WEIGHTED RESOLVENT ESTIMATES FOR VOLterra
OPERATORS ON UNBOUNDED INTERVALS

By

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1. Introduction

Let \((X, \|\|)\) be a Banach space of real functions on a given interval \(I\) of the real line, and let \(V\) be a Volterra integral operator on \(X\), defined by

\[
(Vf)(\xi) = \int_{(\eta \in I, \eta \leq \xi)} V(\xi, \eta)f(\eta)d\eta, \xi \in I.
\]  

(1.1)

The interval may be bounded or unbounded, including the case of the whole real line, and the kernel \(V(\xi, \eta)\) is supposed to be such that \(V\) maps \(X\) into \(X\) (we shall make precise assumptions later).

The case of a bounded interval \(I\) with \(X = C(I)\) and \(V(\xi, \eta)\) continuous for \((\xi, \eta) \in I \times I\) and \(\xi > \eta\) is classical. In this case, the operator \(V\) is bounded and its spectrum consists of \(0\). In other words, the spectral radius of \(V\) is zero. A standard proof of these two facts is to prove that for \(\lambda \neq 0\) the Neumann series for the inverse of \((\lambda I - V)\) converges with respect to the operator norm.

The purpose of this note is to study the case of a possibly unbounded interval \(I\). In this case we introduce a weight function \(w\) on the interval \(I\) and formulate an assumption and similar conclusions on \(V\). At the same time, we are interested in weighted resolvent estimates, that is to say, estimates on the \(w\)-norm of \((\lambda I - V)^{-1}\). Such weighted resolvent estimates play a key role in the study of spectral properties of Schrödinger operators. In fact, this role motivated our interest in the present problem, [DMR1], [DMR2], [R1], [R2]. In Section 2 we introduce, for a given weight function \(w\), a norm for functions defined on the interval \(I\), and so define a Banach space \(B(w)\). Second, we define the Love-Erdelyi-Olver bound of \(V\), \(\|V\|(LEO, w)\). Third, in Theorem 2.1, we show that if this bound is finite, then so is the \(B(w)\) operator norm of \(V\) and of the inverse of \((\lambda I - V)\). In conclusion (2.6) of Theorem 2.1 we estimate this norm. Theorem 2.1 is not new. In fact, essentially, it is due to Erdelyi [E] and Olver [O]. However, we give a new proof of the key conclusion (2.6).

To put theorem 2.1 in a general context we need some general facts about possibly non-Volterra operators,

\[
Kf(\xi) = \int_{I} K(\xi, \eta)f(\eta)d\eta, \xi \in I.
\]  

(1.2)

In Lemma 2.2 we formulate a simple boundedness criterion for \(K\). Then, in Corollary 2.3, we observe that the original theorem of Love [L] on the invertibility of \(\lambda I - K\) is a corollary of Lemma 2.2.

In Section 3 we show that the key conclusion (2.6) of Theorem 2.1 is implied by Reid’s version of Gronwall’s inequality with integrable coefficients [CL], [R].

In Section 4, for completeness we prove all of the conclusions of Theorem 2.1.
Appendix I is by W. S. Loud and it gives an example illustrating that Theorem 2.1 is optimal.

In Appendix II we illustrate the fact that the assumption of Lemma 2.2 is strictly weaker than the assumption of Theorem 2.1.

It is a pleasure to thank Professors McCarthy, Sibuya and Dr. Zheng for valuable conversations.

2. Formulation of the Results.

Let $\mathcal{I}$ be a given subinterval of $\mathbb{R}$ and let $w$ be a continuous positive weight function on $\mathcal{I}$. Given a continuous function $f$ on $\mathcal{I}$, we define its $w$-norm by

$$
\|f\|(w) = \sup_{\xi \in \mathcal{I}} (|f(\xi)|w(\xi)^{-1}).
$$

Then the set $X$ of all continuous functions on $\mathcal{I}$ whose $w$-norm is finite is a Banach space, $\mathcal{B}(w)$.

