ORDER PARAMETER MODELS OF ELASTIC BARS AND PRECURSOR OSCILLATIONS

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Order parameter models of elastic bars and precursor oscillations

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Abstract

In this paper we consider a model of solid-solid phase transitions that includes elastic effects and an order parameter. Our purpose is to show that such a model can exhibit small amplitude oscillations in the strain before transition from Austenite to Martensite. The model under investigation falls within the general framework of models developed by Fried and Gurtin, but we examine the special case of a triple-well free energy with a central Austenite well flanked by two symmetric Martensite wells. (The problem is posed in one space dimension, so the terms “Austenite” and “Martensite” are simply meant to be suggestive.) The relative height of the wells and the elastic modulus of the Austenite well vary with temperature. We examine the static problem and, as we would see in a pure elasticity problem, there is a uniform Austenite solution at high temperature and highly oscillatory “twinned” Martensite solutions at low temperature. However, we show that the combination of the microlocal forces from the order parameter and a softening of the Austenite modulus cause precursor oscillations in the Austenite phase to bifurcate from the uniform mode above the transition temperature. These branches of solutions connect to Martensite oscillations of the same wavelength below the transition temperature.

1 Introduction

Many shape-memory alloys exhibit fine microstructures during structural phase transformations from a high-temperature, high-symmetry Austenite phase to a low-temperature, lower-symmetry Martensite phase. (Here the term “microstructure” is being used to describe high frequency plane-wave oscillations between two Martensitic variants.) In this paper we examine a closely related

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phenomenon. There have been a number of observations of small-amplitude "precursor" oscillations that take place above the temperature for transition to Martensite (See, e.g. [15, 16, 17, 18, 20, 21, 22]). These oscillations often have a characteristic "tweedy" pattern and take place on roughly the same wavelength as the eventual high-amplitude Martensite oscillations.

Why would such oscillations take place? A local instability of the homogeneous Austenite ground state is not predicted by typical elastic energy minimization problems, and refining elasticity by adding higher-order terms like capillarity would only seem to stabilize a uniform Austenite state. So far, the most successful explanations focus on material inhomogeneities (either quenched in [12] or reactive [19]) that force the material out of the Austenite well and into some oscillatory mixture of Austenite and Martensite. However, one common interpretation of the relevant experimental observations is that the oscillations take place "within the Austenite well," and there is no mixture of Austenite and Martensite states. This paper is intended to show that order parameter models can exhibit a mechanism by which oscillations occur within the Austenite phase.

In particular, we show that a simplified one-dimensional mathematical problem based on a model of the type considered by Fried and Gurtin [9, 10] exhibits a bifurcation picture that has branches of steady-state solutions that start as small-amplitude oscillations in the Austenite phase and become "twinned" Martensite oscillations further along the branch. We should say at the outset that the problem considered here is clearly a mathematical "cartoon" of the real physical situation: the problem is posed in one space dimension; in our global analysis the elastic part of the free energy is assumed to be piecewise quadratic. Thus, we are taking a great deal of liberty in using such terms as "Austenite," "Martensite," and "twinned." We do so both to draw attention to the analogy we wish to make and to avoid the use of less felicitous (though more accurate) terms such as "central-well," "lateral-well," and "locally symmetric."

2 The mathematical model

In this section we examine Fried and Gurtin's general formulation of elastic phase transitions characterized by an order parameter [9, 10] and make specific assumptions about the constitutive models to be considered in this paper. The exposition here is greatly simplified by the restriction to one space dimension.

2.1 Kinematics and forces

We consider a bar of elastic material whose reference configuration has an axis of centroids of cross sections lying on an interval on the $x$-axis. We wish to describe longitudinal motions of the bar. We assume that the uniform temperature $\theta$ enters the problem only as a parameter. The state of the bar is described by the following scalar fields of reference position $x \in [0, l]$ and time $t \in [0, \infty)$. 
\[ u \quad \text{longitudinal displacement} \]
\[ v := \dot{u} \quad \text{longitudinal velocity} \]
\[ w := u_x \quad \text{longitudinal strain} \]
\[ \phi \quad \text{order parameter} \]
\[ h := \dot{\phi} \quad \text{order parameter rate of change} \]
\[ p := \phi_x \quad \text{order parameter gradient} \]

Here and in the following, a superposed dot indicates a partial derivative with respect to time. We do not put restrictions on the strain that would ensure that the material does not interpenetrate. Such restrictions can be difficult to treat analytically (see, e.g. [1]) and the issues raised by their treatment are largely separate from the phase transition problems considered here.

