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IMA Preprint Series # 1996

(October 2004)

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INVERSE SCATTERING FOR VOWEL ARTICULATION WITH FREQUENCY-DOMAIN DATA

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Abstract: An inverse scattering problem is analyzed for vowel articulation in the human vocal tract. When a unit amplitude, monochromatic, sinusoidal volume velocity is sent from the glottis towards the lips, various types of scattering data are used to examine whether the cross sectional area of the vocal tract can uniquely be determined by each data set. Among the data sets considered are the absolute value of the pressure measured at a microphone placed at some distance from the lips, the pressure at the lips, and the transfer function from the glottis to the lips. In case of nonuniqueness, it is indicated what additional information may be used for the unique determination.

PACS (2003): 02.30.Zz, 43.72.Ct
Mathematics Subject Classification (2000): 34A55, 35R30, 76Q05
Keywords: Vowel articulation, Shape of vocal tract, Inverse scattering, Webster's horn equation
Short title: Inverse scattering for vowel articulation

1. INTRODUCTION

The fundamental inverse problem for vowel articulation is concerned [1-5] with the determination of the geometry of the human vocal tract from some data. In this paper, we consider various types of scattering data in the frequency domain resulting from a unit amplitude, monochromatic, sinusoidal volume velocity sent from the glottis, and we analyze whether each data set uniquely determines the shape of the vocal tract, or else, what additional information may be used for the unique recovery.

Let us use x to denote the distance from the glottis and l for the length of the vocal tract. Hence, the lips are located at a distance l from the glottis. Typically, l varies between 14 cm and 20 cm, usually smaller for children than for adults and smaller for females than for males [1,2,5]. Even though the vocal tract is not a right cylinder, to a good approximation it can be treated as one [3,4].

We will let $A(x)$ denote the cross sectional area as a function of the distance from the glottis, and we suppose that $A(x)$ is positive on $[0, l]$. Assuming that the propagation is lossless and planar (these assumptions are known [3,4] to be reasonable), the acoustics in the vocal tract is governed [1-5] by the first-order linear system of partial differential equations

$$\begin{cases} A(x)\, p_x(x,t) + \mu\, v_t(x,t) = 0, \\ A(x)\, p_t(x,t) + \mu\, c^2\, v_x(x,t) = 0, \end{cases} \tag{1.1}$$

where t is the time variable, the subscripts x and t denote the respective partial derivatives, μ is the air density, c is the speed of sound, $v(x,t)$ is the volume velocity of the air flow, and $p(x,t)$ is the pressure at location x and at time t.

The volume velocity is equal to the product of the cross sectional area with the average velocity of the air molecules crossing that area. The pressure is the force per unit cross sectional area and is exerted by the moving air molecules. The air density at room temperature is $\mu = 1.2 \times 10^{-3}$ gm/cm^3. The speed of sound varies slightly with temperature, and $c = 3.43 \times 10^4$ cm/sec in air at room temperature. In our analysis of the inverse problem, we assume that the values of μ and c are already known. There is no loss of generality to start the time at $t = 0$.

By using $v_{xt} = v_{tx}$, we can eliminate v in (1.1) and obtain Webster's horn equation

$$\frac{1}{A(x)}[A(x)\, p_x(x,t)]_x - \frac{1}{c^2}\, p_{tt}(x,t) = 0, \qquad x \in (0,l), \quad t > 0.$$

Letting

$$\Phi(x,t) := \sqrt{A(x)}\, p(x,t), \tag{1.2}$$

we find that Φ satisfies the plasma-wave equation

$$\Phi_{xx}(x,t) - \frac{1}{c^2}\, \Phi_{tt}(x,t) = Q(x)\, \Phi(x,t), \qquad x \in (0,l), \quad t > 0, \tag{1.3}$$

where we have defined

$$Q(x) := \frac{[\sqrt{A(x)}]''}{\sqrt{A(x)}}, \tag{1.4}$$

with the prime denoting the x-derivative. The quantity Q is called the relative concavity of the vocal tract or the potential. Separating the variables as

$$\Phi(x,t) := \psi(k,x)\, e^{ikct}, \tag{1.5}$$

we find that $\psi(k,x)$ satisfies the Schrödinger equation

$$\psi''(k,x) + k^2\psi(k,x) = Q(x)\, \psi(k,x), \qquad x \in (0,l). \tag{1.6}$$

The frequency ν is measured in Hertz and related to k as $\nu = \dfrac{kc}{2\pi}$. Informally, we can refer to k as the frequency even though the proper term for k is the angular wavenumber.