Let $V$ be the Volterra operator defined formally by (1.1), with $V(\xi, \eta)$ integrable in the triangular region $\xi \geq \eta, \xi, \eta \in \mathcal{I}$. Our goal is to give conditions on $V$ with respect to the given weight function $w$ such that:

(i) $V$ maps $X$ into itself.

(ii) $V$ is a bounded operator.

(iii) $V$ has zero spectral radius.

We also wish to obtain sharp estimates for the $\mathcal{B}(w)$-operator norms of $V$ and $(I-V)^{-1}$. As usual, we denote the algebra of everywhere defined bounded operators on $\mathcal{B}(w)$ by $\mathcal{B}(\mathcal{B}(w))$.

To formulate such estimates, following Love [L] Erdelyi [E] and Olver [O], we define

$$
\|V\|(LEO, w) = \int_{\mathcal{I}} \sup_{\xi > \eta} w(\xi)^{-1}w(\eta)|V(\xi, \eta)|d\eta.
$$

2.1. Let the Volterra operator $V$ be given by the definition (1.1) and let the space $\mathcal{B}(w)$ be given with the help of the definition (2.1). Suppose that

$$
\|V\|(LEO, w) < \infty.
$$

Then,

$$
V \in \mathcal{B}(\mathcal{B}(w)) \text{ and } \|V\|(w) \leq \|V\|(LEO, w).
$$

Furthermore, for each $\lambda \neq 0$ in $\mathbb{C}$,

$$
(\lambda I - V)^{-1} \in \mathcal{B}(\mathcal{B}(w))
$$
and
\[ \|(\lambda I - V)^{-1}\| \|w\| \leq |\lambda|^{-1} \exp(|\lambda|^{-1} \|V\|\|(LEO, w)\|). \tag{2.6} \]

Next, we formulate a weaker version of assumption (2.3), which is strong enough to imply conclusion (2.4). For this purpose, we follow Love [L] and define
\[ \|K\|(L, w) = \sup_{\xi \in I} w(\xi)^{-1} \int_{I} w(\eta) |K(\xi, \eta)| d\eta. \tag{2.7} \]

**Lemma 2.2.** Let the (possibly non-Volterra) operator $K$ be given by the definition (1.2). Suppose that
\[ \|K\|(L, w) < \infty. \tag{2.8} \]
Then, for this operator, conclusion (2.4) of Theorem 2.1 holds, and
\[ \|K\|(w) \leq \|K\|(L, w). \tag{2.9} \]

To prove Lemma 2.2 note that insertion of the definition (1.2) into the definition (2.1) yields,
\[ \|Kf\|(w) = \sup_{\xi \in I} w(\xi)^{-1} \int_{I} K(\xi, \eta) f(\eta) d\eta. \tag{2.10} \]
By applying the definition (2.1) to the function $f$, we find
\[ |f(\eta)| \leq \|f\|(w) \cdot w(\eta), \quad \text{for all} \quad \eta \in I. \tag{2.11} \]
Since the absolute value of an integral is majorized by the integral of the absolute value, by inserting estimate (2.11) into relation (2.10) we find,
\[ \|Kf\|(w) \leq \|f\|(w) \cdot \sup_{\xi \in I} w(\xi)^{-1} \int_{I} |K(\xi, \eta)| w(\eta) d\eta. \tag{2.12} \]

Remembering the definition (2.7), we see that estimate (2.12) yields
\[ \|Kf\|(w) \leq \|K\|(L, w) \cdot \|f\|(w), \]
and so conclusion (2.9) follows. This completes the proof of Lemma 2.2.

