ERROR ANALYSIS OF A PENDULOUS INTEGRATING GYRO ACCELEROMETER

IMA Mathematical Modeling Workshop

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1. INTRODUCTION

Inertial navigation and guidance systems depend on the measurement of linear accelerations and subsequent integrations of these to obtain vehicle velocity and position. Linear acceleration is measured by precision accelerometers, which can be divided into two categories: Pendulous and non-pendulous accelerometers. Of the pendulous accelerometers, the pendulous integrating gyro accelerometer (PIGA) offers superior accuracy compared to floated and flexure-supported pendulums. This superiority is based on the measurement of the mechanical movement of the servo-driven turntable on which the accelerometer is mounted as opposed to the measurement of a weak current generated by the movement of a coil and an iron rotor mounted on the gimbal axis of floated and flexure-supported pendulums.

A PIGA consists of a pendulous gyro mounted on a gimbal, which in turn is mounted on a servo-driven turntable. A diagrammatic representation of the gyro, gimbal, and turntable assembly is depicted in Figure 1. The gimbal assembly, relative to the turntable on which it is mounted, has one degree of freedom in its motion, which is rotation about its horizontal axis. The gimbal assembly is essentially a pendulum since its center of mass does not lie on its axis of rotation. Thus an acceleration

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of the PIGA apparatus can result in a torque being applied to the gimbal assembly. The PIGA is designed to be operated in a mode where the center of mass of the gimbal assembly lies in the horizontal plane containing the axis of rotation of the gimbal assembly. Ideally, only acceleration of the PIGA in the vertical direction will result in a torque being applied about the gimbal axis.

In order to maintain the gimbal in its horizontal position, a counter-torque must be applied. This is accomplished by a control system which rotates the turntable on which the gimbal assembly is mounted. The gyroscope, mounted transversally on the gimbal, reacts to this rotation, applying a new torque about the gimbal's axis. Thus, in principle, a careful motion of the turntable can result in the gimbal apparatus staying horizontal even when the PIGA is subject to accelerating motion. Net rotations of the turntable are a result of a change in velocity in the apparatus's vertical direction. Thus the angle of rotation of the turntable is used to estimate the velocity of the apparatus. In order to detect three-dimensional motion, three such devices mounted transversally are required.

The sensors for the gimbal rotation and the turntable position are not perfect, however. Mathematical models can be used to predict performance and improve sensor design. To formulate a more tangible goal for our modeling efforts, an objective of determining the design precision and accuracy to produce a PIGA that yields an error of at most $10^{-5}G$ over an operating range of 0 to 10 G, where G is force of gravity on Earth.

This report is organised in the following manner. We first present a nominal model of a PIGA. In this model we assume perfect knowledge and exact sensing of the dynamics. The model is linear and consequently amenable to mathematical analysis. A set of nominal parameter values for which the behavior of the nominal model falls within various operational guidelines were computed. These guidelines include, for example, accuracy of the velocity estimation and maximum rotation rate of the turntable. The design (i.e., the choice of parameters and control) of the nominal system results from these considerations and an analysis of the nominal model. Next, we present an off-nominal model. This model takes into account nonlinearities which
FIGURE 1. Sketch of a PIGA.
arise from not maintaining the gimbal assembly in a horizontal position, uncertainty in the physical parameters of the system, and limits in sensor accuracy. Numerical simulations are presented which indicate that the off-nominal model also has behavior which lies within the operational guidelines. We conclude this report with a discussion of other behaviors in the system that were not modeled and indicate areas of analysis where future work may be fruitful. Some of the mathematical results used in the text are included in the appendices.

2. THE NOMINAL MODEL

In this section we present a nominal model for a PIGA. The components of the model are pictured diagrammatically in Figure 2. For this model we choose the simplest elements for each of the components. We shall now discuss each component of the model.

The gimbal dynamics are modeled by the linearized equations of motion of a pendulum. If \( \theta \) is the angle of the gimbal apparatus measured away from the horizontal (and positive when the center of mass of the gimbal is below the horizontal plane), then we write

\[
I_y \ddot{\theta} + C \dot{\theta} = Pa_\tau(t) - H\omega(t),
\]

where \( I_y, C, P \) and \( H \) are physical parameters of apparatus, \( a_\tau(t) \) is the acceleration in the vertical direction and \( \omega(t) \) is the angular velocity of the turntable. The physical parameter \( I_y \) is the moment of inertia of the gimbal apparatus about its axis of rotation, \( C \) is the damping coefficient, \( P \) is the pendulosity, and \( H \) is the angular momentum of the gyroscope. The term \( Pa_\tau(t) \) represents the torque applied to the gimbal by the acceleration of the apparatus in the vertical direction and is the zero-order approximation to \( P(a_\tau(t)\cos(\theta) + a_y(t)\sin(\theta)) \) for small \( \theta \), where \( a_y \) is acceleration of the apparatus in a direction transverse to both the gimbal's axis of rotation and the vertical. This transverse acceleration results in a torque about the gimbal's axis of rotation of \( Pa_y(t)\sin(\theta) \). The term \( H\omega(t) \) represents the torque due to the interaction of the rotation of the turntable and the gyroscope.
Figure 2. Diagram of the PIGA model.
Moving on to describe the second box in Figure 2, we assume for the nominal model that we have perfect sensing of the angle of the gimbal $\theta$ and its rate of change $\theta'$. Thus we assume that the exact values of these quantities are available to the controller of the turntable rotation. The controller is the third box and implements a proportional-integral-differential (PID) control scheme. The sole purpose of the control scheme is to attempt to keep the gimbal apparatus horizontal, i.e., $\theta \approx 0$.

The motivation for keeping $\theta = 0$ comes from the fact that if we succeed perfectly then the above differential equation becomes

$$0 = Pa_x(t) - H\omega(t) .$$

Thus the change in velocity is given by

$$v_x(t) - v_x(0) = \frac{H}{P} \int_0^t \omega(t)dt .$$

So if $\phi$ is the angle of the turntable at time $t$ then $\phi(t) = \phi(0) + \int_0^t \omega(t)dt$ and the change in velocity is given by

$$v_x(t) - v_x(0) = \frac{H}{P}(\phi(t) - \phi(0)) .$$

Now when $\theta$ is not identically zero, Equation (1)

$$\omega(t) = (Pa_x(t) - I_v\theta'' - C\theta')/H .$$

So

$$\phi(t) - \phi(0) = \frac{P}{H}(v_x(t) - v_0(t)) - \frac{I_v}{H}(\theta'(t) - \theta'(0)) - \frac{C}{H}(\theta(t) - \theta(0)) .$$

Since we estimate the change in velocity as $\Delta v_x$measured $= \frac{H}{P}\Delta \phi$, where $\Delta v = v(t) - v(0)$ and similarly, we have the error in estimated velocity at time $t$ given by

$$\Delta v_x$estimated $- \Delta v_x = -\frac{I_v}{P}\Delta \theta' - \frac{C}{P}\Delta \theta .$$

So here we explicitly see the dependence of the error in the calculated change in velocity as a function of how well we manage to keep the gimbal assembly horizontal.

