CONVERGENCE TO EQUILIBRIUM OF A MULTISCALE MODEL FOR SUSPENSIONS

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Convergence to equilibrium of a multiscale model for suspensions

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Abstract

We consider a multiscale model describing the flow of a concentrated suspension. The model couples the macroscopic equation of conservation of momentum with a nonlinear nonlocal kinetic equation describing at the microscopic level the rheological behaviour of the fluid. We study the long-time limit of the time-dependent solution. For this purpose, we use the entropy method to prove the convergence to equilibrium of the kinetic equation.

1 Setting of the problem

The rheological behaviour of complex fluids is the topic of many studies, both theoretical and experimental. For concentrated suspensions, one model available in the literature is the Hébraud-Lequeux model [5]. In this model, which aims at reproducing simple shear experiments (planar Couette flows), the material is divided in mesoscopic blocks. Their size is sufficiently large so that the stress and strain tensors may be defined for each block, but small compared to the characteristic length scale of the macroscopic stress field. It is assumed that the blocks behave as follows:

1. at low shear, each block keeps the same neighbors, and behaves as an Einstein elastic solid ;
2. beyond a given stress threshold σ_c, deformation induces local reorganization and the block flows as an Eyring fluid: the configuration reached by shearing the suspension relaxes with a characteristic time T_0 towards a completely relaxed state, where no stress is stored ;
3. lastly, the coupling between the flow of neighboring blocks is taken into account by the introduction of a diffusion process in the stress state space (see below), with a diffusion coefficient proportional to the number of reorganizations per unit time.

In the mathematical description of the model, each block carries a given shear stress σ (σ is a real number; it is in fact an extra-diagonal term of the stress tensor in convenient coordinates). The stress state in the material is described through a probability density $p(t,\sigma)$ which represents the distribution of shear stress in the assembly of blocks at time t and obeys the Fokker-Planck equation:

$$\partial_t p = -b(t)\,\partial_\sigma p + D(p(t))\,\partial^2_{\sigma\sigma} p - \frac{\mathbb{1}_{\mathbf{R}\setminus[-\sigma_c,\sigma_c]}(\sigma)}{T_0}\,p + \frac{D(p(t))}{\alpha}\,\delta_0(\sigma) \qquad (1)$$

where for $f \in L^1(\mathbb{R})$, we denote

$$D(f) = \frac{\alpha}{T_0} \int_{|\sigma|>\sigma_c} f(\sigma)\, d\sigma \,. \tag{2}$$

The macroscopic stress is then given by the average

$$\tau(t) = \int_{\mathbf{R}} \sigma\, p(t,\sigma)\, d\sigma. \tag{3}$$

In equation (1), $1\!\!1_{\mathbf{R}\setminus[-\sigma_c,\sigma_c]}$ denotes the characteristic function of the open set $\mathbb{R} \setminus [-\sigma_c, \sigma_c]$ and δ_0 the Dirac delta function on $\mathbb{R}$. The terms arising in the right-hand side of equation (1) (HL equation in short) model the three physical features described above. When a block is submitted to a shear rate $\dot\gamma(t)$, the stress of this block evolves with a variation rate $b(t) = G_0\dot\gamma(t)$ where G_0 is an elastic shear modulus. When the absolute value of the shear stress of a block overcomes the critical value σ_c, the block becomes unstable and may relax into a state with zero stress after a characteristic relaxation time T_0. This property is expressed by the last two terms in (1). This relaxation phenomenon induces a rearrangement of the other blocks and this is modelled through the diffusion term $D(p(t))\ \partial^2_{\sigma\sigma}p$. The diffusion coefficient $D(p(t))$ as given by (2) is assumed to be proportional to the density of blocks that rearrange during time T_0. The parameter α depends on the microscopic properties of the sample and models how fragile the material is. For more details on the physical relevance of the model, we refer to the original article [5].

In the original Hébraud-Lequeux model, the shear rate $\dot\gamma$ is assumed to be homogeneous in space. It has however been experimentally observed that $\dot\gamma$ is not homogeneous for planar Couette flows of real concentrated suspensions. In order to better describe the coupling between the macroscopic flow and the evolution of the mesostructure, we have proposed in [4] the following multiscale model:

$$\left\{\begin{array}{rcl} \rho\partial_t u(t,y) &=& \partial_y\tau(t,y) + \mu\partial_{yy}u(t,y) \\ \partial_t p(t,y,\sigma) &=& -G_0\,\partial_y u(t,y)\,\partial_\sigma p(t,y,\sigma) \;+\; D(p(t,y))\,\partial^2_{\sigma\sigma}p(t,y,\sigma) \\ && -\,\dfrac{1\!\!1_{\mathbf{R}\setminus[-\sigma_c,\sigma_c]}}{T_0}p(t,y,\sigma) \;+\; \dfrac{1}{T_0}\left(\displaystyle\int_{|\sigma'|>1} p(t,\sigma',y)\,d\sigma'\right)\delta_0(\sigma) \\ \tau(t,y) &=& \displaystyle\int_{\mathbf{R}} \sigma\, p(t,y,\sigma)\, d\sigma. \end{array}\right. \tag{4}$$

In the above equations, $u(t,y)$ denotes the component along e_x of the velocity field (the flow being laminar and incompressible, the velocity field is of the form $\vec{u} = u(t,y)\,e_x$), ρ is the volumic mass of the fluid and μ some non-negative viscosity coefficient. This system is complemented by the no-slip boundary conditions

$$\left\{\begin{array}{ll} u(t,0) = 0 & \text{for almost all } t\,; \\ u(t,L) = V(t) & \text{for almost all } t. \end{array}\right. \tag{5}$$

Such a mathematical setting typically models a rheometer, where $V(t)$ describes the velocity of the inner cylinder, while the outer cylinder is kept immobile (thus the boundary conditions (5)). As the radii of the cylinders are almost the same, and large, and as the streamlines are expected to be cylinders as well, this justifies geometrically the approximation by a one dimensional flow between two straight lines.

In a recent work [4], the Cauchy problem for the multiscale model (8) has been shown to be well posed. The proof relies upon a previous study [3] devoted to the Cauchy problem for the single equation (6) below, with $b(t)$ arbitrarily given in L^2_{loc}. In the present article, we turn to the long-time behaviour of the solution $(u(t,y), p(t,y,\sigma))$ to system (8). To date, we

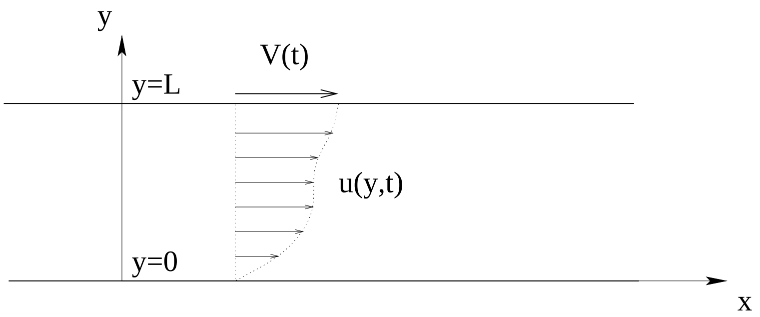

Figure 1: Planar Couette flow

have not been able to deal with the specific case of system (8). We will only deal here with the special case when $\rho = 0$ in the first equation of (8) (this amounts to neglecting inertial effects).

Without loss of generality, we may set $T_0 = 1$, $G_0 = 1$ and $\sigma_c = 1$ (this is a simple matter of nondimensionalization of the equations). Consequently, we shall deal henceforth with the kinetic equation

$$\frac{\partial p}{\partial t}(t,\sigma) = -b(t)\frac{\partial p}{\partial \sigma}(t,\sigma) + D(p)\,\frac{\partial^2 p}{\partial \sigma^2}(t,\sigma) - H_1 p(t,\sigma) + \frac{D(p)}{\alpha}\delta_0 \tag{6}$$

with $H_1 = \mathbb{1}_{\mathbf{R}\setminus[-1,1]}$ and

$$D(p) = \alpha \int_{|\sigma|\geq 1} p(t,\sigma)\, d\sigma, \tag{7}$$

and with the multiscale model

$$\left\{ \begin{array}{rcl} 0 & = & \mu\,\dfrac{\partial^2 u}{\partial y^2}(t,y) + \dfrac{\partial \tau}{\partial y}(t,y) \\ \tau(t,y) & = & \displaystyle\int_{\mathbf{R}} \sigma\, p(t,y,\sigma)\, d\sigma \\ \dfrac{\partial p}{\partial t}(t,y,\sigma) & = & -\dfrac{\partial u}{\partial y}(t,y)\dfrac{\partial p}{\partial \sigma}(t,y,\sigma) + D(p)\,\dfrac{\partial^2 p}{\partial \sigma^2}(t,y,\sigma) \\ & & -H_1 p(t,y,\sigma) + \dfrac{D(p)}{\alpha}\delta_0 \end{array} \right. \tag{8}$$

coupled with the expression (7) for the diffusion coefficient, the boundary conditions (5) for the first equation, and the initial condition

$$u(0,y) = u_0(y), \qquad p(0,y,\sigma) = p_0(y,\sigma)$$

with $u_0 \in H^1(0,L)$ such that $u_0(0) = 0$, $u_0(L) = V(0)$, and

$$p_0 \geq 0 \text{ a.e.}, \quad p_0(y,\cdot) \in L^1(\mathbb{R}), \quad \int_{\mathbf{R}} p_0(y,\sigma)\, d\sigma = 1 \text{ for a.a. } y \in]0,L[. \tag{9}$$

