LAURICELLA - SARAN TRIPLE HYPERGEOMETRIC FUNCTIONS OF MATRIX ARGUMENTS -II

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ABSTRACT

The present paper carries ahead our previous studies [7,8] of the Lauricella-Saran functions of matrix arguments. The Lauricella-Saran functions F_N, F_P and F_S have been defined for the matrix arguments case. Six results have been proved here- one each for the functions F_N, F_P and F_S, two for the function F_G, and a transformation relation for the function F_K of matrix arguments.

INTRODUCTION

We have already defined the Lauricella- Saran functions F_K, F_G, F_E, F_F, F_M and F_T of matrix arguments. Now, we are defining the Lauricella-Saran functions F_N, F_P and F_S with matrix arguments. Some properties of these functions will be studied besides discussing some properties of the functions F_G and F_K. All the matrices appearing in this paper are $(p \times p)$ real symmetric positive definite matrices and the meanings of all the other symbols used are the same as in the works of Mathai [2,3].

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1. Preliminary Definitions

DEFINITION 1.1: The Lauricella-Saran function F_N of matrix arguments

$$F_N = F_N(a_1, a_2, a_3, b_1, b_2, b_1; c_1, c_2, c_2; -X, -Y, -Z)$$

is defined as that class of functions which has the following matrix transform (M-transform):

$$M(F_N) = \int_{X>0}\int_{Y>0}\int_{Z>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} \times$$
$$|Z|^{\rho_3-(p+1)/2} F_N(a_1, a_2, a_3, b_1, b_2, b_1; c_1, c_2, c_2; -X, -Y, -Z) \times$$
$$dXdYdZ$$
$$= \frac{\Gamma_p(a_1-\rho_1)\Gamma_p(a_2-\rho_2)\Gamma_p(a_3-\rho_3)\Gamma_p(b_1-\rho_1-\rho_3)}{\Gamma_p(a_1)\Gamma_p(a_2)\Gamma_p(a_3)\Gamma_p(b_1)} \times$$
$$\frac{\Gamma_p(b_2-\rho_2)\Gamma_p(c_1)\Gamma_p(c_2)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(b_2)\Gamma_p(c_1-\rho_1)\Gamma_p(c_2-\rho_2-\rho_3)}$$

for Re $(a_i-\rho_i, b_1-\rho_1-\rho_3, b_2-\rho_2, c_1-\rho_1, c_2-\rho_2-\rho_3, \rho_i)$

$> (p-1)/2,\ i = 1, 2, 3.$ (1.1)

DEFINITION 1.2:

$$F_P = F_P(a_1, a_2, a_1, b_1, b_1, b_2; c_1, c_2, c_2; -X, -Y, -Z)$$

$$M(F_P) = \int_{X>0}\int_{Y>0}\int_{Z>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} \times$$
$$|Z|^{\rho_3-(p+1)/2} F_P(a_1, a_2, a_1, b_1, b_1, b_2; c_1, c_2, c_2; -X, -Y, -Z) \times$$
$$dXdYdZ \tag{1.2}$$
$$= \frac{\Gamma_p(a_1-\rho_1-\rho_3)\Gamma_p(a_2-\rho_2)\Gamma_p(b_1-\rho_1-\rho_2)}{\Gamma_p(a_1)\Gamma_p(a_2)\Gamma_p(b_1)} \times$$
$$\frac{\Gamma_p(b_2-\rho_3)\Gamma_p(c_1)\Gamma_p(c_2)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(b_2)\Gamma_p(c_1-\rho_1)\Gamma_p(c_2-\rho_2-\rho_3)}$$

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for Re $(a_1-\rho_1-\rho_3, a_2-\rho_2, b_1-\rho_1-\rho_2, b_2-\rho_3, c_1-\rho_1,$
$c_2-\rho_2-\rho_3, \rho_i) > (p-1)/2,\ i=1,2,3.$

