Complete sets of invariants for dynamical systems that admit a separation of variables

E. G. Kalnins and J. M. Kress
Department of Mathematics, University of Waikato,
Hamilton, New Zealand,
e.kalnins@waikato.ac.nz and jonathan@math.waikato.ac.nz

G. Pogosyan
Laboratory of Theoretical Physics, Joint Institute for Nuclear Research,
Dubna, Moscow Region, 14980, Russia,
pogosyan@thsun1.jinr.dubna.su

and W. Miller, Jr.
School of Mathematics, University of Minnesota,
Minneapolis, Minnesota, 55455, U.S.A.,
miller@ima.umn.edu

February 28, 2002

Abstract

Consider a classical Hamiltonian $H$ in $n$ dimensions consisting of a kinetic energy term plus a potential. If the associated Hamilton-Jacobi equation admits an orthogonal separation of variables, then it is possible to generate algorithmically a canonical basis $Q, P$ where $P_1 = H, P_2, \ldots, P_n$ are the other 2nd-order constants of the motion associated with the separable coordinates, and $\{Q_i, Q_j\} = \{P_i, P_j\} = 0, \{Q_i, P_j\} = \delta_{ij}$. The $2n - 1$ functions $Q_1, Q_2, \ldots, Q_n, P_1, \ldots, P_n$ form a basis for the invariants. We show how to determine for exactly which spaces and potentials the invariant $Q_j$ is a polynomial in the original momenta. We shed light on the general question of exactly when the Hamiltonian admits a constant of the motion that is polynomial in the momenta. For $n = 2$ we go further and consider all cases
where the Hamilton-Jacobi equation admits a 2nd-order constant of the motion, not necessarily associated with orthogonal separable coordinates, or even separable coordinates at all. In each of these cases we construct an additional constant of the motion.