Our main result here (precisely stated in Theorem 3.1 below) states that, under convenient conditions, if $V(t)$ converges exponentially fast to $V_\infty := \lim_{t\longrightarrow+\infty} V(t)$, then the solution $(u(t,y), p(t,y,\sigma))$ of system (8) converges exponentially fast in time to $(u_\infty(y), p_\infty(\sigma))$ where

$$u_\infty(y) = \frac{y}{L}\, V_\infty, \tag{10}$$

and p_∞ solution to

$$0 = -\frac{V_\infty}{L}\,\frac{dp_\infty}{d\sigma} + D(p_\infty)\,\frac{d^2 p_\infty}{d\sigma^2} - H_1 p_\infty + \frac{D(p_\infty)}{\alpha}\delta_0. \tag{11}$$

The pair (u_∞, p_∞) therefore solves the steady-state system consisting of the following three equations

$$\left\{\begin{array}{rcl} 0 &=& \mu \dfrac{\partial^2 u_\infty}{\partial y^2}(y) + \dfrac{\partial \tau_\infty}{\partial y}(y) \\ \tau_\infty(y) &=& \displaystyle\int_{\mathbb{R}} \sigma\, p_\infty(y,\sigma)\, d\sigma \\ 0 &=& -\dfrac{\partial u_\infty}{\partial y}\dfrac{\partial p_\infty}{\partial \sigma} + D(p_\infty)\,\dfrac{\partial^2 p_\infty}{\partial \sigma^2} - H_1 p_\infty + \dfrac{D(p_\infty)}{\alpha}\delta_0, \end{array}\right. \tag{12}$$

with the boundary conditions

$$u_\infty(y=0) = 0, \;\; u_\infty(y=L) = V_\infty, \tag{13}$$

for the first equation. System (12) is obviously the static version of system (11).

Note that as we do not know whether (12)-(13) admits a unique solution, we will have to assume that the initial condition on (u,p) is close enough to (u_∞, p_∞) given by (10)-(11), which is one of the possibly many solutions to (12).

To prove our result, we proceed in two steps. First we consider equation (6) where we let the drift $b(t)$ go to a limit b_∞ and prove then the convergence of the solution p to the solution p_∞ of the corresponding stationary equation. This is completed using an entropy method as in the general linear case described in [6] (other references are listed therein, and we would like to point out [1]). The nonlinearity however imposes, already in the simple case under study here, some adaptations of the general strategy that are worthwhile detailing. In a second step, we treat the coupled system (8).

2 Long-time behaviour of the kinetic equation

We concentrate in this section on the long-time behaviour of the solution to the microscopic part of the model, namely that of the kinetic equation (6) that we rewrite here for convenience

$$\frac{\partial p}{\partial t} = -b(t)\frac{\partial p}{\partial \sigma} + D(p)\,\frac{\partial^2 p}{\partial \sigma^2} - H_1 p + \frac{D(p)}{\alpha}\delta_0 \tag{14}$$

where $b(t)$ is some given scalar function depending only on time t. Equation (14) is set on the whole real line $\mathbb{R}$ and is complemented by the initial condition (9). We assume in this section that the function $b(t)$ remains close to a value b_∞ and converges as time goes to infinity to this value, in the following sense

$$|b(t) - b_\infty| \le \mathcal{C} f(t) \quad \text{with} \quad |f(t)| \le 1 \quad \text{and} \quad f(t) \underset{t \to +\infty}{\longrightarrow} 0, \tag{15}$$

with $\mathcal{C}$ small (see Theorem 2.3 below). In particular, the special case of a convergence at exponential rate, i.e.

$$|b(t) - b_\infty| \le \mathcal{C}\, e^{-\mu\, t}, \tag{16}$$

for a positive constant μ will be examined.

We also assume (see Proposition 2.2 below) that the parameter α and the constant b_∞ are chosen in such a way that there exists a unique solution $p_\infty^{\alpha, b_\infty}$ (denoted by p_∞ in the sequel), of the steady-state equation

$$0 = -b_\infty \frac{dp_\infty}{d\sigma} + D(p_\infty)\,\frac{d^2 p_\infty}{d\sigma^2} - H_1 p_\infty + \frac{D(p_\infty)}{\alpha}\delta_0. \tag{17}$$

We henceforth denote by

$$D_\infty^{\alpha, b_\infty} = D(p_\infty), \quad M_\infty^{\alpha, b_\infty} = \int_{|\sigma|>1} p_\infty,$$

so that $D_\infty^{\alpha,b_\infty} = \alpha\, M_\infty^{\alpha,b_\infty}$.

We recall from [3] that

Proposition 2.1 (cf. Theorem 1.1 in [3]). *Let the initial data p_0 satisfy the conditions*

$$p_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})\,, \quad p_0 \geq 0\,, \quad \int_{\mathbb{R}} p_0 = 1 \quad \text{and} \int_{\mathbb{R}} |\sigma|\, p_0 < +\infty\,, \tag{18}$$

and assume that $D(p_0) > 0$. Assume also that $b \in L^2_{loc}(]0,+\infty[)$. Then there exists a unique global-in-time solution $p \geq 0$ to equation (14) with initial datum p_0, that belongs to $L^\infty_{loc}(]0,+\infty[, L^1_\sigma \cap L^2_\sigma) \cap L^2_{loc}(]0,+\infty[, H^1_\sigma)$. In addition, $p \in L^\infty_{loc}(]0,+\infty[, L^\infty_\sigma) \cap C^0([0,+\infty[, L^1_\sigma \cap L^2_\sigma)$, $\int_{\mathbb{R}} p(t,\sigma)\, d\sigma = 1$ for all $t > 0$, $D(p) \in C^0([0,+\infty[)$ and for every $T > 0$, $\min_{0 \leq t \leq T} D(p(t)) > 0$. Finally, $\sigma\, p \in L^\infty_{loc}(]0,+\infty[, L^1_\sigma)$ so that the average stress $\tau(t)$ defined by (3) belongs to $L^\infty_{loc}(]0,+\infty[)$.

Regarding the steady states, i.e. the solutions to (17), we also recall

Proposition 2.2 (cf. Proposition 5.1 in [3]) *When $b_\infty = 0$ and $\alpha > \frac{1}{2}$, there exists a unique stationary solution p_∞ of (17) corresponding to a positive value of the diffusion coefficient*

$$D_\infty^{\alpha,0} = \alpha - \frac{1}{2}\sqrt{4\alpha - 1}.$$

On the other hand, when $b_\infty \neq 0$, for any $\alpha > 0$, there exists a unique stationary solution p_∞ to (17), and it corresponds to a positive value for $D_\infty^{\alpha,b_\infty}$.

Remark 2.1 *In either case, the explicit expression of p_∞ is available, see [3] for the details.*

We are now in position to state the main result of this section.

Proposition 2.3 *There exists $\alpha_c > \frac{1}{2}$ such that for any $\alpha \geq \alpha_c$, there exists $b_\infty(\alpha) > 0$ and $C > 0$ such that: for all $|b_\infty| \leq b_\infty(\alpha)$ and $b(t)$ satisfying (15), for all p_0 close enough to p_∞ (in a sense that will be made precise in equation (44)), the solution $p(t,\cdot)$ to (14) converges (at least in $L^2_{loc}(\mathbb{R}) \cap L^1(\mathbb{R})$) as time t goes to infinity to the unique stationnary state p_∞ solution to*

$$-b_\infty \frac{dp_\infty}{d\sigma} + D(p_\infty)\frac{d^2 p_\infty}{d\sigma^2} - H_1 p_\infty + \frac{D(p_\infty)}{\alpha}\delta_0 = 0$$

(if $b_\infty = 0$, one needs to impose the condition $D(p_\infty) > 0$).

In addition, if $b(t)$ converges exponentially, the same holds (with different constants) for the convergence of $p(t,\cdot)$.

In the rest of this section, we give the main ingredients and the outline of the formal manipulations necessary to perform the proof of Proposition 2.3. The key quantity to be considered is the relative entropy

$$S_2(t) = \int_{\mathbb{R}} \left(\frac{p(t,\sigma)}{p_\infty(\sigma)} - 1\right)^2 p_\infty(\sigma)\, d\sigma. \tag{19}$$

In the proof below, we shall assume that such an integral over the whole space, along with other integrals of the same type, do exist. This is indeed the case, but making the proof rigorous in that respect would require to first perform all the manipulations below on a truncation of such integrals and in a second step let the truncation go to infinity in the final estimate (51). A possible way to perform the truncation consists in the following. We truncate the initial condition p_0 by multiplying it by the characteristic function of $[-R, +R]$. Then we manipulate the solution p_R of

$$\frac{\partial p_R}{\partial t} = -b(t)\frac{\partial p_R}{\partial \sigma} + D(p)\,\frac{\partial^2 p_R}{\partial \sigma^2} - H_1 p_R + \frac{D(p)}{\alpha}\delta_0$$

where $D(p)$ is the diffusion coefficient corresponding to the solution p of (14). It may be observed that p_R exhibits a Gaussian fall-off while p_∞ decreases exponentially fast, so that the ratio p_R/p_∞ has a Gaussian fall-off. For the sake of brevity, and also because we believe it somehow makes obscure the main ingredients of the proof, we will omit the truncation henceforth.