DEFINITION 1.3:

$$F_S = F_S(a_1,a_2,a_2,b_1,b_2,b_3;c_1,c_1,c_1;-X,-Y,-Z)$$

$$M(F_S)=\int_{X>0}\int_{Y>0}\int_{Z>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times$$
$$|Z|^{\rho_3-(p+1)/2}F_S(a_1,a_2,a_2,b_1,b_2,b_3;c_1,c_1,c_1;-X,-Y,-Z)\times$$
$$dXdYdZ$$
$$=\frac{\Gamma_p(a_1-\rho_1)\Gamma_p(a_2-\rho_2-\rho_3)\Gamma_p(b_1-\rho_1)}{\Gamma_p(a_1)\Gamma_p(a_2)\Gamma_p(b_1)}\times$$
$$\frac{\Gamma_p(b_2-\rho_2)\Gamma_p(b_3-\rho_3)\Gamma_p(c_1)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(b_2)\Gamma_p(b_3)\Gamma_p(c_1-\rho_1-\rho_2-\rho_3)}$$

for Re $(a_1-\rho_1, a_2-\rho_2-\rho_3, b_1-\rho_1, b_2-\rho_2, b_3-\rho_3, c_1-\rho_1$
$-\rho_2-\rho_3, \rho_i) > (p-1)/2,\ i=1,2,3.$ (1.3)

2. Results

THEOREM 2.1:

$$F_N(a_1,a_2,a_3,b_1,b_2,b_1;c_1,c_2,c_2;-X,-Y,-Z)$$
$$=\frac{1}{\Gamma_p(a_3)\Gamma_p(b_1)\Gamma_p(b_2)}\int_{R_1>0}\int_{R_2>0}\int_{R_3>0}e^{-\mathrm{tr}(R_1+R_2+R_3)}\times$$
$$|R_1|^{a_3-(p+1)/2}|R_2|^{b_1-(p+1)/2}|R_3|^{b_2-(p+1)/2}\,{}_1F_1(a_1;c_1; \tag{2.1}$$
$$-R_2^{1/2}XR_2^{1/2})\Phi_3(a_2;c_2;-R_3^{1/2}YR_3^{1/2},-R_2^{1/2}R_1^{1/2}ZR_1^{1/2}R_2^{1/2})\times$$
$$dR_1dR_2dR_3$$

for Re $(a_3,b_1,b_2) > (p-1)/2.$

PROOF: Taking the M-transform of the right side of eq.(2.1) with respect to the variables X, Y, Z and the parameters ρ_1, ρ_2, ρ_3 respectively, we obtain,

$$\begin{aligned}&\int_{X>0}\int_{Y>0}\int_{Z>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}|Z|^{\rho_3-(p+1)/2}\times\\&{}_1F_1(a_1;c_1;-R_2^{1/2}XR_2^{1/2})\Phi_3(a_2;c_2;-R_3^{1/2}YR_3^{1/2},\\&-R_2^{1/2}R_1^{1/2}ZR_1^{1/2}R_2^{1/2})dXdYdZ\end{aligned} \tag{2.2}$$

Applying the transformations,

$$X_1 = R_2^{1/2}XR_2^{1/2}, Y_1 = R_3^{1/2}YR_3^{1/2}, Z_1 = R_2^{1/2}R_1^{1/2}ZR_1^{1/2}R_2^{1/2};$$

$$\text{with, } dX_1 = |R_2|^{(p+1)/2}dX, dY_1 = |R_3|^{(p+1)/2}dY, dZ_1 = |R_2|^{(p+1)/2}|R_1|^{(p+1)/2}dZ; \text{ and, } |X_1| = |R_2||X|, |Y_1| = |R_3||Y|,$$