In order to recover A, we will consider various types of data for $k \in \mathbf{R}^+$ resulting from the glottal volume velocity $v(0,t)$ given in (4.1). As our data sets, we consider the absolute value of the impedance at the lips, the absolute value of the pressure measured at a microphone placed at some distance from the lips, the absolute value of the pressure at the lips, the absolute value of the transfer function from the glottis to the lips, the absolute value of the impedance at the glottis, the absolute value of a Green's function for (1.3) measured at the lips, and the real or imaginary part of the reflectance at the glottis.

The inverse problem of recovery of A can be analyzed either as an inverse spectral problem or as an inverse scattering problem. In the inverse-spectral formulation, in addition to a boundary condition at the the glottis such as (2.1), a boundary condition is also imposed at the lips. The imposition of the boundary conditions at both ends of the vocal tract results in standing waves that are related to an infinite sequence of discrete frequencies. It was established by Borg [6] that Q can be recovered by using two such infinite sequences of discrete frequencies corresponding to two sets of boundary conditions. It then follows [3,4,7-11] that A can be recovered from two infinite sequences of constants. For example, such sequences can be chosen as the zeros and poles [7,8] of the input impedance or the poles and residues [9] of the input impedance.

In the the inverse-scattering formulation, a boundary condition is imposed at only one end of the vocal tract—either at the glottis or at the lips. Then, the measurement of the acoustic data used in the recovery of A is performed at the same end or at the opposite end. If the boundary condition and the measurement occur at the same end of the tract, the corresponding inverse problem is usually known as a reflection problem. On the other hand, if the boundary condition and the measurement occur at different ends, then we have a transmission problem. The methods based on the inverse scattering formulation may be applied either in the time domain or in the frequency domain, where the data set is a function of t in the former case and of k in the latter. We refer the reader to [3,4,12-15] for some approaches as time-domain reflection problems and to [16] for an approach as a time-domain transmission problem. Our approach in this paper is a frequency-domain approach, where the analysis in Sections 6-10 may be viewed as that for a transmission problem and the analysis in Sections 11 and 12 may be viewed as that for a reflection problem.

Our paper is organized as follows. In Section 2 we review some preliminary material related to the Schrödinger equation and introduce the selfadjoint boundary condition involving $\cot\alpha$ given in (2.1), the Jost solution f, the Jost function F_α, and the scattering coefficients T, L, and R. In Section 3 we briefly review the recovery of Q, $\cot\alpha$, F_α, f, T, L, and R from the data $\{|F_\alpha(k)| : \ k\in\mathbf{R}^+\}$. In Section 4 we obtain some explicit expressions for the pressure and the volume velocity in the vocal tract in terms of A, f, and F_α, and we also show that $\cot\alpha$ appearing in (2.1) is directly related to the physical parameters $A(0)$ and $A'(0)$. In Section 5 we introduce the relative area $[\eta(x)]^2$ and express it in terms of the Jost solution, $\cot\alpha$, and the scattering coefficients. In Sections 6-12 we analyze the recovery of Q, η, and A from various data sets. The data set used in Section 6 is the absolute value of the output impedance at the lips. In Section 7 it includes the absolute value of the pressure measured at a microphone placed at some distance from the lips. In Section 8 it is the absolute value of the pressure measured at the lips. In Section 9 the data set includes the absolute value of the transfer function from the glottis to the lips. In Section 10 it is the absolute value of an analog of the Green's function introduced in [17] for (1.3), in Section 11 the absolute value of the input impedance at the glottis, and in Section 12 the real or imaginary part of the reflectance. Finally, in Section 13 we present some examples to illustrate the theoretical results presented in the earlier sections.

2. PRELIMINARIES

In this section we review the scattering data related to the potential Q appearing in the Schrödinger equation on the half line $\mathbf{R}^+$ with the selfadjoint boundary condition [18-21]

$$\sin\alpha\cdot\varphi'(k,0)+\cos\alpha\cdot\varphi(k,0)=0, \tag{2.1}$$

where α is a number in the interval $(0,\pi)$ identifying the boundary condition at $x=0$. We can relate the half-line Schrödinger equation to (1.6) by assuming that $Q(x)\equiv 0$ for $x>l$. Note that the mapping $\alpha\mapsto\cot\alpha$ is one-to-one and from $(0,\pi)$ onto $\mathbf{R}$.