Let us first of all split (19) into the following two parts

$$S_2^{int}(t) = \int_{|\sigma|<1} \left(\frac{p(t,\sigma)}{p_\infty(\sigma)} - 1\right)^2 p_\infty(\sigma)\, d\sigma$$

and

$$S_2^{ext}(t) = \int_{|\sigma|>1} \left(\frac{p(t,\sigma)}{p_\infty(\sigma)} - 1\right)^2 p_\infty(\sigma)\, d\sigma.$$

Step 1: Preliminary dissipation inequality for the entropy

Multiplying (14) by $2\left(\dfrac{p}{p_\infty} - 1\right)$ and integrating over the real line, we have

$$\begin{aligned}\frac{dS_2}{dt} &= \int_{\mathbf{R}} 2\,\frac{\partial p}{\partial t}\left(\frac{p}{p_\infty} - 1\right)\\ &= -\,2b(t)\int_{\mathbf{R}} \frac{\partial p}{\partial \sigma}\frac{p}{p_\infty} + 2D(p)\int_{\mathbf{R}} \frac{\partial^2 p}{\partial \sigma^2}\frac{p}{p_\infty}\\ &\quad -2\int_{\mathbf{R}} \frac{H_1 p^2}{p_\infty} + 2\frac{D(p)}{\alpha}\frac{p(0)}{p_\infty(0)},\end{aligned} \tag{20}$$

where we have used the fact that $\displaystyle\int_{\mathbf{R}} p\, d\sigma = 1$ for all time. Since

$$2\int_{\mathbf{R}} \frac{\partial p}{\partial \sigma}\frac{p}{p_\infty} = \int_{\mathbf{R}} \frac{1}{p_\infty}\frac{\partial p^2}{\partial \sigma} = -\int_{\mathbf{R}} p^2 \frac{\partial}{\partial \sigma}\left(\frac{1}{p_\infty}\right) = \int_{\mathbf{R}} \frac{p^2}{p_\infty^2}\frac{\partial p_\infty}{\partial \sigma},$$

we thus obtain

$$\frac{dS_2}{dt} = -b(t)\int_{\mathbf{R}} \frac{\partial p_\infty}{\partial \sigma}\left(\frac{p}{p_\infty}\right)^2 + 2D(p)\int_{\mathbf{R}} \frac{\partial^2 p}{\partial \sigma^2}\frac{p}{p_\infty} - 2\int_{\mathbf{R}} \frac{H_1 p^2}{p_\infty} + 2\frac{D(p)}{\alpha}\frac{p(0)}{p_\infty(0)}. \tag{21}$$

On the other hand, multiplying (17) by $\left(\dfrac{p}{p_\infty}\right)^2$ and integrating over the real line, we have

$$\begin{aligned}0 = &-b_\infty\int_{\mathbf{R}} \frac{\partial p_\infty}{\partial \sigma}\left(\frac{p}{p_\infty}\right)^2 + D_\infty^{\alpha,b_\infty}\int_{\mathbf{R}} \frac{\partial^2 p_\infty}{\partial \sigma^2}\left(\frac{p}{p_\infty}\right)^2\\ &-\int_{\mathbf{R}} H_1 p_\infty\left(\frac{p}{p_\infty}\right)^2 + \frac{D_\infty^{\alpha,b_\infty}}{\alpha}\left(\frac{p(0)}{p_\infty(0)}\right)^2.\end{aligned}$$

We multiply the latter equation by $\dfrac{D(p)}{D_\infty^{\alpha,b_\infty}}$, and substract it to (21). This yields

$$\begin{aligned}\frac{dS_2}{dt} = &-\left((b(t)-b_\infty)+b_\infty(1-\frac{D(p)}{D_\infty^{\alpha,b_\infty}})\right)\int_{\mathbf{R}}\frac{\partial p_\infty}{\partial\sigma}\left(\frac{p}{p_\infty}\right)^2\\ &+D(p)\left[2\int_{\mathbf{R}}\frac{\partial^2 p}{\partial\sigma^2}\frac{p}{p_\infty}-\int_{\mathbf{R}}\frac{\partial^2 p_\infty}{\partial\sigma^2}\left(\frac{p}{p_\infty}\right)^2\right]\\ &+\left[\left(\frac{D(p)}{D_\infty^{\alpha,b_\infty}}-2\right)\int_{\mathbf{R}}\frac{H_1p^2}{p_\infty}+2\frac{D(p)}{\alpha}\frac{p(0)}{p_\infty(0)}-\frac{D(p)}{\alpha}\left(\frac{p(0)}{p_\infty(0)}\right)^2\right],\end{aligned}$$

where in the second term of the right-hand side we remark

$$\begin{aligned}&2\int_{\mathbf{R}}\frac{\partial^2 p}{\partial\sigma^2}\frac{p}{p_\infty}-\int_{\mathbf{R}}\frac{\partial^2 p_\infty}{\partial\sigma^2}\left(\frac{p}{p_\infty}\right)^2\\ &\qquad=2\int_{\mathbf{R}}\frac{\partial^2 p}{\partial\sigma^2}\frac{p}{p_\infty}-2\int_{\mathbf{R}}\frac{\partial^2 p_\infty}{\partial\sigma^2}\left(\frac{p}{p_\infty}\right)^2+\int_{\mathbf{R}}\frac{\partial^2 p_\infty}{\partial\sigma^2}\left(\frac{p}{p_\infty}\right)^2\\ &\qquad=2\int_{\mathbf{R}}\frac{\partial^2 p}{\partial\sigma^2}\frac{p}{p_\infty}-2\int_{\mathbf{R}}\frac{\partial^2 p_\infty}{\partial\sigma^2}\left(\frac{p}{p_\infty}\right)^2-\int_{\mathbf{R}}\frac{\partial p_\infty}{\partial\sigma}\frac{\partial}{\partial\sigma}\left(\left(\frac{p}{p_\infty}\right)^2\right)\\ &\qquad=2\left[\int_{\mathbf{R}}\frac{\partial^2 p}{\partial\sigma^2}\frac{p}{p_\infty}-\int_{\mathbf{R}}\frac{\partial^2 p_\infty}{\partial\sigma^2}\left(\frac{p}{p_\infty}\right)^2-\int_{\mathbf{R}}\frac{\partial p_\infty}{\partial\sigma}\left(\frac{p}{p_\infty}\right)\frac{\partial}{\partial\sigma}\left(\frac{p}{p_\infty}\right)\right]\\ &\qquad=2\left[\int_{\mathbf{R}}\frac{\partial^2 p}{\partial\sigma^2}\frac{p}{p_\infty}-\int_{\mathbf{R}}\frac{p}{p_\infty}\frac{\partial}{\partial\sigma}\left(\frac{p}{p_\infty}\frac{\partial p_\infty}{\partial\sigma}\right)\right]\\ &\qquad=2\left[-\int_{\mathbf{R}}\frac{\partial p}{\partial\sigma}\frac{\partial}{\partial\sigma}\left(\frac{p}{p_\infty}\right)+\int_{\mathbf{R}}\frac{\partial}{\partial\sigma}\left(\frac{p}{p_\infty}\right)\frac{p}{p_\infty}\frac{\partial p_\infty}{\partial\sigma}\right]\\ &\qquad=-2\int_{\mathbf{R}}\left(\frac{\partial p}{\partial\sigma}-\frac{p}{p_\infty}\frac{\partial p_\infty}{\partial\sigma}\right)\frac{\partial}{\partial\sigma}\left(\frac{p}{p_\infty}\right)\\ &\qquad=-2\int_{\mathbf{R}}p_\infty\left|\frac{\partial}{\partial\sigma}\left(\frac{p}{p_\infty}\right)\right|^2.\end{aligned}$$

We have thus obtained

$$\begin{aligned}\frac{dS_2}{dt} = &-\left((b(t)-b_\infty)+b_\infty(1-\frac{D(p)}{D_\infty^{\alpha,b_\infty}})\right)\int_{\mathbf{R}}\frac{\partial p_\infty}{\partial\sigma}\left(\frac{p}{p_\infty}\right)^2\\ &-2\,D(p)\left[\int_{\mathbf{R}}p_\infty\left|\frac{\partial}{\partial\sigma}\left(\frac{p}{p_\infty}\right)\right|^2\right]\\ &+\left[\left(\frac{D(p)}{D_\infty^{\alpha,b_\infty}}-2\right)\int_{\mathbf{R}}\frac{H_1p^2}{p_\infty}+2\frac{D(p)}{\alpha}\frac{p(0)}{p_\infty(0)}-\frac{D(p)}{\alpha}\left(\frac{p(0)}{p_\infty(0)}\right)^2\right].\end{aligned}\tag{22}$$

Remark 2.2 *In the particular case when $b\equiv 0$ and $\sigma_c=0$, one has $D(p)\equiv\alpha$ (see [4]) and equation (22) reads*

$$\frac{dS_2}{dt}+2\,\alpha\left[\int_{\mathbf{R}}p_\infty\left|\frac{\partial}{\partial\sigma}\left(\frac{p}{p_\infty}\right)\right|^2\right]+S_2+\left(\frac{p(0)}{p_\infty(0)}-1\right)^2=0.$$

It immediately follows that $S_2(t)$ converges exponentially fast to zero. In the rest of the section, we extend this argument to the general case.

For this purpose, let us first rewrite (22), with obvious notations,

$$\frac{dS_2}{dt}=A+B+C.\tag{23}$$

We keep term B as such and now treat terms A and C.