Our PID controller is given by

$$\omega_{control}(t) = K_0\theta'(t) + K_1\theta(t) + K_2\int_0^t \theta(\tau)d\tau .$$
The term $K_1 \theta(t)$ is called the proportional control. The idea is that the greater the deflection of the gimbal angle $\theta$ the faster we need to spin the turntable to provide a counter-torque.

The problem with only using proportional control lies in the solution of the differential equation

$$I_y \theta'' + C \theta' = Pa_x - H\omega,$$
$$\omega = K_1 \theta.$$

Here the steady state solution to a constant acceleration input has $\theta_{ss}' = 0$, but $\theta_{ss} \neq 0$, which leads to measurement errors in velocity as shown in Equation 2.

The term $K_2 \int_0^t \theta(\tau)d\tau$ is the integral control. By adding this to the proportional control we get the following ODE

$$I_y y''' + Cy'' = Pa_x - H\omega,$$
$$\omega = K_1 y' + K_2 y,$$

where $y' = \theta$. The solution of this ODE has the property that its asymptotic solution for a constant acceleration has $\theta_{ss}' = \theta_{ss} = 0$.

The term $K_0 \theta'$ is the differential control term. In a realistic system, the damping must be quite small in order to make the gimbal sensitive to small accelerations. This causes the transient effects to die out slowly. The differential control term can be thought of as introducing additional damping which is necessary to force the transient behavior to decay more rapidly.

Returning now to our discussion of the boxes in Figure 2, we next assume in the nominal model that the motor drives the turntable at the exact rotation rate requested by the controller, thus forcing the gyroscope to apply the desired torque. Finally, we assume that we can detect the rotation of the turntable exactly and use the linear factor $H/P$ to convert the rotation to a change in velocity.
Putting this all together we get the nominal model equations:

\[
\ddot{\theta} + \frac{C + HK_0}{I_y} \dot{\theta} + \frac{HK_1}{I_y} \theta + \frac{HK_2}{I_y} y^+ = \frac{P}{I_y} a_x(t),
\]

\[
y' = \theta.
\]

If we write this as a third order equation in \( y \) the nominal model becomes

\[
y''' + \frac{C + HK_0}{I_y} y'' + \frac{HK_1}{I_y} y' + \frac{HK_2}{I_y} y = \frac{P}{I_y} a_x(t),
\]  

(3)

where \( \theta = y' \). Furthermore, we can write the expression for the error in acceleration as a function of time as

\[
a_x \text{ measured}(t) - a_x(t) = \frac{H}{P} \omega(t) - a_x(t) = \frac{HK_0}{P} \theta' + \frac{HK_1}{P} \theta + \frac{HK_2}{P} y - a_x(t)
\]

\[
= \frac{HK_0}{P} y'' + \frac{HK_1}{P} y' + \frac{HK_2}{P} y - a_x(t)
\]

\[
= -\frac{I_y}{F} y''' - \frac{C}{P} y''.
\]

This can be integrated to get the accumulated error in velocity.

We shall now discuss the properties of the solutions of the inhomogeneous linear ODE which models the nominal system. As seen from the general solutions presented in Appendix 2, the dynamics of the homogeneous solution are essentially determined by the eigenvalues of the linear differential operator representing the ODE. Assuming that the eigenvalues (which are functions of the parameters) are given by \( \alpha, \beta \pm i \gamma \), then we need \( \alpha, \beta < 0 \) in order for the homogeneous solution \( \theta_h = 0 \) to be stable. If so, then the homogeneous solution will decay exponentially at a rate of \( e^{-\min(|\alpha|,|\beta|)t} \). The particular solution which depends on \( a(t) \) will give the nontransient behavior. We shall consider the explicit solutions for three different inputs: a constant acceleration, a linear acceleration, and a periodic acceleration.
We consider first a constant acceleration $a_x(t) = a_0$, then

$$y_p(t) = \frac{Pa_0}{HK_2}$$

$$\theta(t) \rightarrow 0$$

$$\theta'(t) \rightarrow 0$$

$$a_x \text{ measured} - a_x \rightarrow 0.$$

For a linear acceleration $a_x(t) = a_1 t$,

$$y_p(t) = \frac{Pa_1 t}{HK_2} - \frac{PK_1 a_1}{K_2^2}$$

$$\theta(t) \rightarrow \frac{Pa_1}{HK_2}$$

$$\theta'(t) \rightarrow 0$$

$$a_x \text{ measured} - a_x \rightarrow 0$$

For a periodic acceleration $a_x(t) = e^{i\Omega t}$, then

$$y_p(t) = \frac{P I_y e^{i\Omega t}}{(HK_2 - (C + HK_0)\Omega^2 + i(HK_1\Omega - I_y\Omega^3))}$$

$$\theta(t) \rightarrow i\Omega P I_y e^{i\Omega t} / (HK_2 - (C + HK_0)\Omega^2 + i(HK_1\Omega - I_y\Omega^3))$$

$$\theta'(t) \rightarrow -\Omega^2 P I_y e^{i\Omega t} / (HK_2 - (C + HK_0)\Omega^2 + i(HK_1\Omega - I_y\Omega^3))$$

$$a_x \text{ measured} - a_x \rightarrow (I_y C\Omega^2 + iI_y^2\Omega^3 e^{i\Omega t}) / (HK_2 - (C + HK_0)\Omega^2 + i(HK_1\Omega - I_y\Omega^3)).$$

3. The off-nominal model

We now introduce the off-nominal model which takes into account the nonlinearity of the problem, the errors in sensing, and uncertainties in parameter values. To present this model we shall once again use the diagram in Figure 2 and simply replace the components we used for the nominal model with more complicated ones as necessary.

We begin by assuming there is some measurement error in our nominal parameter values:
<table>
<thead>
<tr>
<th>Quantity</th>
<th>Measured value</th>
<th>True value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gimbal assembly moment of inertia</td>
<td>$I_y$</td>
<td>$I_y(1 + \epsilon_1)$</td>
</tr>
<tr>
<td>Damping coefficient</td>
<td>$C$</td>
<td>$C(1 + \epsilon_2)$</td>
</tr>
<tr>
<td>Pendulosity</td>
<td>$P$</td>
<td>$P(1 + \epsilon_3)$</td>
</tr>
<tr>
<td>Gyro angular momentum</td>
<td>$H$</td>
<td>$H(1 + \epsilon_4)$</td>
</tr>
</tbody>
</table>

Our differential equations becomes:

$$I_y(1 + \epsilon_1)\theta'' + C(1 + \epsilon_2)\theta' = P(1 + \epsilon_3)(a_x(t)\cos \theta + a_y(t)\sin \theta) - H(1 + \epsilon_4)\omega(t) + \epsilon_5.$$

(4)

The parameters $\epsilon_1, \ldots, \epsilon_4$ are introduced to model the fact that the actual physical parameters of the system are only approximated by the estimated values $I_y, C, P, H$. The torque on the gimbal assembly due to acceleration is modeled by the nonlinear term $P(1 + \epsilon_3)(a_x(t)\cos \theta + a_y(t)\sin \theta)$ which takes into account a transverse acceleration $a_y$ and the fact that the gimbal is not always perfectly horizontal.