$|Z_1| = |R_2||R_1||Z|,$ to the above expression followed by writing the M-transforms of the ${}_1F_1$ and Φ_3 functions generates,

$$\begin{aligned}&|R_2|^{-\rho_1-\rho_3}|R_3|^{-\rho_2}|R_1|^{-\rho_3}\frac{\Gamma_p(a_1-\rho_1)\Gamma_p(c_1)\Gamma_p(\rho_1)}{\Gamma_p(a_1)\Gamma_p(c_1-\rho_1)}\times\\&\frac{\Gamma_p(a_2-\rho_2)\Gamma_p(c_2)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(a_2)\Gamma_p(c_2-\rho_2-\rho_3)}\end{aligned} \tag{2.3}$$

Substituting this expression on the right side of eq.(2.1), subsequently, integrating out R_1, R_2 and R_3 by using a Gamma integral yields $M(F_N)$ as given by eq.(1.1).

THEOREM 2.2:

$$F_P(a_1, a_2, a_1, b_1, b_1, b_2; c_1, c_2, c_2; -X, -Y, -Z)$$

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$$=\frac{1}{\Gamma_p(a_1)\Gamma_p(b_1)}\int_{R_1>0}\int_{R_2>0} e^{-\mathrm{tr}(R_1+R_2)}\left|R_1\right|^{a_1-(p+1)/2}\times$$

$$\left|R_2\right|^{b_1-(p+1)/2}\ {}_0F_1(\ ;c_1;-R_2^{1/2}R_1^{1/2}XR_1^{1/2}R_2^{1/2})\times \tag{2.4}$$

$$\Phi_2(a_2,b_2;c_2;-R_2^{1/2}YR_2^{1/2},-R_1^{1/2}ZR_1^{1/2})dR_1dR_2$$

for Re $(a_1,b_1)>(p-1)/2$.

PROOF: Taking the M-transform of the right side of eq.(2.4) with respect to the variables X,Y,Z and the parameters ρ_1,ρ_2,ρ_3 respectively, we get,

$$\int_{X>0}\int_{Y>0}\int_{Z>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}|Z|^{\rho_3-(p+1)/2}\times$$

$${}_0F_1(\ ;c_1;-R_2^{1/2}R_1^{1/2}XR_1^{1/2}R_2^{1/2})\Phi_2(a_2,b_2;c_2;-R_2^{1/2}YR_2^{1/2}, \tag{2.5}$$

$$-R_1^{1/2}ZR_1^{1/2})dXdYdZ$$

The application of the transformations,

$$X_1=R_2^{1/2}R_1^{1/2}XR_1^{1/2}R_2^{1/2},Y_1=R_2^{1/2}YR_2^{1/2},Z_1=R_1^{1/2}ZR_1^{1/2};$$

to the above expression followed by writing of the M-transforms of the ${}_0F_1$ and Φ_2 functions yields,

$$\left|R_2\right|^{-\rho_1-\rho_2}\left|R_1\right|^{-\rho_1-\rho_3}\frac{\Gamma_p(c_1)\Gamma_p(\rho_1)}{\Gamma_p(c_1-\rho_1)}\frac{\Gamma_p(a_2-\rho_2)}{\Gamma_p(a_2)\Gamma_p(b_2)}\times$$

$$\frac{\Gamma_p(b_2-\rho_3)\Gamma_p(c_2)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(c_2-\rho_2-\rho_3)} \tag{2.6}$$

Putting back this expression on the right side of eq.(2.4) consequently, integrating out R_1 and R_2 by using a Gamma integral yields $M(F_P)$ as given by eq.(1.2).