For term A we remark

$$\begin{aligned}\left|\int_{\mathbf{R}} \frac{\partial p_\infty}{\partial \sigma}\left(\frac{p}{p_\infty}\right)^2\right| &= \left|\int_{\mathbf{R}} \left(\frac{p^2}{p_\infty^2}-1\right)\frac{\partial p_\infty}{\partial \sigma}\right| \\ &= \left|\int_{\mathbf{R}} \left(\frac{p}{p_\infty}-1\right)\sqrt{p_\infty}\left(\frac{p}{p_\infty}+1\right)\sqrt{p_\infty}\,\frac{1}{p_\infty}\frac{\partial p_\infty}{\partial \sigma}\right| \\ &\le \sqrt{S_2}\left(\int_{\mathbf{R}}\left(\frac{p}{p_\infty}+1\right)^2 p_\infty \left|\frac{1}{p_\infty}\frac{\partial p_\infty}{\partial \sigma}\right|^2\right)^{1/2} \\ &\le \sqrt{S_2}\, C_\infty^{\alpha,b_\infty}\left(\int_{\mathbf{R}}\left(\frac{p}{p_\infty}+1\right)^2 p_\infty\right)^{1/2}.\end{aligned}$$

with $C_\infty^{\alpha,b_\infty} = \left\|\frac{1}{p_\infty}\frac{\partial p_\infty}{\partial \sigma}\right\|_{L^\infty}$, a quantity that can be evaluated with the explicit expression of p_∞ provided in [3]. Next a simple calculation shows

$$\int_{\mathbf{R}}\left(\frac{p}{p_\infty}+1\right)^2 p_\infty = S_2+4,$$

It follows that

$$A \le C_\infty^{\alpha,b_\infty}\left(|b(t)-b_\infty| + |b_\infty|\,|1-\frac{D(p)}{D_\infty^{\alpha,b_\infty}}|\right)\sqrt{S_2+4}\,\sqrt{S_2}. \tag{24}$$

On the other hand, for term C, we remark that

$$\begin{aligned} S_2^{ext} &= \int_{|\sigma|>1}\left(\frac{p}{p_\infty}-1\right)^2 p_\infty \\ &= \int_{\mathbf{R}} \frac{H_1 p^2}{p_\infty} - 2\frac{D(p)}{\alpha} + \frac{D_\infty^{\alpha,b_\infty}}{\alpha}.\end{aligned}$$

Therefore, using $2x-x^2 = -(x-1)^2+1$, we have

$$\begin{aligned} C &= \left(\frac{D(p)}{D_\infty^{\alpha,b_\infty}}-2\right)\int_{\mathbf{R}}\frac{H_1p^2}{p_\infty} + 2\frac{D(p)}{\alpha}\frac{p(0)}{p_\infty(0)} - \frac{D(p)}{\alpha}\left(\frac{p(0)}{p_\infty(0)}\right)^2 \\ &= -\left(2-\frac{D(p)}{D_\infty^{\alpha,b_\infty}}\right)S_2^{ext} - \frac{D(p)}{\alpha}\left(\frac{p(0)}{p_\infty(0)}-1\right)^2 + \frac{2D_\infty^{\alpha,b_\infty}}{\alpha}\left(\frac{D(p)}{D_\infty^{\alpha,b_\infty}}-1\right)^2.\end{aligned} \tag{25}$$

Inserting (24) and (25) into (23), we obtain

$$\begin{aligned} &\frac{dS_2}{dt} + 2\,D(p)\int_{\mathbf{R}} p_\infty\left|\frac{\partial}{\partial\sigma}\left(\frac{p}{p_\infty}\right)\right|^2 + \left(2-\frac{D(p)}{D_\infty^{\alpha,b_\infty}}\right)S_2^{ext} + \frac{D(p)}{\alpha}\left(\frac{p(0)}{p_\infty(0)}-1\right)^2 \\ &\le \frac{2D_\infty^{\alpha,b_\infty}}{\alpha}\left(1-\frac{D(p)}{D_\infty^{\alpha,b_\infty}}\right)^2 + C_\infty^{\alpha,b_\infty}\left(|b(t)-b_\infty| + |b_\infty|\,|1-\frac{D(p)}{D_\infty^{\alpha,b_\infty}}|\right)\sqrt{S_2+4}\,\sqrt{S_2},\end{aligned} \tag{26}$$

At this stage, it is useful to make the following comments in order to better understand the sequel of the proof. Our goal is to deduce from (26) an ordinary differential inequation of the type

$$\frac{dS_2}{dt} + \varepsilon S_2 \le C\,|b(t)-b_\infty|\,(\sqrt{S_2}+S_2), \tag{27}$$

for some constant C and small positive constant ε, from which we shall easily deduce the convergence of S_2 to 0 when $b(t)$ converges to b_∞ according to (15). Then this provides the basis of the so-called entropy-decay method in the framework of kinetic equation (see [6]). The convergence of p to p_∞ follows from the convergence of the entropy by another type of argument (see again [6] for the general strategy, and below for more details in our specific setting). For this purpose, there are two difficulties in the estimate (26): first the diffusion coefficient $D(p)$ depends on p, and second we have to bound from below one of the terms of the left hand side so that S_2, and not only S_2^{ext}, appears there. If the coefficient $D(p)$ was freezed to the constant value $D_\infty^{\alpha,b_\infty}$, the above estimation (26) would read

$$\frac{dS_2}{dt} + 2\,D_\infty^{\alpha,b_\infty} \int_{\mathbf{R}} p_\infty \left|\frac{\partial}{\partial\sigma}\left(\frac{p}{p_\infty}\right)\right|^2 + S_2^{ext} + \frac{D_\infty^{\alpha,b_\infty}}{\alpha}\left(\frac{p(0)}{p_\infty(0)} - 1\right)^2$$
$$\leq C_\infty^{\alpha,b_\infty}\,|b(t) - b_\infty|\,\sqrt{S_2 + 4}\,\sqrt{S_2},$$

and thus imply

$$\frac{dS_2}{dt} + 2\,D_\infty^{\alpha,b_\infty} \int_{\mathbf{R}} p_\infty \left|\frac{\partial}{\partial\sigma}\left(\frac{p}{p_\infty}\right)\right|^2 + S_2^{ext} \leq C_\infty^{\alpha,b_\infty}\,|b(t) - b_\infty|\,\sqrt{S_2 + 4}\,\sqrt{S_2}, \tag{28}$$

and then the next step would be to estimate from below the term

$$\int_{\mathbf{R}} p_\infty \left|\frac{\partial}{\partial\sigma}\left(\frac{p}{p_\infty}\right)\right|^2$$

by the entropy S_2, in the spirit of what is usually done in order to establish Log-Sobolev inequalities.

In the present case, where $D(p)$ does depend on p, our proof will precisely follow the pattern that we have described. We first need to estimate (in Step 2) how $D(p)$ differs from its (tentative) asymptotic value $D_\infty^{\alpha,b_\infty}$, and next circumvent (in Step 3) the Log-Sobolev type inequality by taking benefit from the one-dimensional setting that allows for a simple estimate.

Step 2: Estimating the difference $D(p) - D_\infty^{\alpha,b_\infty}$

For this purpose, we remark

$$\left|1 - \frac{D(p)}{D_\infty^{\alpha,b_\infty}}\right| = \frac{\alpha}{D_\infty^{\alpha,b_\infty}}\left|\int_{|\sigma|>1}\left(\frac{p}{p_\infty} - 1\right)\sqrt{p_\infty}\,\sqrt{p_\infty}\right|$$
$$\leq \frac{\alpha}{D_\infty^{\alpha,b_\infty}}\left(\int_{|\sigma|>1} p_\infty\right)^{1/2}\left(\int_{|\sigma|>1}\left(\frac{p}{p_\infty} - 1\right)^2 p_\infty\right)^{1/2} \tag{29}$$

Therefore

$$\left|1 - \frac{D(p)}{D_\infty^{\alpha,b_\infty}}\right| \leq \sqrt{\frac{S_2^{ext}}{M_\infty^{\alpha,b_\infty}}}. \tag{30}$$

Likewise,

$$\left|1 - \frac{D(p)}{D_\infty^{\alpha,b_\infty}}\right| = \frac{\alpha}{D_\infty^{\alpha,b_\infty}}\left|\int_{|\sigma|<1}\left(\frac{p}{p_\infty} - 1\right)\sqrt{p_\infty}\,\sqrt{p_\infty}\right|$$
$$\leq \frac{\alpha}{D_\infty^{\alpha,b_\infty}}\left(\int_{|\sigma|<1} p_\infty\right)^{1/2}\left(\int_{|\sigma|<1}\left(\frac{p}{p_\infty} - 1\right)^2 p_\infty\right)^{1/2} \tag{31}$$

Hence

$$\left|1 - \frac{D(p)}{D_\infty^{\alpha,b_\infty}}\right| \leq \sqrt{\frac{(1 - M_\infty^{\alpha,b_\infty})S_2^{int}}{(M_\infty^{\alpha,b_\infty})^2}}. \tag{32}$$

Actually, denoting by

$$K_\infty^{\alpha,b_\infty} = \min \left(\frac{1}{\sqrt{M_\infty^{\alpha,b_\infty}}} \, , \, \sqrt{\frac{1-M_\infty^{\alpha,b_\infty}}{(M_\infty^{\alpha,b_\infty})^2}} \right),$$

estimate (30) and (32) may be collected in

$$\left| 1 - \frac{D(p)}{D_\infty^{\alpha,b_\infty}} \right| \le K_\infty^{\alpha,b_\infty} \sqrt{S_2}. \tag{33}$$

Step 3: A Poincaré type inequality

On the other hand, we need to bound from below $\int_{|\sigma|<1} p_\infty \left| \frac{\partial}{\partial \sigma} \left(\frac{p}{p_\infty} \right) \right|^2$, i.e. in fact the term B from (23). In order to deal with this term, we first note that

$$\begin{aligned} S_2^{int} &= \int_{|\sigma|<1} \left(\frac{p}{p_\infty} - 1 \right)^2 p_\infty \\ &\le \int_{|\sigma|<1} \left(\left(\frac{p}{p_\infty} - \frac{p(0)}{p_\infty(0)} \right) + \left(\frac{p(0)}{p_\infty(0)} - 1 \right) \right)^2 p_\infty \\ &\le \int_{|\sigma|<1} \left(2 \left(\frac{p}{p_\infty} - \frac{p(0)}{p_\infty(0)} \right)^2 + 2 \left(\frac{p(0)}{p_\infty(0)} - 1 \right)^2 \right) p_\infty \\ &\le 2 \int_{|\sigma|<1} \left(\frac{p}{p_\infty} - \frac{p(0)}{p_\infty(0)} \right)^2 p_\infty + 2 \left(\frac{p(0)}{p_\infty(0)} - 1 \right)^2 \int_{|\sigma|<1} p_\infty \, . \end{aligned} \tag{34}$$