The only unusual field described above is the order parameter \( \phi \). The physical interpretation of this field differs for various constitutive choices. Under appropriate choices it can take the role of the “phase field” in which different phases of the material are described by different arbitrary values of the parameter (e.g., \( \phi = 0 \) indicates liquid and \( \phi = 1 \) indicates solid). This is typical in some of the work of Caginalp [6, 7] and numerous others. While the general formulation described here includes models of this type, we consider a different constitutive choice when modeling precursor oscillations. Our use of \( \phi \) is more consistent with its interpretation as a concentration (see, e.g. [8]) or as a measure of atomic shuffles (see, e.g. [2, 11]). However, we feel that for the purposes of this paper, a specific physical identification of the order parameter is unnecessary. Our intent is to show that a generic order parameter can provide a mathematical mechanism for inducing precursors in a given constitutive regime.

To describe the dynamics of the kinematic quantities above we make use of the following fields.

\[ \rho \quad \text{mass density per unit reference length} \]
\[ \sigma \quad \text{resultant contact force (stress) across a cross section} \]
\[ f \quad \text{resultant external body force per unit reference length} \]
\[ \xi \quad \text{resultant microlocal stress across a cross section} \]
\[ \pi \quad \text{resultant internal microlocal body force per unit reference length} \]
\[ \gamma \quad \text{resultant external microlocal body force per unit reference length} \]
\[ \psi \quad \text{free energy per unit reference length} \]

All of the fields in this list are assumed to be functions of reference position and time, except for \( \rho \) which is independent of time. Once again, all of the terms are familiar except for the three microlocal force terms: \( \xi, \pi, \) and \( \gamma \). We think of the balance of microlocal forces (described below) as “governing” the dynamics of the order parameter in the way that the balance of linear momentum “governs” the dynamics of displacement.
2.2 Balance Laws

We now describe a set of axioms relating the fields described above. We specify two balance laws and one inequality, which we assume hold for all processes of the material bar. For any $0 \leq a \leq b \leq l$ and $t > 0$ we require the following equations to hold.

**Balance of linear momentum**

$$
\int_a^b \rho(x)v(x,t) \, dx = \sigma(b,t) - \sigma(a,t) + \int_a^b f(x,t) \, dx \tag{1}
$$

**Balance of microlocal forces**

$$
\xi(b,t) - \xi(a,t) + \int_a^b (\pi(x,t) + \gamma(x,t)) \, dx = 0 \tag{2}
$$

**Dissipation inequality**

$$
\int_a^b \left( \psi + \frac{1}{2} \rho |v|^2 \right) \, dx \leq \sigma(b,t)v(b,t) - \sigma(a,t)v(a,t) + \xi(b,t)h(b,t) - \xi(a,t)h(a,t) + \int_a^b (fv + \gamma h) \, dx \tag{3}
$$

A few comments on these laws are in order.

- We have followed Fried and Gurtin in ignoring any “inertial effects” of microlocal forces. In fact, for a number of interpretations of the order parameter one could argue that inertial effects are physically important (e.g., if one interprets the order parameter as a transformation strain). Maugin and Muschik [14] give an extensive discussion of the role of inertial terms in internal variable and order parameter models.

Fortunately, we could have easily included inertial effects without any change in the constitutive restrictions we derive. Thus, since we study only static problems below, the point is moot in this paper.

- The dissipation inequality says that the rate of energy increase cannot exceed the total power expended. Note that it is assumed that internal microlocal body forces do no net work though they do enter the balance law.

If the fields are sufficiently smooth we can derive the following local versions of the laws.

$$
\rho \dot{v} = \sigma_u + f \tag{4}
$$

$$
\xi_x + \pi + \gamma = 0 \tag{5}
$$

$$
\psi - \sigma \dot{w} - \xi \dot{h} + \pi h \leq 0 \tag{6}
$$
2.3 Constitutive Restrictions

We now consider a fairly general constitutive class for the bars: we assume that the bars are homogeneous and that the constitutive functions depend on the list

$$\Gamma(x, t) := (w(x, t), \phi(x, t), p(x, t), h(x, t)).$$

Thus, we assume the following constitutive relations

$$\begin{align*}
\psi(x, t) &= \dot{\psi}(w(x, t), \phi(x, t), p(x, t), h(x, t)) \\
\sigma(x, t) &= \dot{\sigma}(w(x, t), \phi(x, t), p(x, t), h(x, t)) \\
\pi(x, t) &= \dot{\pi}(w(x, t), \phi(x, t), p(x, t), h(x, t)) \\
\xi(x, t) &= \dot{\xi}(w(x, t), \phi(x, t), p(x, t), h(x, t)).
\end{align*}$$

We ignore questions of smoothness of the constitutive functions. Below, we consider examples that are piecewise real analytic.

The quantities $f$ and $\gamma$ are assumed to come from sources external to the body and are therefore not given by constitutive laws. Instead, we assume that these quantities can take on any values consistent with the balance laws.