Next, we show that assumption (2.8) is weaker than assumption (2.3). More specifically, we show that for a Volterra operator
\[ \|V\|(L, w) \leq \|V\|(LEO, w). \tag{2.13} \]
To see this, note that application of the definition (2.7) to the operator of the definition (1.1) yields

$$\|V\|(L, w) = \sup_{\xi \in I} w(\xi)^{-1} \int_{(\eta \in I, \eta < \xi)} |V(\xi, \eta)| w(\eta) d\eta. \quad (2.14)$$

Since the supremum of an integral with respect to a parameter is majorized by the integral of the supremum with respect to that parameter, we find

$$\sup_{\xi \in I} \int_{(\eta \in I, \eta < \xi)} w(\xi)^{-1} w(\eta) |V(\xi, \eta)| d\eta \leq \int_{I} \sup w(\xi)^{-1} w(\eta) |V(\xi, \eta)| d\eta. \quad (2.15)$$

By combining estimate (2.15) with relation (2.14) and with the definition (2.2) we find estimate (2.13).

The original theorem of Love [L] is a straightforward corollary of Lemma 2.2.

**Corollary 2.3.** Let \( \lambda \) in \( \mathbb{C} \) be given and suppose that

$$\|K\|(L, w) < |\lambda|. \quad (2.16)$$

Then, for this operator, conclusion (2.9) of Lemma 2.2 holds.

Furthermore,

$$\|(\lambda I - K)^{-1}\|(w) \leq |\lambda|^{-1} \cdot (1 - |\lambda|^{-1} \|K\|(L, w))^{-1}. \quad (2.17)$$

To see that Lemma 2.2 implies Corollary 2.3, note that assumption (2.16) implies that \( \lambda \neq 0 \), and so

$$\lambda (I - K) = \lambda (I - \lambda^{-1} K). \quad (2.18)$$

This formula, together with another application of assumption (2.16), allows us to invert the second factor of formula (2.18) with the help of a Neumann series \([NS],[S]\). Then conclusion (2.17) follows in the usual manner \([NS],[S]\).

We conclude this section by formulating a condition which is necessary for \( K \) to map \( \mathcal{B}(w) \) into itself.

**Remark 2.4.** Suppose that the operator \( K \) of the definition (1.2) maps \( \mathcal{B}(w) \) into itself. Then

$$\sup_{\xi \in I} | \int_{I} w(\xi)^{-1} w(\eta) K(\xi, \eta) d\eta | < \infty. \quad (2.19)$$

To prove conclusion (2.19) note that application of the definition (2.1) to the function \( w \) yields,

$$\|w\|(w) = 1 \quad \text{and so} \quad w \in \mathcal{B}(w).$$
This relation allows us to apply the operator $K$ to $w$. Then the definitions (1.2) and (2.1) together, yield
\[ \|Kw\|(w) = \sup_{\xi \in \mathcal{I}} \left| \int_{\mathcal{I}} w(\xi)^{-1}w(\eta)K(\xi,\eta)d\eta \right|. \] (2.20)

Since by assumption the left member of formula (2.20) is finite, conclusion (2.19) follows. This completes the proof of Remark 2.4.

3. Conclusion (2.5) Implies Conclusion (2.6) via Gronwall.

We start this section by recalling the Gronwall-Reid inequality [CL], [R]. For this purpose, let $q = q(\eta)$ be a given positive, locally integrable function on $\mathcal{I}$ and let $b \geq 0$ be a given constant. Suppose that the given function $f$ is continuous on $\mathcal{I}$ and its absolute value satisfies
\[ |f(\xi)| \leq b + \int_{(\eta \in \mathcal{I}, \eta < \xi)} q(\eta)|f(\eta)|d\eta, \quad \text{for all} \quad \xi \in \mathcal{I}. \] (3.1)

Then
\[ |f(\xi)| \leq b \exp\left( \int_{(\eta \in \mathcal{I}, \eta < \xi)} q(\eta)d\eta \right), \quad \text{for all} \quad \xi \in \mathcal{I}. \] (3.2)

To show that conclusion (2.5) implies conclusion (2.6) note that it clearly implies the following. To each $f$ in $B(w)$, there is a $y$ in $B(w)$ such that
\[ (I - V)y = f. \] (3.3)