The first term of the right-hand side of (34) is now bounded from above by the use of a simple Poincare-type inequality. Indeed, for any $u \in H^1(-1,1)$, we have

$$\begin{aligned} &\int_{-1}^{1} (u(\sigma) - u(0))^2 \, p_\infty(\sigma) \, d\sigma \\ &= \int_{-1}^{0} \left(\int_\sigma^0 u'(t) \, dt \right)^2 p_\infty(\sigma) \, d\sigma + \int_0^1 \left(\int_0^\sigma u'(t) \, dt \right)^2 p_\infty(\sigma) \, d\sigma \\ &= \int_{-1}^{0} \left(\sqrt{p_\infty(\sigma)} \int_\sigma^0 u'(t) \, dt \right)^2 d\sigma + \int_0^1 \left(\sqrt{p_\infty(\sigma)} \int_0^\sigma u'(t) \, dt \right)^2 d\sigma \end{aligned}$$

and thus, as the function $p_\infty(\sigma)$ is nondecreasing on the interval $[-1,0]$ and nonincreasing on the interval $[0,1]$ (see [3]), we have

$$\begin{aligned} \int_{-1}^{1} (u(\sigma) - u(0))^2 \, p_\infty(\sigma) \, d\sigma &\le \int_{-1}^{0} \left(\int_\sigma^0 |u'(t)| \sqrt{p_\infty(t)} \, dt \right)^2 d\sigma \\ &\quad + \int_0^1 \left(\int_0^\sigma |u'(t)| \sqrt{p_\infty(t)} \, dt \right)^2 d\sigma \\ &\le \int_{-1}^{0} \left(\int_\sigma^0 |u'(t)|^2 p_\infty(t) \, dt \right) |\sigma| \, d\sigma \\ &\quad + \int_0^1 \left(\int_0^\sigma |u'(t)|^2 p_\infty(t) \, dt \right) \sigma \, d\sigma \\ &\le \frac{1}{2} \int_{-1}^{0} |u'(\sigma)|^2 p_\infty(\sigma) \, d\sigma + \frac{1}{2} \int_0^1 |u'(\sigma)|^2 p_\infty(\sigma) \, d\sigma \\ &= \frac{1}{2} \int_{-1}^{1} |u'(\sigma)|^2 p_\infty(\sigma) \, d\sigma. \end{aligned}$$

Applying this to $u = p - p_\infty$, we obtain

$$S_2^{int} \leq \int_{|\sigma|<1} p_\infty \left| \frac{\partial}{\partial \sigma} \left(\frac{p}{p_\infty} \right) \right|^2 + 2 \left(1 - M_\infty^{\alpha, b_\infty} \right) \left(\frac{p(0)}{p_\infty(0)} - 1 \right)^2 . \tag{35}$$

Step 4: Entropy dissipation inequality

We now use (30), (32) and (35) to treat inequality (26). Because of (30), we have

$$M_\infty^{\alpha, b_\infty} \left(1 - \frac{D(p)}{D_\infty^{\alpha, b_\infty}} \right)^2 \leq S_2^{ext}, . \tag{36}$$

On the other hand, we have

$$\begin{aligned} M_\infty^{\alpha, b_\infty} \left(1 - \frac{D(p)}{D_\infty^{\alpha, b_\infty}} \right)^2 &\leq \frac{1 - M_\infty^{\alpha, b_\infty}}{M_\infty^{\alpha, b_\infty}} S_2^{int} \\ &\leq \frac{(1 - M_\infty^{\alpha, b_\infty})}{M_\infty^{\alpha, b_\infty}} \int_{|\sigma|<1} p_\infty \left| \frac{\partial}{\partial \sigma} \left(\frac{p}{p_\infty} \right) \right|^2 \\ &\quad + \frac{2(1 - M_\infty^{\alpha, b_\infty})^2}{M_\infty^{\alpha, b_\infty}} \left(\frac{p(0)}{p_\infty(0)} - 1 \right)^2 , \end{aligned} \tag{37}$$

successively using (32) and (35). Adding (36) and (37) with the respective weights 1/2 and 3/2, we get

$$\begin{aligned} 2\, M_\infty^{\alpha, b_\infty} \left(1 - \frac{D(p)}{D_\infty^{\alpha, b_\infty}} \right)^2 &\leq \frac{1}{2} S_2^{ext} \\ &\quad + \frac{3}{2} \frac{(1 - M_\infty^{\alpha, b_\infty})}{M_\infty^{\alpha, b_\infty}} \int_{|\sigma|<1} p_\infty \left| \frac{\partial}{\partial \sigma} \left(\frac{p}{p_\infty} \right) \right|^2 \\ &\quad + \frac{3}{2} \frac{2(1 - M_\infty^{\alpha, b_\infty})^2}{M_\infty^{\alpha, b_\infty}} \left(\frac{p(0)}{p_\infty(0)} - 1 \right)^2 . \end{aligned}$$

We now add this inequality to (26) and obtain

$$\begin{aligned} &\frac{dS_2}{dt} + \left(2\, D(p) - \frac{3(1 - M_\infty^{\alpha, b_\infty})}{2\, M_\infty^{\alpha, b_\infty}} \right) \int_{\mathbf{R}} p_\infty \left| \frac{\partial}{\partial \sigma} \left(\frac{p}{p_\infty} \right) \right|^2 \\ &\quad + \left(\frac{3}{2} - \frac{D(p)}{D_\infty^{\alpha, b_\infty}} \right) S_2^{ext} \\ &\quad + \left(\frac{D(p)}{\alpha} - 3 \frac{(1 - M_\infty^{\alpha, b_\infty})^2}{M_\infty^{\alpha, b_\infty}} \right) \left(\frac{p(0)}{p_\infty(0)} - 1 \right)^2 \\ &\quad \leq C_\infty^{\alpha, b_\infty} \left(|b(t) - b_\infty| + |b_\infty| \left| 1 - \frac{D(p)}{D_\infty^{\alpha, b_\infty}} \right| \right) \sqrt{S_2 + 4}\, \sqrt{S_2}, \end{aligned} \tag{38}$$

thus, using (33) to bound from above the right-hand side,

$$\begin{aligned} &\frac{dS_2}{dt} + \left(2\, D(p) - \frac{3}{2} \frac{(1 - M_\infty^{\alpha, b_\infty})}{M_\infty^{\alpha, b_\infty}} \right) \int_{\mathbf{R}} p_\infty \left| \frac{\partial}{\partial \sigma} \left(\frac{p}{p_\infty} \right) \right|^2 \\ &\quad + \left(\frac{3}{2} - \frac{D(p)}{D_\infty^{\alpha, b_\infty}} \right) S_2^{ext} \\ &\quad + \left(\frac{D(p)}{\alpha} - \frac{3(1 - M_\infty^{\alpha, b_\infty})^2}{M_\infty^{\alpha, b_\infty}} \right) \left(\frac{p(0)}{p_\infty(0)} - 1 \right)^2 \\ &\quad \leq C_\infty^{\alpha, b_\infty} \left(|b(t) - b_\infty| + |b_\infty|\, K_\infty^{\alpha, b_\infty} \sqrt{S_2} \right) \sqrt{S_2 + 4}\, \sqrt{S_2} . \end{aligned} \tag{39}$$

We now claim that we may choose the parameter α large enough and the shear b_∞ small enough in such a way that

$$\left\{\begin{array}{lcl} 2\,\alpha\, M_\infty^{\alpha,b_\infty} - \dfrac{3(1-M_\infty^{\alpha,b_\infty})}{2\,M_\infty^{\alpha,b_\infty}} & \geq & \dfrac{1}{2}, \\ M_\infty^{\alpha,b_\infty} - \dfrac{3(1-M_\infty^{\alpha,b_\infty})^2}{M_\infty^{\alpha,b_\infty}} & \geq & 1-M_\infty^{\alpha,b_\infty}. \end{array}\right. \tag{40}$$

Indeed, we recall from [3] that

$$M_\infty^{\alpha,b_\infty\equiv 0} = \frac{D_\infty^{\alpha,0}}{\alpha} = 1 - \frac{1}{2\,\alpha}\sqrt{4\alpha-1}. \tag{41}$$

It is simple to see that $M_\infty^{\alpha,0}$ is a strictly increasing function of α for $\alpha \geq 1/2$, that is zero when $\alpha = 1/2$ and goes to 1 as $\alpha \longrightarrow +\infty$. Thus for

$$\alpha \geq \alpha_c, \tag{42}$$

it is clear that (40) holds in the case $b_\infty = 0$, and thus, by continuity argument, when

$$|b_\infty| \leq b_\infty^\alpha, \tag{43}$$

(where the critical value b_∞^α may indeed depend on α) the bounds (40) also hold for b_∞.

