On superintegrable symmetry-breaking potentials in $\mathcal{N}$-dimensional Euclidean space

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Abstract
We give a graphical prescription for obtaining and characterising all separable coordinates for which the Schrödinger equation admits separable solutions for one of the superintegrable potentials

\[
V = \frac{1}{2} \sum_{\ell=1}^{n} \left( \frac{k_{\ell}^2 - \frac{1}{4}}{x_{\ell}^2} + \omega^2 x_{\ell}^2 \right) + 2\omega^2 x_{n+1}^2
\]

or

\[
V = -\frac{1}{2} \left( \frac{2\alpha}{\sqrt{x_1^2 + \ldots + x_{n+1}^2}} + \sum_{\ell=1}^{n} \frac{1}{4} - k_{\ell}^2 \right).
\]

Here $x_{n+1}$ is a distinguished cartesian variable. The algebra of second order symmetries of the resulting Schrödinger equation is given and, for the first potential,
the closure relations of the corresponding quadratic algebra. These potentials are particularly interesting because they occur in all dimensions $n \geq 1$, the separation of variables problem is highly nontrivial for them, and many other potentials are limiting cases.