This in turn, clearly implies that
\[ |y(\xi)| \leq \int_{(\eta \in \mathcal{I}, \eta < \xi)} |V(\xi,\eta)||y(\eta)|d\eta + |f(\xi)|, \quad \text{for all} \quad \xi \in \mathcal{I}. \]

The assumption that $w(\xi)$ is strictly positive allows us to divide both sides of this inequality by $w(\xi)$ and so the definition
\[ g(\xi) = \frac{|y(\xi)|}{w(\xi)} \] (3.4)
yields
\[ g(\xi) \leq \int_{(\eta \in \mathcal{I}, \eta < \xi)} w(\xi)^{-1}|V(\xi,\eta)||w(\eta)g(\eta)d\eta + \|f\|(w) \]
which, in turn, yields
\[ g(\xi) \leq \int_{(\eta \in \mathcal{I}, \eta < \xi)} \{\sup_{\xi > \eta} w(\xi)^{-1}|V(\xi,\eta)||\}w(\eta)g(\eta)d\eta + \|f\|(w). \] (3.5)
By defining
\[ q(\eta) = \{\sup_{\xi > \eta} w^{-1}(\xi) |V(\xi, \eta)|\} w(\eta) \quad \text{and} \quad b = \|f\|(w), \tag{3.6} \]
we see from the inequality (3.5) that the function \( f = g \) satisfies assumption (3.1) of the Gronwall-Reid inequality. Hence it also satisfies conclusion (3.2), and so
\[ |g(\xi)| \leq b \exp(\int_{(\eta \in \mathcal{I}, \eta < \xi)} q(\eta)d\eta), \quad \text{for all} \quad \xi \in \mathcal{I}. \tag{3.7} \]

By inserting the definitions (3.4) and (2.2) into this estimate and taking the supremum with respect to \( \xi \), we find
\[ \|g\|(w) \leq \|f\|(w) \cdot \exp(\int_{\mathcal{I}} q(\eta)d\eta). \tag{3.8} \]

By inserting, in turn, the definitions (3.6) and (2.2) into this estimate, we find
\[ \|y\|(w) \leq \|f\|(w) \cdot \exp(\|V\|(LEO, w)). \tag{3.9} \]

Combining conclusion (2.3) with equation (3.3) we see that
\[ y = (I - V)^{-1} f. \tag{3.10} \]

Finally, by combining relations (3.9) and (3.10), we arrive at conclusion (2.4). In other words, we have shown that conclusion (2.5) of Theorem 2.1 implies conclusion (2.6) via the Gronwall-Reid inequality.

4. The Proof of Theorem 2.1.

In this section, for completeness, we prove all the conclusions of Theorem 2.1.

We prove conclusion (2.4) by combining estimate (2.13) with conclusion (2.9) of Lemma 2.2.

We start the proof of conclusion (2.5) by noting that if the operator \( V \) satisfies assumption (2.3) then so does any scalar multiple of it. By combining this fact with formula (2.18), we see that it is no loss of generality to assume that \( \lambda = 1 \). We continue the proof of conclusion (2.5) by claiming that
\[ \|V\|(w)^n \leq \frac{1}{n!}(\int_{\mathcal{I}} q(\eta)d\eta)^n, \quad n \in \mathbb{Z}^+. \tag{4.1} \]

We prove estimate (4.1) by induction on \( n \). For \( n = 1 \), we prove it by combining estimates (2.15) and (2.12) with the definitions (3.6),(2.2) and (1.2). This clearly yields
\[ \|V\|(w) \leq \int_{\mathcal{I}} q(\eta)d\eta. \tag{4.2} \]
To prove estimate (4.1) for \( n = 2 \) note that these definitions also yield

\[
\|V^2 f\| (w) = \sup_{\xi_1 \in I} w(\xi_1)^{-1} \int_{(\xi_2 \in I, \xi_2 < \xi_1)} V(\xi_1, \xi_2) \int_{(\xi_3 \in I, \xi_3 < \xi_2)} V(\xi_2, \xi_3)f(\xi_3)d\xi_3d\xi_2. \quad (4.3)
\]

