Lorenz Equations Part II:
“Randomly” Rotated Homoclinic Orbits and Chaotic Trajectories

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Abstract. The Lorenz equations are a system of ordinary differential equations

\[ x' = s(y - x), \quad y' = Rx - y - xz, \quad z' = xy - qz \]

where \( s, \ R, \) and \( q \) are positive parameters. We show that for each non-negative integer \( N, \) there are positive parameters \( s, q, \) and \( R \) such that the Lorenz system has homoclinic orbits associated with the origin (i.e., orbits that tend to the origin as \( t \to \pm \infty \)) which can rotate around the \( z \)-axis \( N/2 \) times; namely, the \( x \)-component changes sign exactly \( N \) times, the \( y \)-component changes sign exactly \( N + 1 \) times, and the zeros of \( x \) and \( y \) are simple and interlace. Also, we show the existence of “randomly” rotated homoclinic orbits (i.e., homoclinic orbits that rotate around the \( z \)-axis for arbitrary prescribed number of times, and during each time interval in which trajectories do not cross the \( \{x = 0\} \) plane, they rotate around the vertical line \( x = y = \sqrt{q(R-1)} \) or \( x = y = -\sqrt{q(R-1)} \) certain number of times determined by arbitrary prescribed finite integer sequences). Furthermore, we establish the existence of chaotic trajectories (i.e., trajectories that rotate around the \( z \)-axis infinitely many times and during each time interval in which the trajectories do not cross the \( \{x = 0\} \) plane, the trajectories rotate around the vertical line \( x = y = \sqrt{q(R-1)} \) or \( x = y = -\sqrt{q(R-1)} \) certain number of times determined by arbitrary prescribed bounded integer sequences).

Keywords. Lorenz equations, homoclinic orbits, chaotic behavior.


1. Introduction.

The Lorenz equations we studied here are a system of ordinary differential equations

\[ \begin{align*}
    x' &= s(y - x), \\
    y' &= Rx - y - xz, \\
    z' &= xy - qz
\end{align*} \tag{1.1} \]

where \( \frac{d}{dt} \) and \( s, R, \) and \( q \) are positive parameters. This system was first presented in 1963 by E. N. Lorenz [11] in studying fluid convection in a two dimensional layer heated from below. In the last decades, there has been an immense amount of interest generated by these equations due to the fact that for some parameter values, numerical computed solutions oscillate in the pseudo-random way which people call “chaotic”. For

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