Note that under these constitutive assumptions, the dissipation inequality implies that for every sufficiently smooth process we have

$$0 \leq |\dot{\sigma} - \dot{\psi}_w| \dot{w} + |\dot{\xi} - \dot{\psi}_p| \dot{p} - |\dot{\pi} + \dot{\psi}_h| h + \dot{\psi}_h \dot{h}$$

We now follow the procedure due to Coleman and Noll for employing this form of the second law of thermodynamics to derive restrictions on the constitutive functions. We note that by choosing appropriate values of $f$ and $\gamma$ we can construct processes such that $w, \dot{w}, \phi, h, p, \dot{h}$, and $\dot{p}$ can be chosen independently and arbitrarily at a given point and time. Thus, at each point and time, (7) must hold for arbitrary prescribed values of the quantities listed above. From this we deduce the following constitutive restrictions.

$$\begin{align*}
\dot{\psi}_h &= 0 \\
\dot{\sigma} &= \dot{\psi}_w \\
\dot{\xi} &= \dot{\psi}_p.
\end{align*}$$

In addition, we must have

$$\langle \dot{\pi}(\Gamma) + \dot{\psi}_f(w, \phi, p) \rangle h \leq 0.$$  

If we set

$$\dot{\pi}(\Gamma) = -\dot{\psi}_f(w, \phi, p) - \dot{b}(\Gamma) h,$$

where

$$\dot{b}(\Gamma) \geq 0,$$
then (11) follows.

If we incorporate these constitutive restrictions into our balance laws, the balance of linear momentum (1) becomes

\[ \rho \ddot{u} = \dot{\psi}_w(u_x, \phi, \phi_x)_x + f. \]  

(14)

The balance of microlocal forces (2) becomes

\[ \dot{b}(\Gamma) \dot{\phi} = \dot{\psi}_p(u_x, \phi, \phi_x)_x - \dot{\psi}_\phi(u_x, \phi, \phi_x) + \gamma. \]  

(15)

2.4 Particular problems

For the purposes of exhibiting precursor oscillations we examine only static problems. We choose the following boundary conditions. At the left end of the bar we impose a fixed displacement boundary condition

\[ u(0) = 0 \]  

(16)

while at the right end we consider a free end

\[ \sigma(l) = 0. \]  

(17)

Boundary conditions for the order parameter are more problematic since we have placed such a weak physical interpretation on the variables in question. Fortunately, for the purpose of exhibiting the existence of precursors, almost any reasonable choice of linear boundary conditions works. For concreteness we examine Dirichlet boundary conditions in this paper.

\[ \phi(0) = \phi(l) = 0. \]  

(18)

We now consider a particular set of constitutive choices that highlights the ability of the more general model to predict precursor oscillations and the onset of microstructure. We do not make any further assumptions about the density \( \rho \) and the function \( \dot{b} \) since they do not enter our static calculations. Our only task is to specify a constitutive relation for the free energy \( \dot{\psi} \). We define

\[ \dot{\psi}(w, \phi, p, \theta) := G(w, \theta) + \frac{\kappa}{2} |\phi|^2 - \kappa w \phi + \frac{\varepsilon^2}{2} |p|^2 \]  

(19)

where the temperature \( \theta \) is used as a scalar parameter. We think of \( G \) as a symmetric triple-well potential (see Figure 1) with a central “Austenite” well flanked by two “Martensite” wells. When we do our global analysis below, we prescribe a particular piecewise quadratic function, but while we proceed, we think of the relative heights of the wells as being controlled by temperature with the Austenite well rising as the temperature decreases. We think of the Martensite wells as having a relatively hard modulus that changes little with temperature while the Austenite modulus is assumed to get softer as the temperature decreases.
As a consequence of (19), (9), (10), and (12) we have
\begin{align*}
\dot{\sigma}(u_x, \phi, \phi_x, \theta) &= \sigma(u_x, \theta) - \kappa \phi, \\
\dot{\xi}(u_x, \phi, \phi_x) &= \epsilon^2 \phi_x, \\
\dot{\mathcal{F}}(u_x, \phi_x, \phi) &= \kappa u_x - \alpha \phi - \dot{b} \phi,
\end{align*}
where the elastic part of the stress $\dot{\sigma}$ is defined to be
\begin{equation}
\sigma(w, \theta) := G_w(w, \theta).
\end{equation}

Thus, under these assumptions, the static local balance laws with zero forcing functions $(f \equiv 0, \gamma \equiv 0)$ reduce to
\begin{equation}
\sigma(u_x, \theta)_x - \kappa \phi_x = 0,
\end{equation}
and
\begin{equation}
\epsilon^2 \phi_{xx} - \alpha \phi + \kappa u_x = 0,
\end{equation}
which we consider with boundary conditions (16), (17), and (18).