THEOREM 2.3:

$$\begin{aligned}
&F_S(a_1,a_2,a_2,b_1,b_2,b_3;c_1,c_1,c_1;-X,-Y,-Z)\\
&=\frac{1}{\Gamma_p(a_1)\Gamma_p(a_2)\Gamma_p(b_1)\Gamma_p(b_2)\Gamma_p(b_3)}\int_{R_1>0}\cdots(5)\cdots\int_{R_5>0}\times\\
&e^{-tr(R_1+\cdots+R_5)}\left|R_1\right|^{a_1-(p+1)/2}\left|R_2\right|^{a_2-(p+1)/2}\times\\
&\left|R_3\right|^{b_1-(p+1)/2}\left|R_4\right|^{b_2-(p+1)/2}\left|R_5\right|^{b_3-(p+1)/2}\times\\
&{}_0F_1(\,;c_1;-R_3^{1/2}R_1^{1/2}XR_1^{1/2}R_3^{1/2}-R_4^{1/2}R_2^{1/2}YR_2^{1/2}R_4^{1/2}\\
&-R_5^{1/2}R_2^{1/2}ZR_2^{1/2}R_5^{1/2})dR_1\cdots dR_5
\end{aligned}\tag{2.7}$$

for Re $(a_1,a_2,b_1,b_2,b_3)>(p-1)/2.$

PROOF: Taking the M-transform of the right side of eq.(2.7) with respect to the variables X,Y,Z and the parameters ρ_1,ρ_2,ρ_3 respectively, we acquire,

$$\begin{aligned}
&\int_{X>0}\int_{Y>0}\int_{Z>0}\left|X\right|^{\rho_1-(p+1)/2}\left|Y\right|^{\rho_2-(p+1)/2}\left|Z\right|^{\rho_3-(p+1)/2}\times\\
&{}_0F_1(\,;c_1;-R_3^{1/2}R_1^{1/2}XR_1^{1/2}R_3^{1/2}-R_4^{1/2}R_2^{1/2}YR_2^{1/2}R_4^{1/2}\\
&-R_5^{1/2}R_2^{1/2}ZR_2^{1/2}R_5^{1/2})dXdYdZ
\end{aligned}\tag{2.8}$$

On using the transformations,

$$\begin{aligned}
&X_1=R_3^{1/2}R_1^{1/2}XR_1^{1/2}R_3^{1/2},\ Y_1=R_4^{1/2}R_2^{1/2}YR_2^{1/2}R_4^{1/2},\\
&Z_1=R_5^{1/2}R_2^{1/2}ZR_2^{1/2}R_5^{1/2};
\end{aligned}$$

in the last expression and then applying the theorem (3.3) page 55 of Mathai [3], produces,

$$\left|R_3\right|^{-\rho_1}\left|R_1\right|^{-\rho_1}\left|R_4\right|^{-\rho_2}\left|R_2\right|^{-\rho_2-\rho_3}\left|R_5\right|^{-\rho_3}\times \frac{\Gamma_p(c_1)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(c_1-\rho_1-\rho_2-\rho_3)} \qquad (2.9)$$

Substitution of this expression on the right side of eq.(2.7) followed by the use of a Gamma integral to integrate out $R_1,\cdots,R_5$ ultimately leads to $M(F_S)$ as given in eq.(1.3).

THEOREM 2.4:

$$F_G(a,a,a,b_1,b_2,b_3;c_1,c_2,c_2;-X,-Y,-Z)$$

$$=\frac{1}{\Gamma_p(a)}\int_{U>0}e^{-tr(U)}\left|U\right|^{a-(p+1)/2}{}_1F_1(b_1;c_1;-U^{1/2}XU^{1/2})\times \qquad (2.10)$$

$$\Phi_2(b_2,b_3;c_2;-U^{1/2}YU^{1/2},-U^{1/2}ZU^{1/2})dU$$

for $\text{Re }(a)>(p-1)/2.$

PROOF: Taking the M-transform of the right side of eq.(2.10) with respect to the variables X,Y,Z and the parameters ρ_1,ρ_2,ρ_3 respectively, we achieve,

$$\int_{X>0}\int_{Y>0}\int_{Z>0}\left|X\right|^{\rho_1-(p+1)/2}\left|Y\right|^{\rho_2-(p+1)/2}\left|Z\right|^{\rho_3-(p+1)/2}\times$$