Sensors which measure the angle of the gimbal and its rate of change cannot be expected to behave perfectly. We model their errors with a scale error and a constant bias. Thus, the control will get inputs that approximate $\theta$ and $\theta'$ and are

$$\theta_{\text{measured}} = (1 + \epsilon_5)\theta + \epsilon_6,$$

$$d\theta_{\text{measured}} = (1 + \epsilon_7)\theta' + \epsilon_8.$$

The controller will be as before, but including the sensing errors, resulting in a desired turntable rotation of

$$\tilde{\omega}(t) = K_0 d\theta_{\text{measured}} + K_1 \theta_{\text{measured}} + K_2 y$$

with $y' = \theta_{\text{measured}}$. We shall still assume that the motor can exactly control the turntable, but we need to introduce an error term into the turntable rotation sensor so we now write

$$\phi_{\text{measured}} = \int \omega(\tau)d\tau + \epsilon_9.$$
Then the measured change in velocity is given by

\[ V_{\text{measured}} = \frac{H}{F} \phi_{\text{measured}}. \]

Putting this all together, we get

\[
\begin{align*}
C(1 + \epsilon_2) + HK_0(1 + \epsilon_4)(1 + \epsilon_7) & \quad y''' \\
\frac{1}{(1 + \epsilon_1)I_y} & \quad y'' \\
HK_1(1 + \epsilon_4)(1 + \epsilon_5) & \quad y' \\
\frac{HK_2(1 + \epsilon_4)(1 + \epsilon_5)}{I_y(1 + \epsilon_1)} & \quad y = \frac{P(a_x \cos \theta + a_y \sin \theta)(1 + \epsilon_5)(1 + \epsilon_3)}{(1 + \epsilon_1)} \\
& \quad - \epsilon_8 \frac{HK_0}{(1 + \epsilon_1)} \frac{1}{I_y}
\end{align*}
\]

where \( \theta \) is now written in terms of \( y \) as

\[ \theta(t) = \frac{y' - \epsilon_6}{1 + \epsilon_5}. \]

We present the results of simulation of this nonlinear model in a later section, but we can study a simplified linear version by once again replacing \( \cos \theta \) and \( \sin \theta \) with 1 and 0, respectively, in the above equations.

We now develop the formula for the error in acceleration as a function of time. In the physical apparatus we are of course getting only velocity readings. One can get the velocity error by integrating the acceleration error. This is accurate to within \( \frac{H}{F} \epsilon_9 \) which represents the error due to being unable to read the rotation of the turntable.
accurately. The equations describe the error in accelerations are then

\[
\begin{align*}
a_x \text{ measured}(t) - a(t) &= \frac{H}{P} \omega(t) - a(t) \\
&= \frac{H}{P} \left( K_0 \frac{1 + \epsilon_7}{1 + \epsilon_5} y'' + K_1 y' + K_2 y + k_0 \epsilon_8 \right) - a(t) \\
&= \frac{I_y}{P} \frac{1 + \epsilon_1}{(1 + \epsilon_3)(1 + \epsilon_5)} y'' \\
&\quad + \left( \frac{HK_0}{P} \frac{1 + \epsilon_7}{1 + \epsilon_5} (1 - \frac{1 + \epsilon_4}{1 + \epsilon_3}) - C \frac{1 + \epsilon_2}{P} \frac{1 + \epsilon_3}{(1 + \epsilon_3)(1 + \epsilon_5)} y'' \right) \\
&\quad + \frac{HK_1}{P} (1 - \frac{1 + \epsilon_4}{1 + \epsilon_3}) y' \\
&\quad + \frac{HK_2}{P} (1 - \frac{1 + \epsilon_4}{1 + \epsilon_3}) y \\
&\quad + \frac{HK_0}{P} \epsilon_8 (1 - \frac{1 + \epsilon_4}{(1 + \epsilon_3)(1 + \epsilon_5)}) 
\end{align*}
\]

As in the nominal model, we can now use the solutions presented in Appendix 2 to find the asymptotic (or full) solutions for various inputs \(a_x(t)\) for the simplified linear off-nominal model.

We consider only a constant plus linear acceleration \(a_x(t) = a_0 + a_1 t\). In this case

\[
\begin{align*}
y_p(t) &= \frac{Pa_0}{HK_2} \frac{1 + \epsilon_3}{1 + \epsilon_4} \left( t - \frac{K_1}{K_2} \right) \\
&\quad + \frac{Pa_0}{HK_2} \frac{1 + \epsilon_3}{1 + \epsilon_4} \frac{\epsilon_8}{K_0} \\
\theta(t) &\rightarrow \frac{Pa_1}{HK_2} \frac{1 + \epsilon_3}{(1 + \epsilon_4)(1 + \epsilon_5)} - \frac{\epsilon_6}{1 + \epsilon_5} \\
\theta'(t) &\rightarrow 0 \\
a_x \text{ measured} - a_x &\rightarrow \frac{HK_0}{P} \frac{\epsilon_3 - \epsilon_4 \epsilon_8}{(1 + \epsilon_3)(1 + \epsilon_5)} + (a_0 + a_1 t) \frac{\epsilon_3 - \epsilon_4}{1 + \epsilon_4}.
\end{align*}
\]
4. Selection of Parameters

Several system constraints were used in determining the parameter values. These constraints are summarized in Table 1 below

<table>
<thead>
<tr>
<th>Constraint</th>
<th>Symbol</th>
<th>Magnitude/units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Max. measurable acceleration</td>
<td>$a_{\text{max}}$</td>
<td>$10 \text{G} = 9800 \text{ cm/sec}^2$</td>
</tr>
<tr>
<td>Max. turntable speed at 10 G</td>
<td>$\omega_{\text{max}}$</td>
<td>$2 \text{ Hz} = 12.57 \text{ rad/sec}$</td>
</tr>
<tr>
<td>Gyro speed</td>
<td>$\Omega$</td>
<td>$100 \text{ Hz} = 628.3 \text{ rad/sec}$</td>
</tr>
<tr>
<td>Turntable resolution</td>
<td></td>
<td>$16384 \text{ positions} = 3.834 \cdot 10^{-4} \text{ rad/pos}$</td>
</tr>
<tr>
<td>Drive power limit</td>
<td></td>
<td>$100 \text{ W}$</td>
</tr>
<tr>
<td>Gimbal angle stability</td>
<td></td>
<td>$\pm 10^{-4} \text{ rad}$</td>
</tr>
</tbody>
</table>

The gimbal angle stability constraint was imposed to ensure that the cross-coupling term of $\sin \theta a_y = a_y \theta$ was small enough that a 0.1 G lateral acceleration $a_y$ would not contribute more than $10^{-5}$ G error in acceleration, which is the desired system accuracy.