**Remark 2.1** In our constitutive model, the free energy is nonconvex in the strain, convex in the order parameter, and coupled between the strain and order parameter. This is similar to the hypothesis made in Lin and Rogers [13] and Brandon, Lin and Rogers [3] and is distinct from the assumptions made in the usual phase-field models (see [6, 7]) where the central assumption is the nonconvexity of the energy in the order parameter. We note that while Fried
and Gurtin clearly have generalizations of the phase field model in mind when deriving their models, their formulations are general enough to include the type of constitutive choices made here.

3 Local Analysis in the Austenite Well

In this section we look for solutions of (24), (25), (16), (17),and (18) that lie entirely within the Austenite well. We assume that

\[ \bar{\sigma}(0, \theta) = 0 \]  

(26)

so that our nonlinear problem has the trivial solution \( u \equiv \phi \equiv 0 \). We now set

\[ \beta(\theta) := \bar{\sigma}_w(0, \theta) = G_{ww}(0, \theta) \]  

(27)

and examine the linearized problem about the trivial state. We assume

\[ \beta(\theta) > 0, \]  

(28)

and

\[ \frac{\partial \beta}{\partial \theta}(\theta) > 0. \]  

(29)

The linearized problem is simply

\[ \beta(\theta)u_{xx} - \kappa \phi_x = 0 \]
\[ -\epsilon^2 \phi_{xx} + \alpha \phi - \kappa u_x = 0 \]
\[ u(0) = 0 \]
\[ \beta(\theta) u_x(l) - \kappa \phi(l) = 0 \]
\[ \phi(0) = \phi(l) = 0 \]  

(30)

The first equation can be integrated using the free boundary condition to obtain

\[ u_x = \frac{\kappa}{\beta(\theta)} \phi, \]  

(31)

which, together with the left-end Dirichlet condition, yields

\[ u(x) = \frac{\kappa}{\beta(\theta)} \int_0^x \phi(s) \, ds \]  

(32)

Combining (31) with the balance of microlocal forces (30) and rearranging we get

\[ \phi_{xx} + \left( \frac{\kappa^2 - \beta(\theta) \alpha}{\beta(\theta) \epsilon^2} \right) \phi = 0. \]  

(33)
Of course, the linear boundary-value problem composed of (33) and the Dirichlet boundary conditions for $\phi$ has nontrivial, sinusoidal solutions exactly when $\theta = \theta_n$ satisfies

$$\lambda_A(\theta_n) := \frac{\kappa^2 - \beta(\theta_n)\alpha}{\beta(\theta_n)\epsilon^2} = \frac{n^2\pi^2}{l^2}, \quad n = 1, 2, 3, \ldots$$  \hfill (34)

A few remarks are in order,

- It follows from (29) that there is at most one solution $\theta_n$ of (34) for each $n$, and that the sequence $\theta_n$ is monotone decreasing.

- If there is a critical temperature $\bar{\theta}$ such that

$$\lim_{\theta \to \bar{\theta}^+} \beta(\theta) = 0$$

(e.g., if $\beta$ is of the form $\beta(\theta) := \tilde{\beta}(\theta - \bar{\theta})$) then we have

$$\lim_{\theta \to \bar{\theta}^+} \lambda_A(\theta) = \infty$$

and there exists a solution of $\theta_n$ of (34) for every $n$ sufficiently large.

(Of course, if the function $G(w, \theta)$ depends on temperature in the way suggested above, the Austenite well is far above the Martensite wells at temperatures close to $\bar{\theta}$.)

- Close to the Austenite solution, the free energy has a quadratic approximation of the form

$$\hat{\psi}(w, \phi, p) \approx \frac{1}{2} (w, \phi) \begin{pmatrix} \beta(\theta) & -\kappa \\ -\kappa & \alpha \end{pmatrix} \begin{pmatrix} w \\ \phi \end{pmatrix} + \frac{\epsilon^2}{2} |p|^2.$$  

The function $\lambda_A(\cdot)$ becomes positive (and the ode (33) develops oscillations) exactly when the free energy loses local positive definiteness. In other words, the oscillations develop because the interaction of the strain and the order parameter causes the Austenite well to become a saddle point.

- If the parameters $\kappa$ and $\alpha$ are fixed, a sufficiently small Austenite modulus $\beta(\theta)$ causes the free energy to lose local positive definiteness in the central well. However, if the Martensite modulus is sufficiently large or if $G$ grows faster than quadratically, then the free energy is globally coercive in the sense that $\|w, \phi, p\|_{L^2} \to \infty$ implies $\int_0^1 \psi \to \infty$.

- The quantity $\epsilon/l$ acts as a lengthscale in the problem.