$${}_1F_1(b_1;c_1;-U^{1/2}XU^{1/2})\Phi_2(b_2,b_3;c_2;-U^{1/2}YU^{1/2}, \qquad (2.11)$$

$$-U^{1/2}ZU^{1/2})dXdYdZ$$

The application of the transformations,

$$X_1=U^{1/2}XU^{1/2},Y_1=U^{1/2}YU^{1/2},Z_1=U^{1/2}ZU^{1/2};$$

to the above expression, next employing the M-transforms of the ${}_1F_1$ and Φ_2 functions generates,

$$|U|^{-\rho_1-\rho_2-\rho_3}\frac{\Gamma_p(b_1-\rho_1)\Gamma_p(c_1)\Gamma_p(\rho_1)\Gamma_p(b_2-\rho_2)}{\Gamma_p(b_1)\Gamma_p(c_1-\rho_1)}\times \frac{\Gamma_p(b_3-\rho_3)\Gamma_p(c_2)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(b_2)\Gamma_p(b_3)\Gamma_p(c_2-\rho_2-\rho_3)} \tag{2.12}$$

Replacing this expression on the right side of eq.(2.10), then integrating out U by the help of a Gamma integral gives $M(F_G)$ as given by eq.(1.2) of the authors' paper [7].

THEOREM 2.5:

(i)
$$\begin{aligned}&\lim_{\alpha\to\infty} F_G(\alpha,\alpha,\alpha,b_1,b_2,b_3;c_1,c_2,c_2;-\frac{X}{\alpha},-\frac{Y}{\alpha},-\frac{Z}{\alpha})\\ &= e^{-\mathrm{tr}(X)}\lim_{\alpha\to\infty} F_G(\alpha,\alpha,\alpha,c_1-b_1,b_2,b_3;c_1,c_2,c_2;\frac{X}{\alpha},-\frac{Y}{\alpha},-\frac{Z}{\alpha})\end{aligned} \tag{2.13}$$

(ii)
$$\begin{aligned}&= e^{-\mathrm{tr}(Y)}\lim_{\alpha\to\infty} F_G[\alpha,\alpha,\alpha,b_1,c_2-b_2-b_3,b_3;c_1,c_2,c_2;\\ &-\frac{X}{\alpha},\frac{Y}{\alpha},-\frac{(Z-Y)}{\alpha}]\end{aligned} \tag{2.14}$$

(iii)
$$\begin{aligned}&\lim_{\alpha\to\infty} F_G(\alpha,\alpha,\alpha,b_1,b_2,b_3;c_1,c_2,c_2;-\frac{X}{\alpha},-\frac{Y}{\alpha},-\frac{Y}{\alpha})\\ &= e^{-\mathrm{tr}(Y)}{}_1F_1(b_1;c_1;-X){}_1F_1(c_2-b_2-b_3;c_2;Y)\end{aligned} \tag{2.15}$$

(iv)
$$\begin{aligned}&\lim_{\alpha\to\infty} F_G(\alpha,\alpha,\alpha,b_1,b_2,b_3;c_1,c_2,c_2;-\frac{X}{\alpha},-\frac{Y}{\alpha},-\frac{Z}{\alpha})\\ &= e^{-\mathrm{tr}(Z)}\lim_{\alpha\to\infty} F_G[\alpha,\alpha,\alpha,b_1,b_2,c_2-b_2-b_3;c_1,c_2,c_2;\\ &-\frac{X}{\alpha},-\frac{(Y-Z)}{\alpha},\frac{Z}{\alpha}]\end{aligned} \tag{2.16}$$

(v)
$$\begin{aligned}&= e^{-\mathrm{tr}(X+Y)}\lim_{\alpha\to\infty} F_G[\alpha,\alpha,\alpha,c_1-b_1,c_2-b_2-b_3,b_3;c_1,c_2,c_2;\\ &\frac{X}{\alpha},\frac{Y}{\alpha},-\frac{(Z-Y)}{\alpha}]\end{aligned} \tag{2.17}$$