Let f denote the Jost solution [18-23] to the half-line Schrödinger equation. It is uniquely determined by the asymptotic conditions

$$f(k,x)=e^{ikx}[1+o(1)],\quad f'(k,x)=ik\,e^{ikx}[1+o(1)],\qquad x\to+\infty.$$

Since Q vanishes when $x>l$, we have

$$f(k,l)=e^{ikl},\quad f'(k,l)=ik\,e^{ikl}. \tag{2.2}$$

The Jost function F_α associated with the half-line Schrödinger equation with the boundary condition (2.1) is defined as [18-21]

$$F_\alpha(k):=-i[f'(k,0)+\cot\alpha\cdot f(k,0)]. \tag{2.3}$$

Let us emphasize that the subscript in F_α identifies the boundary condition at $x=0$ and it does not indicate any partial derivative. It is known [18-21] that

$$F_\alpha(-k)=-F_\alpha(k)^*,\qquad k\in\mathbf{R}, \tag{2.4}$$

where the asterisk denotes complex conjugation.

We assume that Q is real valued and integrable on $(0,l)$ and that there are no bound states for the half-line Schrödinger equation with the boundary condition (2.1). The absence of bound states is equivalent [18-21] to assuming that $F_\alpha(k)$ has no zeros on $\mathbf{I}^+$, where $\mathbf{I}^+:=i(0,+\infty)$ is the positive imaginary axis in the complex plane. It is known [19-21] that either $F_\alpha(0)\neq 0$ or $F_\alpha(k)$ has a simple zero at $k=0$; the former is known as the generic case and the latter as the exceptional case. The exceptional case corresponds to the threshold where the number of bound states can be changed by one under a small perturbation of the potential.

By using the extension $Q(x)\equiv 0$ when $x<0$, we can relate $f(k,0)$ and $f'(k,0)$ to the scattering coefficients in the full-line Schrödinger equation. We have [19,20,22,24]

$$f(k,0)=\frac{1+L(k)}{T(k)},\quad f'(k,0)=ik\,\frac{1-L(k)}{T(k)}, \tag{2.5}$$

where T and L denote the transmission coefficient and the left reflection coefficient, respectively, associated with Q. The right reflection coefficient R is given by

$$R(k) = -\frac{L(-k)\,T(k)}{T(-k)}. \tag{2.6}$$

It is known [19,20,22,24] that

$$T(-k) = T(k)^*, \quad R(-k) = R(k)^*, \quad L(-k) = L(k)^*, \qquad k \in \mathbf{R}. \tag{2.7}$$

The absence of bound states for the full-line Schrödinger equation is equivalent [19,20,22,24] for $T(k)$ not to have any poles on $\mathbf{I}^+$, and this is also equivalent [25] for $F_{\pi/2}(k)$ not to have any zeros on $\mathbf{I}^+$.

3. RECOVERY OF Q FROM $|F_\alpha|$

In the absence of bound states, the fundamental inverse scattering problem for the half-line Schrödinger equation with the selfadjoint boundary condition (2.1) consists of determining Q and $\cot\alpha$ from various types of scattering data. In this section we review the solution to this inverse problem when the data set is $\{|F_\alpha(k)| : \ k \in \mathbf{R}^+\}$.

Theorem 3.1 *Assume that Q is real valued, measurable, and integrable for $x \in (0, l)$. Then, the data set $\{|F_\alpha(k)| : \ k \in \mathbf{R}^+\}$ uniquely determines $Q(x)$ for $x \in (0, l)$ and $\cot\alpha$. The same data set also uniquely determines the corresponding Jost solution $f(k, x)$ and the scattering coefficients $T(k)$, $R(k)$, and $L(k)$.*

Below we outline some steps involved in the solution to the inverse problem stated in Theorem 3.1. As seen from (2.4), $|F_\alpha(k)|$ is an even function of $k \in \mathbf{R}$, and hence the data sets $\{|F_\alpha(k)| : \ k \in \mathbf{R}^+\}$ and $\{|F_\alpha(k)| : \ k \in \mathbf{R}\}$ are equivalent. By using the data $\{|F_\alpha(k)| : \ k \in \mathbf{R}\}$ as input in the Gel'fand-Levitan method [18-21], we form the kernel function

$$G_\alpha(x, y) := \frac{1}{\pi}\int_{-\infty}^{\infty} dk \left[\frac{k^2}{|F_\alpha(k)|^2} - 1\right](\cos kx)\,(\cos ky)\,,$$

and then solve the Gel'fand-Levitan integral equation

$$h_\alpha(x,y)+G_\alpha(x,y)+\int_0^x dz\, G_\alpha(y,z)\, h_\alpha(x,z)=0, \qquad 0\le y<x. \tag{3.1}$$