- There is nothing inherently “one-dimensional” about the local analysis. A similar procedure could be performed for more realistic physical energies of deformation of three dimensional materials. (This is not true for the global analysis performed below, where ordinary differential equation techniques such as phase-plane methods are crucial.)
• The question of local stability of the bifurcating branches would be determined by higher-order nonlinearity of \( G \). We have not done a dynamic stability analysis, but one can easily check using standard perturbation techniques that

- if \( \tilde{\sigma}_{ww}(0, \theta_n) < 0 \) at a bifurcation point \( \theta_n \) then the bifurcating branch is subcritical, i.e. it “pitchforks” toward the (stable) high temperature branch.

- if \( \tilde{\sigma}_{ww}(0, \theta_n) > 0 \) at a bifurcation point \( \theta_n \) the bifurcating branch is supercritical, i.e. it pitchforks away from the (stable) high temperature branch.

The obvious conjecture is that all subcritical branches are unstable and the first supercritical branch is locally stable.

• In summary, we claim that the order parameter model exhibits a local bifurcation picture consistent with experimental observations of precursor oscillations if one chooses a free energy with the following properties.

- The Austenite modulus \( \beta(\theta) \) is an increasing function of temperature.

- The function \( \lambda_A(\theta) := \frac{\kappa^2 - \beta(\theta)\alpha}{\beta(\theta)\kappa^2} \) becomes positive at a critical temperature \( \theta_C \) substantially greater than the transition temperature \( \theta_T \) at which the Austenite well rises above the Martensite well.

- There is a precursor temperature \( \theta_P \) in the interval \((\theta_T, \theta_C)\) such that

\[
\tilde{\sigma}_{ww}(0, \theta) < 0 \text{ for } \theta > \theta_P \\
\text{and}
\tilde{\sigma}_{ww}(0, \theta) > 0 \text{ for } \theta < \theta_P
\]

- The value of \( \epsilon \) is sufficiently small so that, for \( l \) in range corresponding to realistic samples on which precursors have been observed, we have

\[ l^2\lambda_A(\theta_P) >> 1. \]

This corresponds to very fine oscillations on the first stable branch.

4 Global Analysis

We now examine the global nonlinear problem composed of local balance laws (24) and (25) and the boundary conditions (16), (17) and (18). We describe sufficient conditions for a class of weak solutions of the problem. Our solutions will have the property that \( u, \phi \) and \( \phi_x \) are continuous while \( w = u_x \) and \( \phi_{xx} \) are piecewise continuous.\(^1\)

\(^1\)In fact, using arguments similar to those used in [3], one can show that all weak solutions are in this class. But since our purpose is simply to exhibit precursor oscillations (rather than to classify all solutions) we do not prove this here.
4.1 Constitutive assumptions

We consider a triple-well quadratic potential. Let $w_T < w_M$, $\beta_M$, and $\beta(\theta)$ be positive scalars. We define

$$G(w, \theta) := \begin{cases} 
\frac{\beta_M}{2}(w + w_M)^2 & -\infty < w < -w_T \\
\frac{\beta_M}{2}(w - w_M)^2 & w_T < w < w_M \\
\beta(\theta) - w^2 + \bar{e}_0(\theta) & w_T \leq w < w_T \\
\frac{\beta_M}{2}(w - w_M)^2 & w_T \leq w < \infty 
\end{cases} \tag{35}$$

where

$$\bar{e}_0(\theta) := \frac{1}{2} \left[ \beta_M(w_M - w_T)^2 - \beta(\theta)w_T^2 \right] \tag{36}$$

Here $\pm w_M$ are the “Martensite equilibrium” states, $w = 0$ is the “Austenite equilibrium” state, and $\pm w_T$ are the transition points between the Martensite wells and the Austenite well. The constants $\beta_M$ and $\beta(\theta)$ are the elastic moduli in the Martensite and Austenite wells respectively. Note that this potential is continuous, but that it suffers a jump in its derivative with respect to $w$ at the transition points.

Note that the free energy now depends on the temperature $\theta$ through the parameter $\beta(\theta)$. We think of the constants $w_T$, $w_M$, $\beta_M$, $\alpha$, $\kappa$, and $\epsilon$ as being fixed. If we assume that $\theta \rightarrow \beta(\theta)$ is monotone increasing, then as that temperature decreases, the concavity of the central well (i.e. the Austenite modulus)
decreases and the well rises. The Martensite wells and the points at which the wells are attached remain fixed.

4.2 The relaxed problem

Our solutions will be piecewise classical solutions of (24) and (25). To ensure that our functions are weak solutions we need only require that the total stress be continuous at each point at which \( w \) and \( \phi_{xx} \) undergo jump discontinuities. But in addition, we require that our weak solutions satisfy a “stability” condition. It is a standard result in the calculus of variations that strong local minimizers of the energy

\[
\mathcal{E}(u, \phi) := \int_0^l \psi(u_x, \phi, \phi_x) \, dx
\]

(37)

over the set of piecewise smooth functions satisfying the boundary conditions must be weak solutions of the relaxed version of (24):

\[
(\sigma^R(u_x, \theta) - \kappa \phi)_x = 0,
\]

(38)

Where \( \sigma^R \) is the derivative with respect to \( w \) of the lower convex envelope of \( \mathcal{G} \).

