote the $p, h, u$ and $\lambda$ corresponding to $(\varepsilon_j, \eta_j)$ by $p_j, h_j, u_j, \lambda_j$, respectively. We may assume that

\[
\lambda_j \to \lambda,
\]

\[
p_j \to p \quad \text{weakly } \ast \text{ in } L^\infty(L_1, L_2),
\]

\[
u_j \to u \quad \text{uniformly in } [0, L],
\]

\[
h_j \to h \quad \text{uniformly in } [L_1, L_2],
\]

and

\begin{equation}
\left| \frac{\partial}{\partial x} u_j \right| \leq C \quad \text{in } [0, L].
\end{equation}

For any $\zeta \in C^\infty_c(L_1, L_2)$ we have, by integration by parts,

\[
\int_{L_1}^{L_2} (p_j h_j - \lambda_j) \zeta dx = -\varepsilon \int_{L_1}^{L_2} \left[ p_j \frac{\partial}{\partial x} (\alpha h_j^2 \zeta) + \frac{1}{2} p_j^2 \frac{\partial}{\partial x} (\beta h_j^3 \zeta) \right] dx
\]

Taking $j \to \infty$ and using (4.32), we obtain

\[
\int_{L_1}^{L_2} \left[ p h - \frac{f(h(L_1))}{h(L_1)} \right] \zeta dx = 0.
\]

Hence

\[
p(x)h(x) = \frac{f(h(L_1))}{h(L_1)} \quad \text{for } L_1 \leq x \leq L_2.
\]
Next, multiplying the differential inequality for \( u_j \) by \( \zeta \in C^\infty_c(0,L) \), \( \zeta \geq 0 \) and integrating by parts, we find that \( u \) is a solution of the variational inequality with \( \eta = 0 \) and, by Lemma 3.1,

\[
(5.5) \quad u(x) - g(x) \geq m > 0, \quad L_1 \leq x \leq L_2, \\
h(L_1) \leq L_1 \delta'(L_1) - \delta(L_1).
\]

(Actually \( u \) is initially only known to be in \( C^{0,1} \) (by (5.4)); but by standard regularity for variational inequalities (e.g. [4; Chap 1]) \( u \) is in \( W^{2,p} \) (for any \( p > 1 \)) and satisfies (3.4) a.e.)

The inequality (5.5) implies that (5.2) and (5.3) hold for \( u_j, h_j \) if \( j \) is large enough, which is a contradiction. \( \square \)

6. Boundary layer behavior. Denote the solution asserted in Theorem 1.1 by \( (p^{\varepsilon, \eta}, h^{\varepsilon, \eta}, u^{\varepsilon, \eta}) \) and the corresponding \( \lambda \) by \( \lambda^{\varepsilon, \eta} \). We take a sequence \( (\varepsilon_j, \eta_j) \) such that

\[
\lambda_j = \lambda^{\varepsilon_j, \eta_j} \to \lambda^0, \\
p_j = p^{\varepsilon_j, \eta_j} \to p^0 \quad \text{weakly * in } L^\infty(L_1, L_2), \\
u_j = u^{\varepsilon_j, \eta_j} \to u^0 \quad \text{uniformly in } [0, L].
\]

Then also

\[
h_j = h^{\varepsilon_j, \eta_j} \to h^0 \quad \text{uniformly in } [L_1, L_2].
\]

By Lemma 4.1,

\[
(6.1) \quad \left| \frac{\partial}{\partial x} u_j \right| \leq C \quad \text{in } [0, L].
\]

Since the boundary condition \( p_j(L_2) = 1 \) is lost in the limit, we want to study the behavior of \( p_j \) near \( x = L_2 \).

Making a change of variables

\[
y = \frac{x - L_2}{\varepsilon}, \quad \tilde{p}_j(y) = p_j(x),
\]

it is clear that

\[
(6.2) \quad \tilde{p}_j(y) h_j(L_2 + \varepsilon y) - [\alpha(h_j(L_2 + \varepsilon y))^2 + \beta(h_j(L_2 + \varepsilon y))^3 \tilde{p}_j(y)] \frac{\partial \tilde{p}_j}{\partial y} = \lambda_j \\
\text{for } -\frac{L_2 - L_1}{\varepsilon} < y < 0,
\]

\[
(6.3) \quad \tilde{p}_j(0) = 1.
\]
From the standard ODE theory it follows that \( \frac{\partial \tilde{p}}{\partial y} \) is uniformly bounded in any bounded interval \([-N, 0] \), \( \tilde{p} \rightarrow \tilde{p} \) uniformly in any such interval, and \( \tilde{p} \) satisfies:

\[
\tilde{p}(y) h^0(L_2) - [\alpha(h^0(L_2))^2 + \beta h^0(L_2)] \frac{\partial \tilde{p}}{\partial y} = h^0(L_1)
\]

(6.4)

for \( -\frac{L_2 - L_1}{\varepsilon} < y < 0 \),

\( \tilde{p}(0) = 1 \).

This system has a unique solution:

If \( h^0(L_1) = h^0(L_2) \) then \( \tilde{p}(y) \equiv 1 \).

If \( h^0(L_1) \neq h^0(L_2) \) then \( \tilde{p}(y) \) is given by

\[
y - \beta h_2^2 \tilde{p} - (\beta h_1^2 + \alpha h_2) \log |h_1 - h_2 \tilde{p}| = C_0
\]

(6.5)

where

\[
h_1 = h^0(L_1), \quad h_2 = h^0(L_2),
\]

and \( C_0 \) is the constant which is evaluated by taking \( y = 0, \tilde{p} = 1 \) in (6.5).