The solution to (3.1) is known [19-21] to exist and to be unique. Once $h_\alpha(x,y)$ is obtained, we recover the potential as

$$Q(x)=2\,\frac{d}{dx}h_\alpha(x,x^-), \qquad x\in(0,l),$$

where x^- indicates that the limit from the left must be used in the evaluation. We also recover the boundary condition as

$$\cot\alpha=-h_\alpha(0,0).$$

Alternatively, we can proceed [21] as follows. Let

$$\Lambda_\alpha(k):=-1+\frac{k\,f(k,0)}{F_\alpha(k)}, \qquad k\in\overline{\mathbf{C}^+}, \tag{3.2}$$

where we use $\mathbf{C}^+$ for the upper half complex plane and $\overline{\mathbf{C}^+}$ for $\mathbf{C}^+\cup\mathbf{R}$. Then

$$\text{Re}[\Lambda_\alpha(k)]=-1+\frac{k^2}{|F_\alpha(k)|^2}, \qquad k\in\mathbf{R}. \tag{3.3}$$

From the data $\{|F_\alpha(k)|:\ k\in\mathbf{R}\}$ we first construct the function $\Lambda_\alpha(k)$ via the Schwarz integral formula as

$$\Lambda_\alpha(k)=\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{dt}{t-k-i0^+}\left[-1+\frac{t^2}{|F_\alpha(t)|^2}\right], \qquad k\in\overline{\mathbf{C}^+}, \tag{3.4}$$

where the quantity $i0^+$ indicates that the values for real k should be obtained as limits from $\mathbf{C}^+$. Next, $F_\alpha(k)$ is obtained from $|F_\alpha(k)|$ by using

$$F_\alpha(k)=k\,\exp\left(\frac{-1}{\pi i}\int_{-\infty}^{\infty}dt\,\frac{\log|t/F_\alpha(t)|}{t-k-i0^+}\right), \qquad k\in\overline{\mathbf{C}^+}. \tag{3.5}$$

Then, we have

$$f(k,0)=\frac{1}{k}\,F_\alpha(k)\,[1+\Lambda_\alpha(k)], \qquad k\in\overline{\mathbf{C}^+}, \tag{3.6}$$

$$f'(k,0) = i\,F_\alpha(k)\left[1+\frac{1+\Lambda_\alpha(k)}{k}\lim_{k\to\infty}[k\,\Lambda_\alpha(k)]\right], \qquad k\in\overline{\mathbf{C}^+}. \tag{3.7}$$

$$\cot\alpha = -i\lim_{k\to\infty}[k\,\Lambda_\alpha(k)], \tag{3.8}$$

where the limit in (3.8) can be evaluated in any manner in $\overline{\mathbf{C}^+}$. Having both $f(k,0)$ and $f'(k,0)$ in hand, we can construct all the quantities that are relevant in the scattering theory for the Schrödinger equation. For example, the scattering coefficients for the full-line Schrödinger equation can be obtained as

$$T(k) = \frac{2ik}{ik\,f(k,0)+f'(k,0)}, \quad L(k) = \frac{ik\,f(k,0)-f'(k,0)}{ik\,f(k,0)+f'(k,0)}, \tag{3.9}$$

$$R(k) = \frac{-ik\,f(-k,0)-f'(-k,0)}{ik\,f(k,0)+f'(k,0)}. \tag{3.10}$$

Having obtained such quantities, we can construct the potential by using any one of the various methods available [19,20,22,24]. For example, we can use the Faddeev-Marchenko method [19,20,22,24] and get

$$Q(x) = -2\,\frac{d}{dx}K(x,x^+), \qquad x\in\mathbf{R},$$

where $K(x,y)$ is obtained by solving the (left) Faddeev-Marchenko integral equation

$$K(x,y)+\hat R(x+y)+\int_x^\infty dz\,\hat R(y+z)\,K(x,z)=0, \qquad -\infty<x<y, \tag{3.11}$$

with the kernel

$$\hat R(y) := \frac{1}{2\pi}\int_{-\infty}^\infty dk\,R(k)\,e^{iky}.$$

The Jost solution $f(k,x)$ can directly be obtained from $K(x,y)$ as

$$f(k,x) = e^{ikx}+\int_x^\infty dy\,K(x,y)\,e^{iky}. \tag{3.12}$$

Let us remark that that, in order to obtain $\{\Lambda_\alpha(k):\ k\in\mathbf{R}\}$ from $\{|F_\alpha(k)|:\ k\in\mathbf{R}\}$, instead of using (3.4) we can equivalently construct the real and imaginary parts of $\Lambda_\alpha(k)$ via (3.3) and

$$\operatorname{Im}[\Lambda_\alpha(k)] = \frac{-1}{\pi}\,\text{CPV}\int_{-\infty}^\infty \frac{dt}{t-k}\left[-1+\frac{t^2}{|F_\alpha(t)|^2}\right], \qquad k\in\mathbf{R},$$

where CPV indicates that the integral must be evaluated as a Cauchy principal value. Consequently, $\cot\alpha$ can be recovered from the equivalent form of (3.8) given by

$$\cot\alpha = \lim_{k\to\infty} \left(k\,\mathrm{Im}[\Lambda_\alpha(k)]\right).$$

