PRECONDITIONING IN $H(\text{div})$ AND APPLICATIONS*

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Abstract. We consider the solution of the system of linear algebraic equations which arises from the finite element discretization of boundary value problems associated to the differential operator $I - \text{grad} \; \text{div}$. The natural setting for such problems is in the Hilbert space $H(\text{div})$ and the variational formulation is based on the inner product in $H(\text{div})$. We show how to construct preconditioners for these equations using both domain decomposition and multigrid techniques. These preconditioners are shown to be spectrally equivalent to the inverse of the operator. As a consequence, they may be used to precondition iterative methods so that any given error reduction may be achieved in a finite number of iterations, with the number independent of the mesh discretization. We describe applications of these results to the efficient solution of mixed and least squares finite element approximations of elliptic boundary value problems.

Key words. preconditioner, mixed method, least squares, finite element, multigrid, domain decomposition

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1. Introduction. The Hilbert space $H(\text{div})$ consists of square-integrable vectorfields on a domain $\Omega$ with square integrable divergence. This space arises naturally in the variational formulation of a variety of systems of partial differential equations. The inner product in $H(\text{div})$ is given by

$$\Lambda(u, v) = (u, v) + (\text{div} \; u, \text{div} \; v),$$

where $(\cdot, \cdot)$ is used to denote the inner product in $L^2$. Associated to the inner product $\Lambda$ is a linear operator $\Lambda$ mapping $H(\text{div})$ isometrically onto its dual space, given by the equations

$$(\Lambda u, v) = \Lambda(u, v) \quad \text{for all} \; v \in H(\text{div}).$$

Just as the corresponding operator for the inner product in the Sobolev space $H^1$ may be considered as a realization of the differential operator $I - \Delta$ together with a homogeneous natural boundary condition, $\Lambda$ may be thought of as a realization of the operator $I - \text{grad} \; \text{div}$ with an appropriate boundary condition. More precisely, if $f \in L^2$, then the operator equation $\Lambda u = f$ is equivalent to the differential equation

$$u - \text{grad} \; \text{div} \; u = f \; \text{in} \; \Omega$$

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