If we take other sequences \((\varepsilon_j, \eta_j)\), we may possibly obtain other limits \( p^0, h^0, u^0 \), but the limit \( \tilde{p}(y) \) depends only on the constants in (6.6). (Notice that if we have a symmetric head as in Theorem 2.8, then the limit is unique, and we have the convergence as \((\varepsilon, \eta) \rightarrow (0, 0)\), not just on subsequences).

The above result shows that

\[
p_j(x) \sim \tilde{p} \left( \frac{x - L_2}{\varepsilon} \right) \quad \text{in} \quad [L_2 - C\varepsilon, L_2]
\]

(6.7)

for any constant \( C \), and this describes the boundary layer near \( x = L_2 \) for \((\varepsilon, \eta)\) small.

The boundary layer phenomenon for general differential equations was widely studied in the literature; see for example [10]. From [10] it follows that near \( x = L_1 \) the functions \( p_j(x) \) are uniformly converging to \( p^0(x) \), so that \( p^0(L_1) = 1 \).

As for the \( u_j(x) \), from (6.1) we deduce that the \( u_j(x) \) are uniformly continuous in \( 0 \leq x \leq L \) and \( u^0(L_1) = u^0(L_2) = 0 \). However the boundary conditions \( \partial u_j / \partial x = 0 \) at \( x = 0 \) and \( x = L \) disappear in the limit. The derivatives \( \partial u_j / \partial x \) must therefore exhibit a boundary layer behavior at \( x = 0 \) and \( x = L \). However what is more important for the head-tape interaction is the behavior of the \( u_j(x) \), or the spacing \( h_j(x) \), at \( x = L_2 \). Although there is no boundary layer for the \( u_j(x) \) at \( x = L_2 \), the fact that \( K \) in (1.2) is large tends to magnify the oscillatory behavior of \( p \) near \( x = L_2 \) and to create a boundary layer-like behavior for the \( h_j \).

**Conclusion.** Both experiments and numerical simulations have established boundary layer behavior for the head-tape problem at \( x = L_2 \) [1][5] for a range of physical parameters \( \varepsilon, \eta \); these parameters are small, e.g., \( 10^{-2} \sim 10^{-3} \). In the present paper we give rigorous mathematical proofs of existence of solutions and of the same type of boundary layer behavior provided the parameters \( \varepsilon, \eta \) are sufficiently small.
7. The two-dimensional case. In this section we consider the 2-d model of head-tape interaction. The profile of the head is a function \( \delta(x,y) \), and the steady state system, after non-dimensionalizing, takes the form

\[
\frac{\partial (ph)}{\partial x} - \varepsilon \nabla (\alpha h^2 \nabla p + \beta h^3 p \nabla p) = 0 \quad \text{in } S_1,
\]

\[
-\frac{\partial^2 u}{\partial x^2} + \eta \Delta^2 u = K(p - 1)\chi_{S_1} \quad \text{in } S_2,
\]

\[
u(x,y) = h(x,y) + \delta(x,y) \quad \text{in } S_1
\]

where

\[S_1 = \{(x,y); \; L_1 < x < L_2, \; 0 < y < b\}, \quad S_2 = \{(x,y); \; 0 < x < L, \; 0 < y < b\}, \quad b = O(1),\]

with the boundary conditions

\[
p = 1 \quad \text{on } \partial S_1,
\]

\[
u = u_x = 0 \quad \text{on } x = 0 \text{ and } x = L,
\]

\[
u_{yy} + \sigma u_{xx} = 0, \quad u_{yyy} + (2 - \sigma)u_{xxy} = 0 \quad \text{on } y = 0 \text{ and } y = b,
\]

where \( \sigma \) is the Poisson ratio, \( 0 < \sigma < 1/2 \).

The boundary conditions for \( u \) are those of a plate clamped at \( x = 0 \) and \( x = L \) and free at \( y = 0 \) and \( y = b \); see [9].

Given \( p \), the function \( u \) is determined as a minimizer of the energy functional (see [7; Chap. 11])

\[
\int_{S_2} \left\{ \eta(\Delta u)^2 + 2\eta(1-\sigma)\left[\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 - \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 u}{\partial y^2}\right] + \left(\frac{\partial u}{\partial x}\right)^2 - 2Ku(p - 1)\chi_{S_1}\right\} dx dy
\]

subject to the boundary conditions (7.5); the differential equation (7.2) and the boundary conditions (7.6) are necessary optimality conditions.

In the limit case \( \varepsilon = \eta = 0 \) the system reduces to (cf. (2.4))

\[-u_{xx} = K\left(\frac{u(L_1, y) - \delta(L_1, y)}{u(x, y) - \delta(x, y)} - 1\right)\chi_{S_1} \quad \text{in } S_2
\]

and, with the boundary conditions

\[u(0, y) = u(L, y) = 0,
\]

it can be solved by the shooting method, for each value \( y \).

For \( \varepsilon > 0, \eta > 0 \) we can set up a variational system (analogous to (4.1)-(4.5)). We still have the bound

\[
\int_{S_2} u_x^2 \leq C
\]
but we do not know whether \( \int u^2 \leq C \), much less whether \( |\nabla u| \leq C \), and thus we are unable to carry out rigorously the analysis of §5. On the other hand one can proceed formally to derive boundary layer behaviors for \( p \) near \( x = L_2 \) and near \( y = 0 \) and \( y = b \), assuming that \( u \) remains continuous there; the boundary layers near \( y = 0 \) and \( y = b \) are of the form given in [10; p. 302]. For numerical results we refer to [8].

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