In the nominal case $T_1$ is the torque applied to the gimbal due to the acceleration of the instrument in the vertical direction. Expanding Figure 2 this is represented in the diagram in Figure 3. $T_1$ is expressed at $\theta = 0^\circ$ gimbal angle as

$$T_1 = Pa$$

where $P$ is pendulosity and $a$ acceleration. As shown in Figure 1 the gyroscope is centered within the gimbal and thus contributes nothing to the pendulosity. If we view $M$ as a point mass, the pendulosity is given by

$$P = 4Mg \cdot cm$$
Figure 3: A block diagram of the PIGA gimbal inputs
The torque acting on the gimbal due to the rotation of the turntable is $T_3$. From Figure 3 we get

$$T_3 = H\omega$$

where $\omega$ is the rate of rotation of the turntable and $H$ is the angular momentum of the gyroscope about its axis of rotation.

We must first consider a bound on the ratio $\omega/P$. Acceleration is measured as $a = \frac{H}{P} \omega$. Since $\omega$ must be $\leq 12.57$ rad/sec but accelerations up to 9800 cm/sec$^2$ must be measured, we must have

$$\frac{P}{H} 9800 \text{cm/sec}^2 = \omega_{\text{max}} \leq 12.57$$

Thus

$$\frac{H}{P} \geq 780 \text{cm/sec}$$

It was assumed that the gyro consists of a stainless steel cylinder with a radius of 2 cm and a height of 4 cm. The density of stainless steel is 8.02 g/cm$^3$ (CRC Handbook of Chemistry and Physics). Angular momentum is the moment of inertia in the direction of gyro rotation times the angular velocity of the gyro. The moment of inertia of a body about an axis is the integral of the density times the square distance to the axis over the body. The moment of inertia is thus

$$8.02 \int_{\theta=0}^{2\pi} \int_{h=0}^{4} \int_{r=0}^{2} r^2 rdrdh\,d\theta$$

$$= 2\pi \cdot 4 \cdot \frac{2^4}{4} \cdot 8.02$$

$$= 806.2 \text{cm}^2 \cdot g$$

Thus, with a gyro speed of 628.3 rad/sec this gives us $H = 506600 \text{ g cm}^2 \text{ rad/sec}$

Now we want
780 = \frac{H}{P} = \frac{506600}{4M}

Thus, the pendulous mass $M$ must be $162.4$ g to generate the correct pendulosity of $P = 649.5$ g•cm.

The inertia of the gimbal $I_y$ is the sum of the inertia of the gyro about the gimbal axis $I_{gyro}$ and the inertia due to the pendulous mass $M$.

\[ I_{gyro} = 8.02 \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (x^2 + z^2) dz dy dx = 940.6 \text{ g} \cdot \text{cm}^2 \]

The inertia due to $M$ is $I_M = 4^2 M = 2598 \text{ g} \cdot \text{cm}^2$. Thus $I_y = I_{gyro} + I_M = 3539 \text{ g} \cdot \text{cm}^2$.

The damping coefficient $C$ is defined as proportionally constant between the angular velocity of the gimbal and and the torque acting on the gimbal axis due to damping friction. Thus

\[ T_c = C \dot{\theta} \]

and has units g • cm² • rad/sec.

To calculate the value of the damping coefficient $C$, it is assumed that the gimbal will lose about half of its angular velocity per minute when allowed to spin freely, i.e., with no torques than those generated by damping and inertia. The differential equation for this motion is

\[ I_y \ddot{\theta} + C \dot{\theta} = 0 \]

which has a solution

\[ \dot{\theta} = Me^{\frac{-C}{I_y}} \]

We want

\[ \frac{\dot{\theta}(60)}{\dot{\theta}(0)} = \frac{1}{2} \]
thus
\[
e^{\frac{C}{I_r}} = \frac{1}{2}
\]
so we must choose
\[
C = \frac{I_y \ln(2)}{60} \text{ g} \cdot \text{cm}^2 / \text{sec}
\]
Using the value of \( I_y \) obtained above we get \( C = 40.88 \text{ g} \cdot \text{cm}^2 / \text{sec} \). Assuming that the turntable inertia is twice the gimbal inertia we get \( I_{\text{turnable}} = 7078 \text{ g} \cdot \text{cm}^2 \).

The controller rotating the turntable to counter the torque on the gimbal assembly can be depicted, expanding Figure 1, as a block diagram in Figure 4 and expressed as
\[
I_y \ddot{\theta} + C \dot{\theta} = Pa(t) - \dot{H}(K_0 \ddot{\theta} + K_1 \dot{\theta} + K_2 \int \theta)
\]
where \( Pa(t) = T_i \), i.e., torque on the gimbal due to acceleration of the instrument and \( H(K_0 \ddot{\theta} + K_1 \dot{\theta} + K_2 \int \theta) = T_3 \), i.e., torque on the gimbal. Given the parameters \( C, I_y, H, \) and \( P \), eigenvalues of the linear system are chosen: \( a, b \pm i \gamma \).

The characteristic polynomial of our nominal ODE is obtained from Equation 3. Since this must also equal
\[
(y - \alpha)(y - \beta + i \gamma)(y - \beta - i \gamma)
\]
By equating coefficients we get the following values for \( K_1, K_2, K_3 \)
\[
K_0 = \frac{-(\alpha + 2\beta)I_y + C}{H}
\]
\[
K_1 = \frac{I_y}{H}(\beta^2 + \gamma^2 + 2\alpha\beta)
\]
\[
K_2 = -\alpha \frac{I_y}{H}(\beta^2 + \gamma^2)
\]
Figure 4: A block diagram of the PIGA turntable controller.
We can now do a simulation and solve explicitly for the solutions of the linear ODE. However, we can explicitly solve only for particular choices of $a(t)$, e.g., constant, periodic, etc.

To ensure $|\theta| \leq 10^{-4}$ radians, the parameters $K_0$, $K_1$, and $K_2$ were calculated using the formula

$$\|\theta\|_\infty \leq \frac{P}{I_y} \cdot \|\dot{\theta}\|_\infty$$

where $\theta(t)$ solves

$$I_y \ddot{\theta} + (HK_0 - C) \dot{\theta} + (HK_1) \dot{\theta} + (HK_2) \theta = P \dot{a}(t)$$

with $\theta(0) = \dot{\theta}(0) = \ddot{\theta}(0) = 0$ and $\alpha, \beta + i\gamma, \beta - i\gamma$ are the eigenvalues of the above equation. If we assume that $\alpha = \beta, \gamma = 0$, then

$$\|\theta\|_\infty \leq \frac{P}{I_y} \cdot \|\dot{\theta}\|_\infty = \frac{P}{HK_2} \|\dot{a}\|_\infty$$

because

$$-\alpha^3 = \frac{HK_2}{I_y}$$

Using our values for $P$, $H$, and $\|\dot{a}\|_\infty$ we get $\|\theta\|_\infty \leq 10^{-4}$ radians if $K_2 \geq 838 \text{ 1/sec}^2$.