For our particular piecewise quadratic constitutive functions the relaxed stress \( \sigma^R \) is described as follows. In the case where \( \tilde{c}_0(\theta) < 0 \) (so that the Austenite well drops below the Martensite wells) we seek \( 0 < a < b \), and \( \sigma_0 > 0 \) satisfying the Maxwell line conditions:

\[
\sigma_0 = \tilde{\sigma}(a, \theta)
\]

(39)

\[
\sigma_0 = \tilde{\sigma}(b, \theta)
\]

(40)

\[
\mathcal{G}(a, \theta) - \mathcal{G}(b, \theta) = \sigma_0(b - a)
\]

(41)

For our constitutive function, the solutions are

\[
a = \hat{a}(\theta) := \frac{\beta(\theta)\beta_M w_M - \sqrt{\beta(\theta)\beta_M (\beta(\theta) - \beta_M) + \beta_M w_M}^2}{\beta(\theta)(\beta_M - \beta(\theta))}
\]

\[
b = \hat{b}(\theta) := \frac{\beta_M^2 w_M - \sqrt{\beta(\theta)\beta_M (\beta(\theta) - \beta_M) + \beta_M w_M}^2}{\beta_M (\beta_M - \beta(\theta))}
\]

\[
\sigma_0 = \hat{\sigma}_0(\theta) := \frac{\beta(\theta)\beta_M w_M - \sqrt{\beta(\theta)\beta_M (\beta(\theta) - \beta_M) + \beta_M w_M}^2}{(\beta_M - \beta(\theta))}
\]

In this case we have

\[
\sigma^R(w, \theta) := \begin{cases} 
\beta_M (w + w_M) & -\infty < w < -\hat{b}(\theta), \\
-\hat{\sigma}_0(\theta) & -\hat{b}(\theta) \leq w < -\hat{a}(\theta), \\
\beta(\theta) w & -\hat{a}(\theta) \leq w < \hat{a}(\theta), \\
\hat{\sigma}_0(\theta) & \hat{b}(\theta) \leq w < \hat{b}(\theta), \\
\beta_M (w - w_M) & \hat{b}(\theta) \leq w < \infty.
\end{cases}
\]

(42)
If $\bar{e}_0(\theta) \geq 0$ (so that the Austenite well never drops below the Martensite wells and thus lies above the convex hull of the energy $\mathcal{G}$) we take

\[
\begin{align*}
\hat{a}(\theta) &= 0 \\
\hat{b}(\theta) &= w_M \\
\sigma_0(\theta) &= 0,
\end{align*}
\]

so that

\[
\tilde{\sigma}^R(w, \theta) := \begin{cases} 
\beta_M(w + w_M) & -\infty < w < -w_M, \\
0 & -w_M \leq w < w_M, \\
\beta_M(w - w_M) & w_M \leq w < \infty.
\end{cases}
\] (43)

4.3 Weak solutions

We now look for solutions of the relaxed boundary-value problem. Conditions (38) and (17) imply

\[
\kappa \phi(x) = \tilde{\sigma}^R(u_x(x), \theta),
\] (44)

for all $x \in [0, l]$. Conversely, note that if a continuous function $\phi$ and piecewise continuous function $u_x$ satisfy (44) almost everywhere, then (38) and (17) are satisfied in the weak sense.

To obtain such functions we use an analog of the procedure used to get (31) in the local problem: we construct a solution $u_x$ of (44) by introducing the discontinuous right inverse function

\[
J(\phi, \theta, \kappa) := \begin{cases} 
-w_M + \kappa \phi / \beta_M & -\infty < \phi \leq -\sigma_0(\theta) / \kappa \\
\kappa \phi / \beta(\theta), & -\sigma_0(\theta) / \kappa < \phi \leq \sigma_0(\theta) / \kappa \\
w_M + \kappa \phi / \beta_M & \sigma_0(\theta) / \kappa < \phi < \infty
\end{cases}
\] (45)

Note the following.

- If $\phi$ and $u_x$ satisfy
  \[
  u_x(x) = J(\phi(x), \theta, \kappa),
  \] (46)
  they also satisfy (44) and hence (38) and (17).

- The range of $J$ is the contained in the set
  \[
  (-\infty, \hat{b}(\theta)] \cup [-\hat{a}(\theta), \sigma(\theta)] \cup [b(\theta), \infty)
  \]
  on which $\tilde{a}(\cdot, \theta)$ and $\tilde{\sigma}^R(\cdot, \theta)$ agree. Thus, solutions of (46) also satisfy (24).