Next, multiply and divide the first integral by \( w(\xi_2) \) and the second by \( w(\xi_3) \). Then, by using the definition (3.6), estimate (2.15), and the inequality (2.11) we obtain

\[
\|V^2 f\| (w) \leq \|f\| (w) \cdot \int_{(\xi_2 \in I, \xi_2 < \xi_1)} q(\xi_2) \int_{(\xi_3 \in I, \xi_3 < \xi_2)} q(\xi_3)d\xi_3d\xi_2. \quad (4.4)
\]

Since

\[
\frac{d}{d\xi_1} \int_{(\xi_2 \in I, \xi_2 < \xi_1)} q(\xi_2) \int_{(\xi_3 \in I, \xi_3 < \xi_2)} q(\xi_3)d\xi_3d\xi_2 = \frac{d}{d\xi_1} \frac{1}{2!} \left( \int_{(\xi_2 \in I, \xi_2 < \xi_1)} q(\xi_2)d\xi_2 \right)^2
\]

and

\[
\int_{(\xi_2 \in I, \xi_2 < \xi_1)} q(\xi_2) \int_{(\xi_3 \in I, \xi_3 < \xi_2)} q(\xi_3)d\xi_3d\xi_2 = \frac{1}{2!} \left( \int_{(\xi_2 \in I, \xi_2 < \xi_1)} q(\xi_2)d\xi_2 \right)^2, \text{ for } \xi_1 = 0
\]

we see that these two functions are equal everywhere, that is

\[
\int_{(\xi_2 \in I, \xi_2 < \xi_1)} q(\xi_2) \int_{(\xi_3 \in I, \xi_3 < \xi_2)} q(\xi_3)d\xi_3d\xi_2 = \frac{1}{2!} \left( \int_{(\xi_2 \in I, \xi_2 < \xi_1)} q(\xi_2)d\xi_2 \right)^2. \quad (4.5)
\]

By inserting formula (4.5) into estimate (4.4), we arrive at

\[
\|V^2 f\| (w) \leq \|f\| (w) \cdot \frac{1}{2!} \left( \int_I q(\eta)d\eta \right)^2
\]

which proves estimate (4.1) for \( n = 2 \). At the same time, it is clear that this proof can be extended to a general \( n \). This remark completes the proof of estimate (4.1).

We complete the proof of conclusion (2.5) by noting that assumption (2.3) implies

\[
\int_I q(\eta)d\eta < \infty. \quad (4.7)
\]

Indeed, we see from the definitions (3.6) and (2.2) that

\[
\int_I q(\eta)d\eta = \|V\| (LEO, w), \quad (4.8)
\]
and so estimate (4.7) follows. Now estimates (4.7) and (4.1) allow us to construct the inverse of the operator \( I - V \) with the help of the usual Neumann series [NS],[S]. Thus conclusion (2.5) follows.

To prove conclusion (2.6), we note that the usual Neumann series proof for the existence of the inverse also gives an estimate for the norm [NS],[S]. In fact, this is the way we arrived at the estimate of conclusion (2.6). This completes the proof of Theorem 2.1.

Appendix I

(By W. S. Loud)

An Example Illustrating the Optimality of Theorem 2.1

Let

\[ I = [0,1] \quad \text{and} \quad w \equiv 1. \]

Then, for each function \( f \), the definition (2.2) clearly yields

\[ \|f\|(1) = \sup_{0 \leq \xi \leq 1} |f(\xi)|. \]  \hspace{1cm} (I.1)

Next let \( v \) be a given function such that

\[ \|v\|_1 = \int_0^1 |v(\xi)|d\xi < \infty \quad \text{and} \quad v > 0. \]  \hspace{1cm} (I.2)