We now assume that $S_2(0)$ is small enough so that

$$8\,K_\infty^{\alpha,b_\infty}\sqrt{S_2(0)} \leq \min\left(\frac{1}{2D_\infty^{\alpha,b_\infty}}\,,\,1\,,\,\frac{1-M_\infty^{\alpha,b_\infty}}{M_\infty^{\alpha,b_\infty}}\right), \tag{44}$$

and by continuity introduce the time interval $[0,T]$ of maximal length T such that $S_2(t)$ satisfies

$$8\,K_\infty^{\alpha,b_\infty}\sqrt{S_2(t)} \leq 2\,\min\left(\frac{1}{2D_\infty^{\alpha,b_\infty}}\,,\,1\,,\,\frac{1-M_\infty^{\alpha,b_\infty}}{M_\infty^{\alpha,b_\infty}}\right). \tag{45}$$

On this interval, we have, in view of (33),

$$\left\{\begin{array}{lcl} 2\left|\dfrac{D(p)}{D_\infty^{\alpha,b_\infty}} - 1\right| D_\infty^{\alpha,b_\infty} & \leq & \dfrac{1}{4}, \\ \left|1 - \dfrac{D(p)}{D_\infty^{\alpha,b_\infty}}\right| & \leq & \dfrac{1}{4}, \\ M_\infty^{\alpha,b_\infty}\left|1 - \dfrac{D(p)}{D_\infty^{\alpha,b_\infty}}\right| & \leq & \dfrac{1}{4}(1-M_\infty^{\alpha,b_\infty}), \end{array}\right. \tag{46}$$

and thus, adding (40) and (46), we have

$$\left\{\begin{array}{lcl} 2\,D(p) - \dfrac{3}{2}\dfrac{2(1-M_\infty^{\alpha,b_\infty})}{2\,M_\infty^{\alpha,b_\infty}} & \geq & \dfrac{1}{4}, \\ \dfrac{3}{2} - \dfrac{D(p)}{D_\infty^{\alpha,b_\infty}} & \geq & \dfrac{1}{4}, \\ \dfrac{D(p)}{\alpha} - \dfrac{3(1-M_\infty^{\alpha,b_\infty})^2}{M_\infty^{\alpha,b_\infty}} & \geq & \dfrac{1}{2}(1-M_\infty^{\alpha,b_\infty})\,. \end{array}\right. \tag{47}$$

We then deduce from (39) and (47) that on $[0,T]$,

$$\begin{aligned} &\frac{dS_2}{dt} + \frac{1}{4}\int_{\mathbf{R}} p_\infty \left|\frac{\partial}{\partial\sigma}\left(\frac{p}{p_\infty}\right)\right|^2 + \frac{1}{4}S_2^{ext} \\ &+\frac{1}{2}(1-M_\infty^{\alpha,b_\infty})\left(\frac{p(0)}{p_\infty(0)} - 1\right)^2 \\ &\qquad \leq C_\infty^{\alpha,b_\infty}\left(|b(t)-b_\infty| + |b_\infty|\,K_\infty^{\alpha,b_\infty}\sqrt{S_2}\right)\sqrt{S_2 + 4\sqrt{S_2}}, \end{aligned} \tag{48}$$

thus, from (35), we obtain

$$\frac{dS_2}{dt} + \frac{1}{4}S_2 \le C_\infty^{\alpha,b_\infty} \left(|b(t) - b_\infty| + |b_\infty|\, K_\infty^{\alpha,b_\infty} \sqrt{S_2} \right) \sqrt{S_2 + 4}\, \sqrt{S_2}. \tag{49}$$

This estimate holds for any $t \in [0, T]$ (time interval defined by (45)), provided the parameter α is chosen such that (40) holds and under the initial condition (44).

Inequality (49) is the estimate that replaces the "ideal" estimate (27) we announced above.

Step 5: Connectivity argument

We now choose b_∞ small enough such that, in addition to (43) we have

$$|b_\infty| \le \frac{1}{8\, C_\infty^{\alpha,b_\infty} \left(4(K_\infty^{\alpha,b_\infty})^2 + \frac{1}{16} \right)^{1/2}}. \tag{50}$$

This is indeed possible because when $\alpha > 1/2$ is fixed and $b_\infty \longrightarrow 0$ we have on the one hand

$$M_\infty^{\alpha,b_\infty} \longrightarrow M_\infty^{\alpha,0},$$

that is bounded away from zero in view of (41), thus $K_\infty^{\alpha,b_\infty}$ remains bounded, while on the other hand $C_\infty^{\alpha,b_\infty}$ remains also bounded (see the analytical expression of p_∞ in [3]). Therefore, for such small b_∞ and $t \le T$, (45) and (50) impose

$$C_\infty^{\alpha,b_\infty}\, |b_\infty|\, K_\infty^{\alpha,b_\infty} \sqrt{S_2 + 4} \le \frac{1}{8},$$

and thus (49) yields

$$\frac{dS_2}{dt} + \frac{1}{8}S_2 \le C_\infty^{\alpha,b_\infty}\, |b(t) - b_\infty| \sqrt{S_2 + 4}\, \sqrt{S_2},$$

which, since $S_2 \le \left(\frac{1}{4K_\infty^{\alpha,b_\infty}} \right)^2$ in the time interval we work, gives in turn

$$\frac{d\sqrt{S_2}}{dt} + \frac{1}{16}\sqrt{S_2} \le c_0^{\alpha,b_\infty}\, \frac{C_\infty^{\alpha,b_\infty}}{2}\, |b(t) - b_\infty|, \tag{51}$$

with

$$c_0^{\alpha,b_\infty} = \sqrt{\left(\frac{1}{4\, K_\infty^{\alpha,b_\infty}} \right)^2 + 4}. \tag{52}$$

We now make use of the assumption (15) so that

$$\frac{d\sqrt{S_2}}{dt} + \frac{1}{16}\sqrt{S_2} \le c_0^{\alpha,b_\infty}\, \frac{C_\infty^{\alpha,b_\infty}}{2}\, \mathcal{C}\, f(t), \tag{53}$$

where we recall $|f(t)| \le 1$. Thus, integrating the above estimate from 0 to t we obtain

$$\sqrt{S_2(t)} \le \sqrt{S_2(0)} + 8\, c_0^{\alpha,b_\infty}\, C_\infty^{\alpha,b_\infty}\, \mathcal{C}. \tag{54}$$

In view of (44), this yields

$$\sqrt{S_2(t)} \le \frac{3}{16} \frac{1}{K_\infty^{\alpha,b_\infty}}, \tag{55}$$

as soon as $\mathcal{C}$ is chosen small enough, the bound depending only on c_0^{α,b_∞} and $C_\infty^{\alpha,b_\infty}$, that are uniformly bounded when b_∞ is small; hence, the bound only depends on α.

The estimate (55) for $t = T$ shows that the second bound of (45) is still true for a small interval beyond T. The same holds for the other two bounds, provided $\mathcal{C}$ is small enough, the bound on $\mathcal{C}$ again depending only on α. This shows, by connectivity, that $T = +\infty$ and thus that (53) holds for any $t > 0$. It is then easy to insert the assumption $f(t) \longrightarrow 0$ in the right-hand side of (53) and obtain the convergence

$$S_2(t) \longrightarrow 0, \quad \text{as} \quad t \longrightarrow +\infty.$$

If in addition the exponential convergence (16) of $f(t)$ is assumed, that of $S_2(t)$ easily follows from (53).

From the convergence of $S_2(t)$ follows in turn that of $p(t,\cdot)$ to p_∞, in the same manner. Indeed, it suffices to notice that

$$S_2(t) = \int_{\mathbf{R}} \frac{(p(t,\cdot) - p_\infty)^2}{p_\infty} \geq \left(\inf_{[-R,R]} p_\infty \right) \|p(t,\cdot) - p_\infty\|_{L^2(]-R,R[)},$$

for any finite R, while

$$\|p(t,\cdot) - p_\infty\|_{L^1(\mathbf{R})} = \int_{\mathbf{R}} \frac{|p(t,\cdot) - p_\infty|}{\sqrt{p_\infty}} \sqrt{p_\infty} \leq \sqrt{S_2(t)},$$

by Cauchy-Schwarz inequality (recall that $\|p_\infty\|_{L^1} = 1$). Further convergences in various norms may be bootstrapped from the above convergences. This concludes the proof of Proposition 2.3

3 The coupled problem

3.1 Set up of the problem

We now come back to the coupled problem (8) and reformulate it. For this purpose, we remark that due to the simple one-dimensional setting and to the boundary conditions (5), the first line of (8) may be integrated to obtain

$$\frac{\partial u}{\partial y}(t,y) = -\frac{1}{\mu}\tau(t,y) + \frac{1}{\mu L}\int_0^L \tau(t,y')\,dy' + \frac{1}{L}\,V(t). \tag{56}$$

Inserting this into system (8), we obtain the equivalent formulation

$$\begin{aligned}\frac{\partial p}{\partial t}(t,y,\sigma) = &- \left[-\frac{1}{\mu}\int_{\mathbf{R}} \sigma p(t,y,\sigma)\,d\sigma + \frac{1}{\mu L}\int_0^L \int_{\mathbf{R}} \sigma p(t,y',\sigma)\,d\sigma\,dy' + \frac{1}{L}\,V(t)\right] \frac{\partial p}{\partial \sigma}(t,y,\sigma) \\ &+ D(p)\,\frac{\partial^2 p}{\partial \sigma^2}(t,y,\sigma) - H_1 p(t,y,\sigma) + \frac{D(p)}{\alpha}\delta_0, \end{aligned} \tag{57}$$

which is a family indexed by $y\,\in]0,L[$ of partial differential equations in (t,σ) coupled through the term $\displaystyle\int_0^L \int_{\mathbf{R}} \sigma p(t,y',\sigma)\,d\sigma\,dy'$.

For completeness, and although it is not the focus of the present work, we provide the reader with the following existence and uniqueness result.

Proposition 3.1 *Let the initial data p_0 satisfy for all $y \in]0,L[$ the conditions (18) together with the following assumption*

$$\left\{ \begin{array}{l} \textit{There exists a positive constant } \eta \textit{ such that} \\ \alpha \displaystyle\inf_{\substack{y\in]0,L[\\ \chi\in\mathbf{R}}} \int_{|\sigma+\chi|>1} p_0(y,\sigma)\,d\sigma \geq \eta > 0\ . \end{array} \right. \tag{58}$$

Then there exists a unique global-in-time solution $p(t,y,\sigma)\geq 0$ to equations (57) with initial datum $p_0(y,\sigma)$, that for all $y\in]0,L[$ belongs to $L^\infty_{loc}([0,+\infty[,L^1_\sigma\cap L^2_\sigma)\cap L^2_{loc}([0,+\infty[,H^1_\sigma)$. In addition, for all $y\in]0,L[$, $p\in L^\infty_{loc}([0,+\infty[,L^\infty_\sigma)\cap C^0([0,+\infty[,L^1_\sigma\cap L^2_\sigma)$, $\int_{\mathbf{R}} p(t,y,\sigma)\,d\sigma=1$ for all $t>0$, $D(p)\in C^0([0,+\infty[)$ and for every $T>0$ $\min_{0\leq t\leq T} D(p(t))>0$. In addition $\sigma\, p\in L^\infty_{loc}([0,+\infty[,L^1_\sigma)$.