We use these observations to obtain the following result.
Lemma 4.1 Suppose \( \phi \) is a \( C^1[0,1] \) function with piecewise continuous second derivative that satisfies the boundary conditions (18) and the differential equation

\[
\phi_{xx} + \mathcal{H}(\phi) = 0 \quad (47)
\]
except for at most a finite number of points of discontinuity of \( \phi_{xx} \). Here

\[
\mathcal{H}(\phi) := \frac{1}{\varepsilon^2} \left( \kappa \mathcal{J}(\phi(x), \theta, \kappa) - \alpha \phi \right)
\]

\[
= \begin{cases} 
-\lambda_M (\phi + \phi_M) & -\infty < \phi \leq -\phi_T(\theta) \\
\lambda_A(\theta) \phi, & -\phi_T(\theta) < \phi \leq \phi_T(\theta) \\
-\lambda_M (\phi - \phi_M) & \phi_T(\theta)/\kappa < \phi < \infty 
\end{cases}
\]

and

\[
\lambda_M = \frac{\beta_M \alpha - \kappa^2}{\beta_M \varepsilon^2} \\
\phi_M = \frac{\kappa \beta_M \alpha}{\beta_M \alpha - \kappa^2} \\
\phi _T(\theta) := \frac{\dot{\sigma}_0(\theta)}{\kappa}
\]

Then \( \phi \) and

\[
u(x) := \int_0^x \mathcal{J}(\phi(s), \theta, \kappa) \, ds \quad (48)
\]

are weak solutions of (24), (25), (16), (17) and (18). Furthermore, they satisfy the relaxed equation (38).

The proof is immediate. Equation (48) implies (16) and can be differentiated to give (46). As noted above, this implies (24), (17), and (38) are satisfied. Also, (46) can then be combined with (47) to give (25).

4.4 Phase-plane analysis

We now use phase plane methods to construct solutions of (47) and (18). Again, our purpose is to discuss a specific class of solutions, so we examine only a particular range of parameters. We assume that

\[
0 < \omega_T < \omega_M, \quad (49)
\]

\[
0 < \beta(\theta) < \frac{\kappa^2}{\alpha} < \beta_M, \quad (50)
\]

which ensures that \( \lambda_M \) and \( \lambda_A \) are real. We assume also

\[
\dot{\sigma}_0(\theta) = \frac{1}{2} \left[ \beta_M (\omega_M - \omega_T)^2 - \beta(\theta) \omega_T^2 \right] < 0. \quad (51)
\]
Figure 3: Phase-plane portrait for stationary points.

Note that this ensures that

$$0 < \phi_T < \phi_M$$ \hfill (52)

so that our solutions can contain both Austenite and Martensite.

The \((\phi, \phi_x)\) phase plane for the ODE \((47)\) is now easy to describe. See Figure 3

- The phase plane is divided into three vertical strips by the lines \(\phi = \pm \phi_T(\theta)\) (which we refer to as the transition lines). We refer to the outer strips as the left and right Martensite strips and the center strip as the Austenite strip.

- Within each strip the phase portrait is that of a linear ode. Each strip contains one critical point.
  - The Austenite strip has a center at \(\phi = \phi_x = 0\).
  - The Martensite strips have saddle points at \(\phi_x = 0\), \(\phi = \pm \phi_M\) respectively.

Within the Austenite strip the trajectories lie along the ellipses

$$\phi_x^2 + \lambda_4 \phi^2 = C$$
and within the Martensite strips the trajectories lie along the hyperbolas

$$\phi_x^2 - \lambda_M^2(\phi \pm \phi_M)^2 = C.$$  

- There is a simple jump discontinuity in the vector field defined by (47) at the transition lines. The trajectories obtained by integrating the vector field are continuous and can be interpreted as corresponding to solutions of (47) with $\phi$ and $\phi_x$ continuous but with $\phi_{xx}$ having a simple jump discontinuity whenever a trajectory crosses a transition line.

- Paths of the trajectories are symmetric about both the $\phi$ and $\phi_x$ axes.

- The plane can be divided into five invariant regions, separated by the stable and unstable manifolds of the saddle points and the portions of orbits of the center that connect to these manifolds. Four of the regions are unbounded and one is bounded.

We now note that only the bounded invariant region contains trajectories that cross the $\phi_x$ axis more than once. Since our boundary conditions specify trajectories that begin and end on this axis, we can focus our attention on that region alone.