With the help of this function \( v \) define the Volterra operator \( V \) by

\[ Vf(\xi) = \int_0^\xi v(\eta)f(\eta)d\eta. \]  \hspace{1cm} (I.3)

First, we claim that for this operator conclusion (2.4) of Theorem 2.1 is optimal. More specifically, we claim that the inequality in conclusion (2.4) is replaced by an equality.

\[ \|V\|(1) = \|V\|(LEO,1). \]  \hspace{1cm} (I.4)

To see this, note that for

\[ f(\eta) \equiv 1 \]  \hspace{1cm} (I.5)

the definition (I.3) yields,

\[ |Vf(\xi)| = \int_0^\xi v(\eta)d\eta. \]  \hspace{1cm} (I.6)
Hence, we see from relation (I.1) that

$$\|vf\|(1) = \|v\|_1 \quad \text{and so} \quad \|V\|(1) \geq \|v\|_1. \quad (I.7)$$

It is clear from the definitions (I.2) and (2.2) that

$$\|V\|(LEO, 1) = \|v\|_1. \quad (I.8)$$

By combining relations (I.8) and (I.7) with conclusion (2.4) of Theorem 2.1, we find relation (I.4). This proves our first claim.

Second, we claim that, for the operator of the definition (I.3), conclusion (2.6) of Theorem 2.1 is optimal. More specifically, we claim that for this operator the inequality in conclusion (2.6) is an equality,

$$\|(I - V)^{-1}\|(1) = \exp(\|V\|(LEO, 1)). \quad (I.9)$$

To prove relation (I.9), note that the integral equation

$$(I - V)g = f \quad (I.10)$$

can be solved explicitly. In fact,

$$g(\xi) = f(\xi) + \int_{0}^{\xi} v(\eta) \exp\left[ \int_{\eta}^{\xi} v(\zeta) d\zeta \right] f(\eta) d\eta.$$

By applying this formula to the function of the definition (I.5) we find, after some elementary algebra,

$$g(\xi) = \exp(\int_{0}^{\xi} v(\eta) d\eta).$$

This formula shows that $$\|g\|(1) = \|v\|_1$$, and so

$$\|(I - V)^{-1}\|(1) \geq \exp(\|v\|_1). \quad (I.11)$$

By combining this inequality with relation (I.8) and with conclusion (2.6), we obtain relation (I.9). This proves our second claim.

**Appendix II**

An Example Illustrating that Assumption (2.8) is Strictly
Weaker than Assumption (2.3)
Let the Volterra operator $V$ be given by

$$V f(\xi) = \int_0^\xi (\xi^2 - \eta^2)^{-1/2} f(\eta) d\eta \quad \text{for} \quad 0 < \xi \leq 1$$

and $V f(0) = \frac{\pi}{2} f(0)$. \hfill (II.2)

Then the definition (2.7) yields

$$\|V\|(L, 1) = \sup_{0 \leq \xi \leq 1} \int_0^\xi (\xi^2 - \eta^2)^{-1/2} d\eta.$$ \hfill (II.3)

Since,

$$\int_0^\xi (\xi^2 - \eta^2)^{-1/2} d\eta = \int_0^1 (1 - \zeta^2)^{-1/2} d\zeta = \frac{\pi}{2},$$ \hfill (II.4)

it follows from relation (II.3) that

$$\|V\|(L, 1) \leq \frac{\pi}{2}.$$ \hfill (II.5)

Hence this operator satisfies assumption (2.8).

Since, for each $\eta$ in $[0,1],$

$$\sup_{\xi > \eta} (\xi^2 - \eta^2)^{-1/2} = \infty,$$

the definition (2.3) yields

$$\|V\|(LEO, 1) = \infty.$$

That is to say, this operator does not satisfy assumption (2.3).

We observe that, for this operator, conclusion (2.5) of Theorem 2.1 does not hold either, since $\frac{\pi}{2}$ is an eigenvalue. To see this, note that formula (II.4) shows that the function of the definition (I.5) is a corresponding eigenfunction.

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