Using the formulas

$$3\alpha = \left(\frac{HK_0 - C}{I_y}\right)$$

$$3\alpha^2 = \frac{HK_1}{I_y}$$

$$\alpha^3 = \frac{-HK_2}{I_y}$$
Then we get \( a = -49.3 \) rad/sec. If we use \( a = -63 \) rad/sec instead, then \( K_p = 1.32 \), \( K_i = 83 \) rad/sec, and \( K_2 = 1747 \) rad/sec\(^2\). The reason for choosing \( a = -63 \) rad/sec rather than \( a = -50 \) rad/sec is due to possible errors in \( H \), \( C \), \( I_y \) and measurement errors in \( \theta \) or \( \dot{\theta} \). Any small perturbations in these values will move the eigenvalues slightly away from -63 in the complex plane. The choice of \( a = -63 \) ensures that with any reasonable errors in the above values we will still have \( a, b < -50 \) and thus \( \|\theta\|_\infty \leq 10^{-4} \) radians. Our system parameters and their values are summarized in Table 2 below:

**Table 2: System parameters and their respective values**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pendulosity</td>
<td>( P )</td>
<td>649.4 g ( \cdot ) cm</td>
</tr>
<tr>
<td>Gyro angular momentum</td>
<td>( H )</td>
<td>506600 g ( \cdot ) cm(^2)/sec</td>
</tr>
<tr>
<td>Gimbal inertia</td>
<td>( T_y )</td>
<td>3539 g ( \cdot ) cm(^2)</td>
</tr>
<tr>
<td>Damping coefficient</td>
<td>( C )</td>
<td>40.88 g ( \cdot ) cm(^2)/sec</td>
</tr>
<tr>
<td>Turntable inertia</td>
<td>( I_n )</td>
<td>7078 g ( \cdot ) cm(^2)</td>
</tr>
<tr>
<td>Differential feedback</td>
<td>( K_o )</td>
<td>1.32</td>
</tr>
<tr>
<td>Proportional feedback</td>
<td>( K_i )</td>
<td>83 rad/sec</td>
</tr>
<tr>
<td>Integral feedback</td>
<td>( K_2 )</td>
<td>1747 rad/sec</td>
</tr>
</tbody>
</table>

Lastly, we wish to use the above values to obtain a set of bounds for the error terms in the non-nominal model. Our benchmark test for determining the tolerances on the \( \varepsilon \)'s was a ramped input acceleration going from 1 G to 5 Gs over 60 seconds. This benchmark simulates the acceleration that would be experienced in a launch vehicle as it exhausts its fuel supply and becomes lighter. We must ensure that the \( \varepsilon \)'s are small enough to meet the system design criteria \( \|\theta\|_\infty \leq 10^{-4} \).
radians and \( \varepsilon_{error} \leq 10^{-5} \text{ Gs} = 9.8 \cdot 10^{-5} \text{ Gs cm/sec}^2 \). Using the values from Table ( ) in the steady state solutions for \( \theta \) and \( a_{error} \) given in Equation 5 we obtain

\[
\theta = 4.795 \cdot 10^{-5} \frac{1 + \varepsilon_3}{(1 + \varepsilon_4)(1 + \varepsilon_5)} - \frac{\varepsilon_6}{(1 + \varepsilon_5)}
\]

\[
a_{error} = \frac{1030(\varepsilon_3 + \varepsilon_4)\varepsilon_8}{(1 + \varepsilon_3)(1 + \varepsilon_5)} + \frac{(\varepsilon_3 + \varepsilon_4)(a_0 + a_1 t)}{(1 + \varepsilon_4)}
\]

At \( t = 60 \) seconds our expression for \( a_{error} \) is

\[
a_{error} = \frac{(\varepsilon_3 + \varepsilon_4)4900}{(1 + \varepsilon_4)} + \frac{1030(\varepsilon_3 + \varepsilon_4)\varepsilon_8}{(1 + \varepsilon_3)(1 + \varepsilon_5)}
\]

We initially make the simplifying assumption that \( \varepsilon_3, \varepsilon_4, \) and \( \varepsilon_5 \) are small enough that the terms \( 1 + \varepsilon_3, 1 + \varepsilon_4, \) and \( 1 + \varepsilon_5 \) can essentially be replace by 1, so \( \theta = 4.795 \cdot 10^{-5} \cdot \varepsilon_6 \) and \( a_{error} = 1030(\varepsilon_3 - \varepsilon_4)(\varepsilon_3 \varepsilon_8 + 4900(\varepsilon_3 - \varepsilon_4)). \) To keep \( |\theta| < 10^4 \) we must ensure that \( |\varepsilon_6| \leq 5 \cdot 10^{-5} \text{ radians}. \) Now, if \( \varepsilon_3, \varepsilon_8 \) are small, then \( a_{error} \equiv 4900(\varepsilon_3 - \varepsilon_4) \) so we need \( (\varepsilon_3 - \varepsilon_4) \leq 9.8 \cdot 10^{-3} / 4900 \) or \( (\varepsilon_3 - \varepsilon_4) \leq 2 \cdot 10^{-6}. \) Thus, we need, say \( |\varepsilon_3|, |\varepsilon_4| < 10^{-6}. \) If we ensure that \( |\varepsilon_1|, |\varepsilon_5| < 10^{-3} \) then our assumptions above will be valid.

Note that the error terms \( \varepsilon_1, \varepsilon_2, \) and \( \varepsilon_3 \) do not appear explicitly in in the equations for \( \theta \) or \( a_{error}. \) The non-nominal equations for the control constants for given eigenvalues \( \alpha = \beta + \gamma t = \beta - \gamma t = \lambda, \) \( K_0, K_1, \) and \( K_2, \) are given by
\[ K_0 = \frac{-3\lambda y(1 + \varepsilon_1) - C(1 + \varepsilon_2)}{H(1 + \varepsilon_4)} = 1.32 \left(\frac{1 + \varepsilon_1}{1 + \varepsilon_4}\right) - 8.07 \cdot 10^{-3} \left(\frac{1 + \varepsilon_2}{1 + \varepsilon_4}\right) \]

\[ K_1 = \frac{I_y(1 + \varepsilon_1)}{H(1 + \varepsilon_4)} (3\lambda^2) = 83.16 \left(\frac{1 + \varepsilon_2}{1 + \varepsilon_4}\right) \]

\[ K_2 = -\frac{I_y(1 + \varepsilon_1)}{H(1 + \varepsilon_4)} (\lambda^3) = 1714 \left(\frac{1 + \varepsilon_2}{1 + \varepsilon_4}\right) \]

Since these depend on \( \varepsilon_1 \) and \( \varepsilon_2 \), these errors will affect the true locations of the eigenvalues of our system. Note, however, that only large error values for the \( \varepsilon \)'s will affect the values of \( K_0, K_1, \), or \( K_2 \) significantly. If we ensure that \( |\varepsilon_1| < 10^{-3}, |\varepsilon_2| < 10^{-1} \) then our true eigenvalues will be very close to the chosen values of \( \lambda = 63 \).