Without loss of generality, suppose we begin a trajectory on the positive $\phi_x$-axis. In all cases the trajectory takes us into the right half-plane, across the $\phi$-axis and back to the positive $\phi_x$-axis. We define the “return time” of a point on the positive $\phi_x$-axis to be amount of time (really distance) that the solution needs to traverse the trajectory to the negative $\phi_x$-axis. Using the symmetry of the phase plane we see that a class of solutions of our problem can be characterized by those solutions whose return time is $l/n$ for some positive integer $n$. (Here $l$ is the length of the rod.) Perhaps the main point of considering piecewise quadratic energies is that we can calculate the return time in closed form. Let us consider a trajectory intercepting the positive $\phi_x$-axis at a point $\tilde{v}$ and the positive $\phi$ axis at a point $\tilde{\phi} < \phi_M$. The relationship between $\tilde{v}$ and $\tilde{\phi}$ is

$$\tilde{v}^2 = \lambda_A^2\tilde{\phi}_T^2 + \lambda_M^2[(\phi_T - \phi_M)^2 - (\tilde{\phi} - \phi_M)^2].$$

A routine calculation gives the return time, which we parameterize by $\tilde{\phi}$ and $\theta$.

$$r(\tilde{\phi}, \theta) := \begin{cases} \frac{\pi}{\lambda_A(\theta)} & \tilde{\phi} < \phi_T(\theta), \theta > \theta_T, \\ \frac{2}{\lambda_M(\theta)} \sin^{-1} \left( \frac{\phi_T(\theta)}{\phi_M - \phi} \right) & \phi_T(\theta) < \tilde{\phi}, \theta > \theta_T, \\ + \frac{2}{\lambda_M} \ln \left( \frac{\phi_T(\theta)}{\phi_M - \phi} + \sqrt{\frac{(\phi_M - \phi_T(\theta))^2}{(\phi_M - \phi)^2} - 1} \right) & \tilde{\phi} < \phi_M, \theta < \theta_T \end{cases}$$

Here $\theta_T$ is the transition temperature at which $\tilde{e}_0(\theta_T) = 0$. We make a few observations.
• The return time for Austenite solutions (solutions that start sufficiently close to the origin so that the trajectories lie entirely in the Austenite strip) is $\pi / \lambda_A(\theta)$. Thus, it depends only on the temperature, not on the crossing point $\phi$.

• The return time of the Austenite solutions provides an upper bound for the minimal return time. A lower bound is given by twice the time required to go from the $\phi_x$ axis to the point on the transition line lying on the (straight line) trajectory that connects to the saddle point in the Martensite strip. (That is, the time spent in the Austenite region by the trajectory that bounds the invariant region.)

• The return time goes to infinity as we go to the boundary trajectory of the invariant region described above (the trajectory connecting the $\phi_x$ axis to the saddle).

In Figure 4 we construct a bifurcation diagram by graphing the level curves of the return time at levels $l/n, n = 1, 2, 3, \ldots$ Note that because the problem is linear in the Austenite well, the bifurcation curves are degenerate straight lines in the regions of pure Austenite. At the point where $\phi_{xx}$ undergoes a discontinuity ($\phi = \phi_T(\theta)$), the branches always swing back toward higher temperatures. The branches then swing back toward lower temperatures and are bounded above by $\phi = \phi_M$.

At temperatures below $\theta_T$, the Austenite well lies above the Martensite wells and no longer is involved in the relaxed problem. The problem becomes a classic double-well problem, and the phase-plane now has a degenerate center at $\phi = \phi_x = 0$. This center is not surrounded by ellipses, but by sections of hyperbolae. Because of this, there is no longer a lower bound on the return time of oscillating about the center, and we can get stationary points of the energy with arbitrarily fine oscillations. (This is similar to the results found in [4].)

5 Conclusions

To summarize, our basic result is that a scalar order parameter that is strongly coupled to a strain with a soft modulus can cause a saddle instability in the total energy. Under appropriate constitutive assumptions, this instability causes small amplitude oscillations within the Austenite well to bifurcate from a stable uniform state.

We conclude with a few comments on future work.

• The local analysis can be extended to nonhomogeneous strains and displacements to predict stress-strain curves for the soft moduli. While the boundary conditions for the order parameter played little part in the demonstration of the existence of precursor oscillations, they may have a large effect on the predicted stress-strain laws.
Figure 4: Bifurcation diagram showing stationary points of the order-parameter energy as a function of temperature.
• As we indicated above, it should be possible to extend the local analysis to problems in more than one space dimension. Of course, the main questions here are how many order parameters should be used and how one should couple them to the strain tensor. A number of possibilities suggest themselves.

1. In [12], precursors were modeled using a “quenched in” scalar inhomogeneity in an energy defined for two-dimensional deformations. We could easily replace the quenched inhomogeneity with a “reactive” inhomogeneity generated by a scalar order parameter.

2. There are a number of well-defined proposals for constitutive functions involving “transformation strain” as an internal variable (see e.g. [5]). These might be modified to fit the framework of our order-parameter models.

• The local analysis can also be extended to the dynamic case so that some predictions about the dependence of stress-strain laws on the frequency of input could be made.

• The question of local dynamic stability of the precursor oscillations needs to be addressed.

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