(vi)
$$= e^{-\mathrm{tr}(X+Z)} \lim_{\alpha\to\infty} F_G[\alpha,\alpha,\alpha,c_1-b_1,b_2,c_2-b_2-b_3;c_1,c_2,c_2; \frac{X}{\alpha},-\frac{(Y-Z)}{\alpha},\frac{Z}{\alpha}] \tag{2.18}$$

PROOF: To prove this theorem we will use eq.(1.10) of our paper [7].

(i) This result is obtained by the application of the transformation $I-U=U_1$ to the above equation.

(ii) We utilize the transformations $V_1 = I-V-W, W_1 = W;$ with, $dV_1dW_1 = dVdW;$ in eq.(1.10) of our paper [7] to see this result.

(iii) This result is obtained by putting $Z = Y$ in eq.(2.14), afterwards employing the transformation $W_2 = (I-V_1)^{-1/2}W_1(I-V_1)^{-1/2}$ followed by integrating out of W_2 by using a type-1 Beta integral and interpreting the consequent expression in the light of theorem 2.3.4 page 42 of Mathai [3].

(iv) To deduce this result we apply the transformations, $V_1 = V, W_1 = I-V-W;$ to eq.(1.10) of our paper [7], and suitably interpret the ensuing expression as per the same equation.

(v) This result follows by employing the transformations $U_1 = I-U,$ $V_1 = I-V-W, W_1 = W;$ to eq.(1.10) of our paper [7].

(vi) We have this result in a similar fashion as eq.(2.17) is deduced.

THEOREM 2.6: A transformation theorem-

$$F_K(a_1,a_2,a_2,b_1,b_2,b_1;c_1,c_2,c_3;-X,-Y,-Z)$$
$$= |I+X|^{-b_1} F_K[c_1-a_1,a_2,a_2,b_1,b_2,b_1;c_1,c_2,c_3; (I+X)^{-1/2}X(I+X)^{-1/2},-Y,-(I+X)^{-1/2}Z(I+X)^{-1/2}] \tag{2.19}$$

PROOF: In order to prove this theorem we first define the function F_K through an integral representation:

$$
\begin{aligned}
&F_K(a_1,a_2,a_2,b_1,b_2,b_1;c_1,c_2,c_3;-X,-Y,-Z)\\
&=\frac{\Gamma_p(c_1)\Gamma_p(c_2)\Gamma_p(c_3)}{\Gamma_p(a_1)\Gamma_p(a_2)\Gamma_p(b_2)\Gamma_p(c_1-a_1)\Gamma_p(c_2-b_2)\Gamma_p(c_3-a_2)}\times\\
&\int_0^I\int_0^I\int_0^I |U|^{a_1-(p+1)/2}\,|V|^{a_2-(p+1)/2}\,|T|^{b_2-(p+1)/2}\times\\
&|I-U|^{c_1-a_1-(p+1)/2}\,|I-V|^{c_3-a_2-(p+1)/2}\times\\
&|I-T|^{c_2-b_2-(p+1)/2}\left|I+Y^{1/2}TY^{1/2}\right|^{-a_2}\Big|I+X^{1/2}UX^{1/2}+\\
&(I+Y^{1/2}TY^{1/2})^{-1/2}Z^{1/2}VZ^{1/2}(I+Y^{1/2}TY^{1/2})^{-1/2}\Big|^{-b_1}\times\\
&dUdVdT \qquad\qquad (2.20)
\end{aligned}
$$

for $\mathrm{Re}\,(a_1,a_2,b_2,c_1-a_1,c_2-b_2,c_3-a_2)>(p-1)/2.$

The desired result simply follows by the application of the transformation $U_1=I-U$, to the above equation and interpreting the consequent expression in accordance with eq.(2.20).

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