The error bound on \( \varepsilon_2 \) may seem large, but it should be noted that it is difficult to obtain a true value for a damping coefficient and hence a large error tolerance for \( C \) is desirable. The error bound for \( \varepsilon_1 \) is not critical and we may impose a constraint \( |\varepsilon_1| < 10^{-2} \). The bounds on our non-nominal error terms which allow us to achieve the design goal of an accelerometer accurate to \( 10^{-5} \) Gs are summarized in Table 3 below.
TABLE 3: Non-nominal error term bounds

<table>
<thead>
<tr>
<th>Error term</th>
<th>Bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_y$ measurement error</td>
<td>$</td>
</tr>
<tr>
<td>$C$ measurement error</td>
<td>$</td>
</tr>
<tr>
<td>$P$ measurement error</td>
<td>$</td>
</tr>
<tr>
<td>$H$ measurement error</td>
<td>$</td>
</tr>
<tr>
<td>$\theta$ measurement scale error</td>
<td>$</td>
</tr>
<tr>
<td>$\theta$ measurement bias error</td>
<td>$</td>
</tr>
<tr>
<td>$\dot{\theta}$ measurement scale error</td>
<td>$</td>
</tr>
<tr>
<td>$\dot{\theta}$ measurement bias error</td>
<td>$</td>
</tr>
</tbody>
</table>
5. SIMULATION

We used MATLAB’s ODE45 for our simulations. Our initial conditions were

\[
\begin{bmatrix}
\theta \\
\dot{\theta} \\
y \\
\Delta v
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

where \( \dot{y} = \theta \). The system was solved for a time interval of 60 seconds. Beginning with the nominal model we tested our program with three types of input acceleration: constant, linear, and periodic. For each type of acceleration we looked at the results in four graphs. The top two graphs show the input as acceleration and velocity. The lower left graph is the estimate error in velocity. The lower right graph shows the phase space of \( \theta \). Our simulations included proportional, integral and differential control. For the off-nominal model we looked at maximum errors we could tolerate in the parameters and still meet specification requirements in \( \Delta v \) and \( \theta \).

5.1. Nominal Model. For the simulation with constant acceleration estimate error in velocity is essentially zero. This agrees with the previous analytical calculations with the exception of an initial transient phase lasting less than 1.2 seconds. The length of the transient is controlled by choice of \( k_0, k_1, k_2 \) and is not considered a source of error.

The linear or “ramp” acceleration consists of two parts. First within the interval \( t = [0, 1] \) seconds the acceleration is constant at \( 1g \). The analogy is a launch vehicle sitting at a platform on earth ready for takeoff. After 1 second the system encounters an linear acceleration lasting the next 59 seconds with max acceleration of \( 4g \) at \( t = 60 \). The equation for the acceleration force for \( t = [1, 60] \) is \( a_i = 980 + (t - 1) \times (4 \times 980)/60 \). For this type of input the estimate in velocity error incurs a constant bias error. This error occurs at the beginning of the ramp, \( t = 1 \), and remains constant thereafter.

For a periodic acceleration we used \( \sin \pi t \). Here the error estimate in velocity is not constant. The resulting error is best seen in the phase space as the error in \( \theta \) and \( d\theta \) centered about the origin. The error for periodic acceleration of fixed amplitude
Figure 3. Constant input acceleration
FIGURE 4. Ramp input acceleration
is bounded. In the first two accelerations only the transient incurs large change in $\theta$ whereas for periodic acceleration this change in $\theta$ does not go to zero.

5.2. Off-nominal Model. After verifying the nominal simulation with the analytical results we added error terms to the parameters to create a off-nominal model. We noticed that $\pm \epsilon$ had to be considered in order to avoid cancellation of error. We looked for the $\epsilon$ giving max error allowed to stay within specifications.

\[
\begin{align*}
I_y & \quad \epsilon_1 \\
C & \quad \epsilon_2 \\
H & \quad \epsilon_3 \\
P & \quad \epsilon_4 \\
\theta_{\text{scale}} & \quad \epsilon_5 \\
\theta_{\text{bias}} & \quad \epsilon_6 \\
d\theta_{\text{scale}} & \quad \epsilon_7 \\
d\theta_{\text{bias}} & \quad \epsilon_8 
\end{align*}
\]

These parameters are described in a previous section. For these tests we used a ramp acceleration. The simulation reveals that the most sensitive parameters are $H, P$ and $\theta_{\text{bias}}$. We looked at $|\epsilon_i| < .001$. Our criteria for $H, P$ was $\Delta v < .294$. In order to achieve this accuracy in $\Delta v$ we require $\epsilon_3 < 8.5e - 7$ and $\epsilon_4 < 8.5e - 7$. In order to avoid cancellation in the magnitudes of $\epsilon_3$ and $\epsilon_4$ opposite signs were used. The maximum $\Delta v$ incurred by these parameter variations was 0.2933.

For the error in $\theta_{\text{scale}}$ we used the criteria $\theta_{\text{drift}} < .0001$. This error is the distance from the origin in phase space. To obtain this tolerance $\epsilon_6 < 5e - 5$.

Changes in the other parameter magnitudes $|\epsilon_i| < .001$ showed no increase in the error criteria we observed.
Figure 5. Periodic input acceleration
6. CONCLUSIONS

We began modeling our accelerometer with a description of a PIGA system as well as performance criteria we wished to meet. We put realistic operating constraints on the system for its operation over a range of accelerations bounded by 10G with maximum jerk 4G per minute. These constraints included bounds on how far the gimbal rotated from the horizontal, accuracy and resolution of the gimbal rotation, size and spin rate of the gyroscope, resolution of the turntable rotation pickoff, accelerometer sensitivity, maximum turntable rotation rate and power constraints.

We studied the PIGA design until we had a firm understanding of the operational principles of the system. We then developed a nominal mathematical model for the system. We extended this model to include off-nominal effects to account for manufacturing tolerances and gimbal sensor errors. We then selected a reasonable set of nominal parameters for our system. This selection was done to insure consistency with the performance criteria. We analyzed the effects of introducing the off-nominal parameters, both by simulation and analytically. At this point we determined our control gains and were able to establish a set of manufacturing tolerances and gimbal sensing tolerances which allowed us to meet our performance goals.

