Analysis of Total Variation Flow and Its Finite Element Approximations

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Abstract

We study the gradient flow for the total variation functional, which arises in image processing and geometric applications. We propose a variational inequality weak formulation for the gradient flow, and establish well-posedness of the problem by the energy method. The main idea of our approach is to exploit the relationship between the regularized gradient flow (characterized by a small positive parameter $\varepsilon$, see (1.7)) and the minimal surface flow [19] and the prescribed mean curvature flow [15]. Since our approach is constructive and variational, finite element methods can be naturally applied to approximate weak solutions of the limiting gradient flow problem. We propose a fully discrete finite element method and establish convergence to the regularized gradient flow problem as $h, k \to 0$, and to the total variation gradient flow problem as $h, k, \varepsilon \to 0$ in general cases. Provided that the regularized gradient flow problem possesses strong solutions, which is proved possible if the datum functions are regular enough, we establish practical a priori error estimates for the fully discrete finite element solution, in particular, by focusing on the dependence of the error bounds on the regularization parameter $\varepsilon$. Optimal order error bounds are derived for the numerical solution under the mesh relation $k = O(h^2)$. In particular, it is shown that all error bounds depend on $\frac{1}{\varepsilon}$ only in some lower polynomial order for small $\varepsilon$.

1 Introduction and Summary

One of the best known and most successful noise removal and image restoration model in image processing is the total variation (TV) model due to Rudin, Osher and Fatemi [22]. Let $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ denote the gray level of an image describing a real scene, and $g$ be the observed image of the same scene, which usually is a degradation of $u$. The total variation model recovers the image $u$ by minimizing the total variation functional

$$J(u) := \int_{\Omega} |\nabla u| \, dx$$

on $BV(\Omega)$, the space of functions of bounded variation (see Section 2 for the precise definition), subject to the constraint

$$Au + \eta = g.$$