We only sketch the proof of the above proposition.

Sketch of proof of Proposition 3.1

Step 1.

We fix $T>0$, $y\in]0,L[$, and a function $f\in L^2([0,T])$. We consider the application θ from $L^2([0,T])$ into itself, that to any function $g\in L^2([0,T])$ associates the function

$$\theta(g)(t)=\int_{\mathbf{R}} p(t,\sigma)\sigma\,d\sigma,$$

where $p(t,\sigma)$ is the unique solution to (6) with $b(t)=-\frac{1}{\mu}\,g(t)+\frac{1}{\mu}\,f(t)+\frac{1}{L}\,V(t)$, i.e.

$$\frac{\partial p}{\partial t}(t,\sigma)=-\Big(-\frac{1}{\mu}\,g(t)+\frac{1}{\mu}\,f(t)+\frac{1}{L}\,V(t)\Big)\frac{\partial p}{\partial\sigma}(t,\sigma)+D(p)\,\frac{\partial^2 p}{\partial\sigma^2}(t,\sigma)-H_1p(t,\sigma)+\frac{D(p)}{\alpha}\delta_0,$$

for the initial condition $p(0,\sigma)=p_0(y,\sigma)$. Integrating (formally) the equation once multiplied by σ we get

$$\frac{d}{dt}\theta(g)(t)=\frac{d}{dt}\int_{\mathbf{R}} p\sigma=-\frac{1}{\mu}\,g(t)+\frac{1}{\mu}\,f(t)+\frac{1}{L}\,V(t)-\int_{|\sigma|\geq 1} p\sigma,$$

and then

$$\frac{d}{dt}\theta(g)(t)+\theta(g)(t)=b(t)+\int_{|\sigma|\leq 1} p\sigma. \tag{59}$$

As $p\geq 0$ and $\int_{\mathbf{R}} p=1$, one obtains

$$\frac{d}{dt}\theta(g)(t)+\theta(g)(t)\leq 1+|b(t)|,$$

and consequently

$$|\theta(g)(t)|\leq\Big(1+\int_{\mathbf{R}}|\sigma|\,p_0(y,\sigma)\,d\sigma\Big)+\sqrt{t}\,\|b\|_{L^2(0,t)}$$

Thus,

$$\|\theta(g)\|_{L^2(0,T)}\leq\sqrt{2T}\,\Big(1+\int_{\mathbf{R}}|\sigma|\,p_0(y,\sigma)\,d\sigma\Big)+T\,\Big(\frac{1}{\mu}\|f\|_{L^2(0,T)}+\frac{1}{L}\|V\|_{L^2(0,T)}+\|g\|_{L^2(0,T)}\Big). \tag{60}$$

We can therefore choose T sufficiently small so that θ maps the unit ball of $L^2(0,T)$ into itself. Let us now consider g_1 and g_2 in the unit ball of $L^2(0,T)$ and denote by p_1 and p_2 the associated probability densities. We deduce from (59) that

$$\frac{d}{dt}\left(\theta(g_1)-\theta(g_2)\right)+\left(\theta(g_1)-\theta(g_2)\right)=-\frac{1}{\mu}\,(g_1-g_2)+\int_{|\sigma|\leq 1}(p_1-p_2)\sigma.$$

We now use [4, equation (2.35)] to claim that there exists a constant $C(T)$ depending of $\|p_0\|_{L^\infty}$, $\|f\|_{L^2(0,T)}$, $\|V\|_{L^2(0,T)}$, η, μ and α, which goes to zero when T goes to zero, such that for all $t\in[0,T]$

$$\int_{\mathbf{R}}(p_1(t,\sigma)-p_2(t,\sigma))^2\,d\sigma\leq C(T)\,\|g_1-g_2\|^2_{L^2(0,T)}.$$

It follows that for T small enough, θ is a contraction of the unit ball of $L^2(0,T)$ into itself, and therefore has a unique fixed point. We thus obtain, for almost all $y \in]0,L[$, a local existence and uniqueness result for the equation

$$\frac{\partial p}{\partial t}(t,\sigma) = -\left(-\frac{1}{\mu}\int_{\mathbf{R}} p\sigma + \frac{1}{\mu}f(t) + \frac{1}{L}V(t)\right)\frac{\partial p}{\partial \sigma}(t,\sigma) + D(p)\frac{\partial^2 p}{\partial \sigma^2}(t,\sigma) - H_1 p(t,\sigma) + \frac{D(p)}{\alpha}\delta_0 \tag{61}$$

with initial condition $p(0,\sigma) = p_0(y,\sigma)$. Now, we use equation (59) again to get

$$\frac{d}{dt}\int_{\mathbf{R}} \sigma p + \left(1+\frac{1}{\mu}\right)\int_{\mathbf{R}} \sigma p = \frac{1}{\mu}f + \frac{1}{L}V + \int_{|\sigma|\le 1} p\sigma. \tag{62}$$

This equation shows that the growth of $\int_{\mathbf{R}} \sigma p$ is controled. We can then use the same technique as in [3] to prove the global in time existence and uniqueness of the solution to (61) with initial condition $p(0,\sigma) = p_0(y,\sigma)$.

Step 2.

We now introduce the function Ψ from $L^2([0,T])$ into itself, that to any function $f \in L^2([0,T])$ associates the function

$$\Psi(f)(t) = \frac{1}{L}\int_0^L \int p(t,y,\sigma)\sigma \, d\sigma \, dy,$$

where $p(t,y,\sigma)$ is the unique solution to

$$\begin{aligned}\frac{\partial p}{\partial t}(t,y,\sigma) = & -\left(-\frac{1}{\mu}\int_{\mathbf{R}} p\sigma + \frac{1}{\mu}f(t) + \frac{1}{L}V(t)\right)\frac{\partial p}{\partial \sigma}(t,y,\sigma) \\ & + D(p)\frac{\partial^2 p}{\partial \sigma^2}(t,y,\sigma) - H_1 p(t,y,\sigma) + \frac{D(p)}{\alpha}\delta_0(\sigma)\end{aligned}$$

with initial condition $p(0,y,\sigma) = p_0(y,\sigma)$. Integrating (62) over $]0,L[$, we get

$$\frac{d}{dt}\Psi(f) + \left(1+\frac{1}{\mu}\right)\Psi(f) = \frac{1}{\mu}f + \frac{1}{L}V + \frac{1}{L}\int_0^L \int_{|\sigma|\le 1} p\sigma.$$

We then argue as in Step 1 to show that equation (57) with initial condition $p(0,y,\sigma) = p_0(y,\sigma)$ has a unique global-in-time solution.

3.2 Long time limit

We are now in position to determine the long time behaviour of the solution to (57). Our result is the following.

Theorem 3.1 *For a large enough coefficient α, for a large enough viscosity μ, for V_∞ small enough, for the initial data $p_0 - p_\infty$ small enough (all these conditions will be made explicit in the course of the proof below), we have: if $V(t)$ converges to V_∞ as t goes to infinity while satisfying*

$$|V(t) - V_\infty| \quad \text{small enough for all } t > 0,$$

the unique regular solution $(u(t,\cdot), p(t,\cdot,\cdot))$ to (8) converges as time t goes to infinity to the stationnary state (u_∞, p_∞) defined by (10) and (11).

In addition, if $V(t)$ converges exponentially, the same holds for the convergence of $u(t,\cdot)$ and $p(t,\cdot,\cdot)$.

Proof of Theorem 3.1

We denote by $b_\infty = \dfrac{V_\infty}{L}$ and by $b(t,y) = \dfrac{\partial u}{\partial y}(t,y)$, where u is the solution to (8), so that

$$b(t,y) - b_\infty = -\frac{1}{\mu}\left(\tau(t,y) - \tau_\infty\right) + \frac{1}{\mu} f(t),$$

where

$$f(t) = \frac{1}{L}\int_0^L \left(\tau(t,y') - \tau_\infty\right) dy' + \frac{\mu}{L}\left(V(t) - V_\infty\right).$$

We assume

- that $\alpha \geq \alpha_c$ and $|b_\infty| \leq b_\infty^\alpha$ so that (40) holds together with a second condition, that will be made precise below in (66),
- that $\sup\limits_{y\in]0,L[} S_2(t=0,y)$ is small such that (44) holds for all $y \in]0,L[$,
- that $\mu \geq \mu_c(\alpha, b_\infty)$ for

$$\mu_c(\alpha, b_\infty) = \max(\mu_1(\alpha, b_\infty), \mu_2(\alpha, b_\infty), \mu_3(\alpha, b_\infty)),$$

  where the μ_i that will be made precise below in (68), (71), (75) respectively.

For such a set of parameters and data $(\alpha, b_\infty, S_2(0,y)_{y\in[0,L]}, \mu)$, we assume in addition that p_0 is close enough to p_∞ and that $|V(0) - V_\infty|$ is small enough in such a way that

$$\left|f(0)\right| = \left|\frac{1}{L}\int_0^L \left(\tau(0,y') - \tau_\infty\right) dy' + \frac{\mu}{L}\left(V(0) - V_\infty\right)\right| \leq \frac{1}{2}. \tag{63}$$

Note that for this purpose $p_0(0,y)$ may be chosen independent from y, but it is not necessary. Let us denote by T_f the maximal time such that

$$\left|f(t)\right| \leq 1, \quad \text{on } [0,T_f]. \tag{64}$$

We now fix $y \in [0,L]$ and make use of the arguments of Section 2. All the quantities below are now parameterized by y, but we do not always make this dependence explicit. We introduce the maximal time $T_y \leq T_f$ such that, at point y, (45) holds on $[0,T]$. As indicated, T_y may depend on y, and we are now going to show that $T_y = T_f$ and thus is independent of $y \in [0,L]$.