We conclude this it is possible to design a PIGA which will meet our performance criteria, provided we can achieve certain manufacturing and sensing tolerances.

The PIGA accelerometer is more accurate than the simpler designs that use an electromagnetic device to provide the control torque on the gimbal. This is because we can achieve much finer control of the gimbal torque indirectly by controlling the speed of the turntable than can be achieved using a circuit.

In this analysis we did not consider a number of off-nominal effects which would tend to degrade performance. Effects such as inaccuracies due to thermal expansion, turntable dynamics (delays, lags, etc...) as well as coning effects from sinusoidal transverse accelerations were not considered. We did, however, consider all the dominant effects which are considered by engineers to be the major sources of error in this system.
7. Bibliography

APPENDIX A: BOUNDS FOR $\theta$

Differentiating $I_y \ddot{\theta} + C \dot{\theta} = Pa(t) - (K_0 \dot{\theta} + K_1 \theta + K_2 \int \theta)$ we get

$$I_y \ddot{\theta} + (C + K_0) \dot{\theta} + K_1 \dot{\theta} + K_2 \theta = Pa(t)$$  \hspace{1cm} (1)

Let $\theta(t)$ be the solution to (1) with $\theta(0) = \dot{\theta}(0) = \ddot{\theta}(0) = 0$. Then if $|\dot{\theta}(t)| \leq M$ for all $t > 0$, we have

$$|\theta(t)| \leq \frac{P}{|\text{Re} \lambda_1||\text{Re} \lambda_2||\text{Re} \lambda_3|} M$$  \hspace{1cm} (2)

for all $t > 0$, where $\lambda_1, \lambda_2, \lambda_3$ are eigenvalues of the equation (1) and assuming $\text{Re} \lambda_1, \text{Re} \lambda_2, \text{Re} \lambda_3 < 0$. Dividing (1) by $I_y$ yields

$$\ddot{\theta} + \frac{(C + K_0)}{I_y} \dot{\theta} + \frac{K_1}{I_y} \dot{\theta} + \frac{K_2}{I_y} \theta = \frac{P}{I_y} \dot{\theta}(t)$$  \hspace{1cm} (3)

which is the same as

$$\left( \frac{d}{dt} - \lambda_1 \right) \left( \frac{d}{dt} - \lambda_2 \right) \left( \frac{d}{dt} - \lambda_3 \right) \theta(t) = \frac{P}{I_y} \dot{\theta}(t)$$  \hspace{1cm} (4)

we solve

$$\left( \frac{d}{dt} - \lambda_1 \right) \Phi_1(t) = \frac{P}{I_y} \dot{\theta}(t)$$  \hspace{1cm} (5)

$$\left( \frac{d}{dt} - \lambda_2 \right) \Phi_2(t) = \Phi_1(t)$$  \hspace{1cm} (6)

$$\left( \frac{d}{dt} - \lambda_3 \right) \theta(t) = \Phi_2(t)$$  \hspace{1cm} (7)
The solution to (5) is

\[
\Phi_1(t) = e^{\lambda_1 t} \int_0^t e^{-\lambda_1 s} \frac{P}{I_y} \dot{\alpha}(s) \, ds
\]

(6)

\[
|\Phi_1(t)| = e^{(\text{Re} \lambda_1) t} \int_0^t e^{-(\text{Re} \lambda_1) s} \frac{P}{I_y} |\dot{\alpha}(s)| \, ds \leq e^{(\text{Re} \lambda_1) t} \int_0^t e^{-(\text{Re} \lambda_1) s} \frac{P}{I_y} M ds
\]

(7)

\[
\int_0^t e^{-(\text{Re} \lambda_1) s} \frac{P}{I_y} M ds = e^{(\text{Re} \lambda_1) t} \left( \frac{P}{I_y} M \right) \frac{1}{-\text{Re} \lambda_1} e^{-(\text{Re} \lambda_1) t} \bigg|_0^t
\]

\[
= \left( \frac{P}{I_y} M \right) \frac{1}{-\text{Re} \lambda_1} (1 - e^{-(\text{Re} \lambda_1) t})
\]

Since \( \text{Re} \lambda_1 < 0 \) and \( t > 0 \), \( 1 - e^{-(\text{Re} \lambda_1) t} < 1 \). So,

\[
|\Phi_1(t)| \leq \left( \frac{P}{I_y} M \right) \frac{1}{\text{Re} \lambda_1} = \frac{P}{I_y} \frac{M}{|\text{Re} \lambda_1|}
\]

(8)

We solve (6) and (7) in a similar manner:

\[
\Phi_2(t) = e^{\lambda_2 t} \int_0^t e^{-\lambda_2 s} \Phi_1(s) \, ds
\]

\[
\theta(t) = e^{\lambda_3 t} \int_0^t e^{-\lambda_3 s} \Phi_2(s) \, ds
\]

Using the same kind of estimate we get
\begin{align}
|\Phi_2(t)| & \leq \frac{1}{\text{Re} \lambda_2} \left( \frac{P}{I_y} \frac{M}{|\text{Re} \lambda_1|} \right) = \frac{P}{I_y} \frac{M}{|\text{Re} \lambda_1||\text{Re} \lambda_2|} \quad (9) \\
|\theta(t)| & \leq \frac{i}{\text{Re} \lambda_3} \left( \frac{P}{I_y} \frac{M}{|\text{Re} \lambda_1||\text{Re} \lambda_2|} \right) = \frac{P}{I_y} \frac{M}{|\text{Re} \lambda_1||\text{Re} \lambda_2||\text{Re} \lambda_3|} \quad (10)
\end{align}

Using these solution formulas ensures \( \theta(0) = \dot{\theta}(0) = \ddot{\theta}(0) = 0 \).
APPENDIX B. THE SOLUTION OF THE THIRD ORDER ODE

Consider the third order linear nonhomogeneous ODE,

\[ y''' + ay'' + by' + cy = f(t). \]

It’s characteristic polynomial is \( \lambda^3 + a\lambda^2 + b\lambda + c = 0 \). If we assume the roots are given by \( \alpha, \beta \pm i\gamma \), then the homogeneous solution of the differential equation is given by

\[ y_h = Ae^{\alpha t} + Be^{\beta t} \cos \gamma t + Ce^{\beta t} \sin \gamma t. \]

We compute

\[
\begin{align*}
y_h' &= \alpha Ae^{\alpha t} + (B\beta + C\gamma)e^{\beta t} \cos \gamma t + (C\beta - B\gamma)e^{\beta t} \sin \gamma t \\
y_h'' &= \alpha^2 Ae^{\alpha t}(B\beta^2 + 2C\gamma\beta - B\gamma^2)e^{\beta t} \cos \gamma t + (C\beta^2 - 2B\gamma\beta - C\gamma^2)e^{\beta t} \sin \gamma t.
\end{align*}
\]