At this stage, we may apply the arguments of Section 2 and obtain (49) i.e.

$$\frac{dS_2}{dt} + \frac{1}{4} S_2 \leq C_\infty^{\alpha,b_\infty}\left(|b(t) - b_\infty| + |b_\infty|\, K_\infty^{\alpha,b_\infty}\sqrt{S_2}\right)\sqrt{S_2 + 4}\sqrt{S_2}.$$

Next we remark that in this coupled case

$$\begin{aligned}
\left|\tau - \tau_\infty\right| &= \left|\int_{\mathbf{R}} (p - p_\infty)\sigma\, d\sigma\right| \\
&\leq \sqrt{\int_{\mathbf{R}} p_\infty\, \sigma^2}\sqrt{\int_{\mathbf{R}} \left(\frac{p}{p_\infty}\right)^2 p_\infty} \\
&= \hat{C}_\infty^{\alpha,b_\infty}\sqrt{S_2(t)},
\end{aligned}$$

where $\hat{C}_\infty^{\alpha,b_\infty} = \left(\int_{\mathbf{R}} p_\infty\, \sigma^2\right)^{1/2}$ does not depend on y because p_∞ does not either, thus

$$|b(t) - b_\infty| \leq \frac{1}{\mu}\hat{C}_\infty^{\alpha,b_\infty}\sqrt{S_2(t)} + \frac{1}{\mu}|f(t)|.$$

Therefore the above inequality reads

$$\frac{dS_2}{dt} + \frac{1}{4}\,S_2 \leq C_\infty^{\alpha,b_\infty} \left(\frac{1}{\mu}\,|f(t)| + \big(|b_\infty|\,K_\infty^{\alpha,b_\infty} + \frac{1}{\mu}\,\hat{C}_\infty^{\alpha,b_\infty}\big)\sqrt{S_2}\right)\,\sqrt{S_2+4}\,\sqrt{S_2}.$$

We now remark that for μ large enough and b_∞ small enough, which amounts here to choose V_∞ small enough, we have

$$C_\infty^{\alpha,b_\infty}\,\big(|b_\infty|\,K_\infty^{\alpha,b_\infty} + \frac{1}{\mu}\,\hat{C}_\infty^{\alpha,b_\infty}\big)\,\sqrt{S_2+4} \leq \frac{1}{8}, \tag{65}$$

for $t \in [0,T]$. This is achieved by noting that, on the one hand,

$$C_\infty^{\alpha,b_\infty}\,|b_\infty|\,K_\infty^{\alpha,b_\infty}\,\sqrt{S_2+4} \;\leq\; 2\,C_\infty^{\alpha,b_\infty}\,|b_\infty|\,K_\infty^{\alpha,b_\infty}\sqrt{\frac{1}{64\,(K_\infty^{\alpha,b_\infty})^2}+1} \;\leq\; \frac{1}{16}, \tag{66}$$

for b_∞ small enough. On the other hand, because of (45) we have for $t \in [0,T_y]$,

$$64\,(K_\infty^{\alpha,b_\infty})^2\,S_2(t) \leq 4,$$

thus

$$\frac{1}{\mu}\,C_\infty^{\alpha,b_\infty}\,\hat{C}_\infty^{\alpha,b_\infty}\,\sqrt{S_2+4} \leq \frac{2}{\mu}\,C_\infty^{\alpha,b_\infty}\,\hat{C}_\infty^{\alpha,b_\infty}\,\sqrt{\frac{1}{64\,(K_\infty^{\alpha,b_\infty})^2}+1} \leq \frac{1}{16} \tag{67}$$

the latter inequality holding for

$$\mu \geq \mu_1(\alpha,b_\infty) = 32\,C_\infty^{\alpha,b_\infty}\,\hat{C}_\infty^{\alpha,b_\infty}\,\sqrt{\frac{1}{64\,(K_\infty^{\alpha,b_\infty})^2}+1}. \tag{68}$$

Summing up (66) and (67), we obtain (65). From (65), we can then deduce that, in the time interval we work,

$$\frac{d\sqrt{S_2}}{dt} + \frac{1}{16}\sqrt{S_2} \leq c_0^{\alpha,b_\infty}\,\frac{C_\infty^{\alpha,b_\infty}}{2}\,\frac{1}{\mu}\,|f(t)| \tag{69}$$

$$\leq c_0^{\alpha,b_\infty}\,\frac{C_\infty^{\alpha,b_\infty}}{2}\,\frac{1}{\mu}, \tag{70}$$

with c_0^{α,b_∞} given by (52). This shows, as in Section 2, that

$$\sqrt{S_2(t)} \leq \sqrt{S_2(0)} \;+\; 8\,c_0^{\alpha,b_\infty}\,C_\infty^{\alpha,b_\infty}\,\frac{1}{\mu},$$

and thus that for

$$\mu \geq \mu_2(\alpha,b_\infty) = 128\;c_0^{\alpha,b_\infty}\,C_\infty^{\alpha,b_\infty}\,K_\infty^{\alpha,b_\infty}\,\frac{1}{\min\left(\frac{1}{2D_\infty^{\alpha,b_\infty}}\,,\,1\,,\,\frac{1-M_\infty^{\alpha,b_\infty}}{M_\infty^{\alpha,b_\infty}}\right)}, \tag{71}$$

we have

$$8\,K_\infty^{\alpha,b_\infty}\,\sqrt{S_2(t)} \leq \frac{3}{2}\,\min\left(\frac{1}{2D_\infty^{\alpha,b_\infty}}\,,\,1\,,\,\frac{1-M_\infty^{\alpha,b_\infty}}{M_\infty^{\alpha,b_\infty}}\right),$$

which contradicts the maximality of T_y in view of (45), unless $T_y = T_f$.

Therefore, we have proved that under the above conditions the bounds (45) on $S_2(t,y)$ holds *uniformly* with respect to $y \in]0,L[$.

This now allows us to redo the above argument in the following way. On $[0, T_f]$, we have obtained (69) namely

$$\frac{d\sqrt{S_2}}{dt} + \frac{1}{16}\sqrt{S_2} \leq c_0^{\alpha,b_\infty} \frac{C_\infty^{\alpha,b_\infty}}{2} \frac{1}{\mu} |f(t)|, \tag{72}$$

Now

$$\begin{aligned} |f(t)| &\leq \frac{1}{L} \int_0^L \left|\tau(t,y') - \tau_\infty\right| dy' + \frac{\mu}{L} \left|V(t) - V_\infty\right| \\ &\leq \frac{1}{L} \hat{C}_\infty^{\alpha,b_\infty} \int_0^L \sqrt{S_2(t,y')}\, dy' + \frac{\mu}{L} \left|V(t) - V_\infty\right|. \end{aligned} \tag{73}$$

Thus we have

$$\frac{d\sqrt{S_2}}{dt} + \frac{1}{16}\sqrt{S_2} \leq c_0^{\alpha,b_\infty} \frac{C_\infty^{\alpha,b_\infty}}{2} \frac{1}{\mu} \frac{1}{L} \hat{C}_\infty^{\alpha,b_\infty} \int_0^L \sqrt{S_2(t,y')}\, dy' + c_0^{\alpha,b_\infty} \frac{C_\infty^{\alpha,b_\infty}}{2} \frac{1}{L} \left|V(t) - V_\infty\right|. \tag{74}$$

We now integrate this inequality over $y' \in [0, L]$ to get

$$\begin{aligned} \frac{d}{dt} \int_0^L \sqrt{S_2(t,y')}\, dy' + \frac{1}{16} \int_0^L \sqrt{S_2(t,y')}\, dy' &\leq c_0^{\alpha,b_\infty} \frac{C_\infty^{\alpha,b_\infty}}{2} \frac{1}{\mu} \hat{C}_\infty^{\alpha,b_\infty} \int_0^L \sqrt{S_2(t,y')}\, dy' \\ &\quad + c_0^{\alpha,b_\infty} \frac{C_\infty^{\alpha,b_\infty}}{2} \left|V(t) - V_\infty\right|. \end{aligned}$$

On the above inequality, it is then simple to show that for

$$\mu \geq \mu_3(\alpha, b_\infty) = 16\, c_0^{\alpha,b_\infty}\, C_\infty^{\alpha,b_\infty}\, \hat{C}_\infty^{\alpha,b_\infty}, \tag{75}$$

we obtain

$$\frac{d}{dt} \int_0^L \sqrt{S_2(t,y')}\, dy' + \frac{1}{32} \int_0^L \sqrt{S_2(t,y')}\, dy' \leq c_0^{\alpha,b_\infty} \frac{C_\infty^{\alpha,b_\infty}}{2} \left|V(t) - V_\infty\right| \tag{76}$$

and thus are able to estimate

$$X(t) = \int_0^L \sqrt{S_2(t,y')}\, dy'$$

only in terms of $\left|V(t) - V_\infty\right|$ and of the constants c_0^{α,b_∞} and $C_\infty^{\alpha,b_\infty}$. Indeed, it follows from (76) that

$$X(t) \leq X(0)e^{-t/32} + c_0^{\alpha,b_\infty} \frac{C_\infty^{\alpha,b_\infty}}{2} \int_0^t e^{-(t-s)/32} \left|V(s) - V_\infty\right| ds\,.$$

We then insert this estimate in the right-hand side of (73) and get

$$|f(t)| \leq \frac{1}{L} \hat{C}_\infty^{\alpha,b_\infty} \left(X(0)e^{-t/32} + c_0^{\alpha,b_\infty} \frac{C_\infty^{\alpha,b_\infty}}{2} \int_0^t e^{-(t-s)/32} \left|V(s) - V_\infty\right| ds \right) + \frac{\mu}{L} \left|V(t) - V_\infty\right|$$

and this gives the control on $S_2(t,y)$ *for all* y, uniformly in y. The rest of the proof is easy.

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