The initial conditions are given by,

\[
\begin{pmatrix}
y_h(0) \\
y'_h(0) \\
y''_h(0)
\end{pmatrix} = \begin{pmatrix}
1 & 1 & 0 \\
\alpha & \beta & \gamma \\
\alpha^2 & \beta^2 - \gamma^2 & 2\beta \gamma
\end{pmatrix} \begin{pmatrix}
A \\
B \\
C
\end{pmatrix}.
\]

Let \( d = \gamma(\alpha^2 - 2\alpha \beta + \beta^2 + \gamma^2) \), then we write the arbitrary parameters \( A, B, C \) as functions of the initial conditions by

\[
\begin{pmatrix}
A \\
B \\
C
\end{pmatrix} = \frac{1}{d} \begin{pmatrix}
\gamma(\beta^2 + \gamma^2) & -2\gamma \beta & \gamma \\
\alpha \gamma(\alpha - 2\beta) & 2\gamma \beta & -\gamma \\
\alpha(\beta^2 - \gamma^2 - \beta \alpha) & \alpha^2 - \beta^2 + \gamma^2 & \beta - \alpha
\end{pmatrix} \begin{pmatrix}
y_h(0) \\
y'_h(0) \\
y''_h(0)
\end{pmatrix}.
\]

We now present particular solutions for various forcing functions:

- \( f(t) = f_0 \) \( y_p = \frac{f_0}{c} \),
- \( f(t) = f_1 t \) \( y_p = \frac{f_1}{c} \left( t - \frac{1}{2} \right) \),
- \( f(t) = e^{i\omega t} \) \( y_p = \frac{1}{-i\omega^3 - a\omega^2 + b\omega + c} e^{i\omega t} \),
APPENDIX C. MATLAB CODE

The code we used for the off-nominal model includes $\epsilon$ parameter variations and nonlinearity terms. The model agrees with the nominal simulation when nonlinearity is zeroed out and all $\epsilon_i = 0$. We include the wrapper script run.m which defines all the global variables used by the integrator.

C.1. run.m.

% code to do analysis of PIGA simulation

global Iy C P H K0 K1 K2 eps accel_function alpha beta gamma nlin_on;

% set up the nominal parameters

Iy = 3539;
Iturn = 2*Iy;
C = 40.88;
P = 649.5;
H = 506600;
K0 = -((alpha+2*beta)*Iy+C)/H;
K1 = Iy*(beta*gamma+2*alpha*beta)/H;
K2 = -alpha*Iy*(beta*beta+gamma*gamma)/H;

% set the following to 0 to forget nonlinearity
nlin_on = 0.0;

% set up the error terms

% eps_mag = 1e-2;
eps_mag = 1.0;
eps = eps_mag * zeros(1,9);
eps(1) = 1;

% eigenvalues of linear nominal system
scale = 1.0;
alpha = -63;
beta = alpha;
gamma = 0.0;

% choose an acceleration function
%accel_function = 'accel_constant';
accel_function = 'accel_ramp';
%accel_function = 'accel_periodic';

% choose a model
model_function = 'nlin';

% choose initial conditions and integrate
Y0 = zeros(4,1);
t0 = 0;
tf = 60; % run for tf seconds
[t,Y] = ode45(model_function, t0, tf, Y0);

% now plot some output
%

n=3;
m=2;

% plot theta vs theta_dot
subplot(n,m,1);plot(Y(:,1),Y(:,2),'r'); title('Phase space'); xlabel('theta');ylabel('d(theta)/dt');

% plot time series for error in velocity
subplot(n,m,2);plot(t,Y(:,4),'g.'); title('Error in estimated velocity'); xlabel('time');ylabel('velocity error');

% plot time series for turntable rotation rate
dtheta_meas = (1+eps(7))*Y(:,2) + eps(8)*ones(size(Y(:,1)));
theta_meas = (1+eps(5))*Y(:,1) + eps(6)*ones(size(Y(:,1)));
omega = -K0*dtheta_meas-K1*theta_meas-K2*Y(:,3);
subplot(n,m,3);plot(t,omega);title('Turntable'); ylabel('rotation rate'); xlabel('time');

yyy = (P.*feval.accel_function,t)+H.*omega - C.*Y(:,2)/Iy;

omega_dot = -K0*(1+eps(7))*yyy-K1*(1+eps(5))*Y(:,2)-K2*Y(:,1);
subplot(n,m,4);
plot(t,omega_dot);

subplot(n,m,5);
plot(t,Iturn.*omega.*omega_dot);

K=[K0,K1,K2],
Y(length(Y),1),

C.2. nlin.m.

function [Yprime] = nlin(t,Y)
%
% nlin is the right hand side of a system
% of nonlinear ODEs modelling a PIGA.
%

% the parameters for this model are global variables
global Iy C P H K0 K1 K2 eps accel_function alpha beta gamma nlin_on;

% construct other parameters
Iy_true = Iy * (1+eps(1));
C_true = C * (1+eps(2));
P_true = P * (1+eps(3));
H_true = H * (1+eps(4));

% the input Y is [theta u y dv]
theta = Y(1);
u = Y(2);  % u = theta_dot
y = Y(3);
dv = Y(4);

% calculate some intermediate quantities
a_y = 0.0;  % skip these effects for now
a_x = feval(accel_function, t);

theta_meas = (1+eps(5)) * theta + eps(6);
dtheta_meas = (1+eps(7)) * u + eps(8);
omega = - K0 * dtheta_meas - K1 * theta_meas - K2 * y;

torque_acceleration = P_true / Iy_true * ...
(a_x * cos(nlin_on*theta) + a_y * sin(nlin_on*theta));
torque_turntable = H_true / Iy_true * omega;
% now work out the right hand side
%

% The dynamics of the gimbal:
theta_dot = u;
u_dot = - C_true / Iy_true * theta_dot ...
+ torque_turntable + torque_acceleration;

% the dynamics of the control
y_dot = theta_meas;

% the dynamics of the error in velocity
dv_dot = - H / P * omega - a_x;

Yprime = [theta_dot; u_dot; y_dot; dv_dot]';

C.3. accel-constant.m.
function [a] = accel_constant(t)
%
% constant acceleration
% g = 9.8 m/s
%
a=9.8*100*ones(size(t)); % cm/sec

C.4. accel-ramp.m.
function [a] = accel_ramp(t)
%
% acceleration ramp of 4g/minute
% g = 9.8 m/s
% 4g = 39.2 m/s
for i=1:length(t),
    if (t(i)<1)
        a = 980;
    else
        a = 980 + (t(i)-1)*(4*980)/60;
    end
end

C.5. accel-periodic.m.

function [a] = accel_periodic(t)
%
%
% periodic acceleration
a = sin(pi*t);