4. PRESSURE AND VOLUME VELOCITY

When the vocal tract area function A is known, via (1.4) we can evaluate the potential Q, solve the corresponding Schrödinger equation, and obtain the Jost solution $f(k,x)$. In this section, with the help $f(k,x)$, we express the pressure and volume velocity corresponding to the glottal volume velocity

$$v(0,t) = e^{ikct}, \qquad t > 0. \tag{4.1}$$

It is known [19,20,22] that $f(k,\cdot)$ and $f(-k,\cdot)$ are linearly independent for each $k \in \overline{\mathbf{C}^+} \setminus \{0\}$. Hence, the general solution to (1.6) can be written as a linear combination of $f(k,\cdot)$ and $f(-k,\cdot)$. From (1.2), (1.5), and (1.6), we see that the pressure has the form

$$p(x,t) = P(k,x)\,e^{ikct}, \tag{4.2}$$

with

$$P(k,x) = \frac{1}{\sqrt{A(x)}}\,[a(k)\,f(-k,x) + b(k)\,f(k,x)]\,, \tag{4.3}$$

where $a(k)$ and $b(k)$ are coefficients to be determined. When $x \geq l$, the pressure $p(x,t)$ should be a wave traveling outward from the lips and should not contain the part proportional to $e^{ik(ct+x)}$ traveling into the mouth. Thus, with the help of (2.2) we see that we must have $b(k) \equiv 0$ in (4.3). Hence, (4.3) is reduced to

$$P(k,x) = \frac{1}{\sqrt{A(x)}}\,a(k)\,f(-k,x), \qquad x \in [0,l]. \tag{4.4}$$

Our aim is to determine the value of $a(k)$ in terms of the pressure $P(k,l+r)$ measured by a microphone placed at a radial distance r from the lips. The relationship between the pressure measured at the microphone and the volume velocity at the lips is explicitly known and is given by (cf. (3.1) of [5])

$$p(l+r,t) = \frac{\mu}{4\pi r}\, v_t(l, t-r/c), \tag{4.5}$$

where we recall that μ is the air density and c is the sound speed.

From (4.1), (4.2), and the first line of (1.1), for the x-derivative of the pressure at the glottis we get

$$P'(k,0) = -\frac{ikc\mu}{A(0)}. \tag{4.6}$$

Note that from (4.4) through differentiation we obtain

$$P'(k,0) = \frac{1}{\sqrt{A(0)}}\, a(k) \left[f'(-k,0) - \frac{A'(0)}{2\,A(0)}\, f(-k,0) \right], \tag{4.7}$$

where we have used

$$\frac{[\sqrt{A(x)}]'}{\sqrt{A(x)}} = \frac{A'(x)}{2\,A(x)}. \tag{4.8}$$

A comparison of (4.7) with (2.3) shows that, by choosing

$$\cot\alpha = -\frac{A'(0)}{2\,A(0)} = -\frac{[\sqrt{A(x)}]'\big|_{x=0}}{\sqrt{A(0)}}, \tag{4.9}$$

we can write (4.7) as

$$P'(k,0) = \frac{i}{\sqrt{A(0)}}\, a(k)\, F_\alpha(-k). \tag{4.10}$$

Comparing (4.6) and (4.10) we get

$$a(k) = -\frac{ck\mu}{\sqrt{A(0)}\, F_\alpha(-k)},$$

and hence we can write (4.4) in the equivalent form

$$P(k,x) = -\frac{c\mu k\, f(-k,x)}{\sqrt{A(x)}\,\sqrt{A(0)}\, F_\alpha(-k)}, \qquad x \in [0,l]. \tag{4.11}$$

Using (4.2) and (4.11) in the first line of (1.1), we get

$$v_t(x,t) = \frac{ck\, A(x)\, e^{ikct}}{\sqrt{A(0)}\, F_\alpha(-k)}\, \frac{d}{dx}\left[\frac{f(-k,x)}{\sqrt{A(x)}}\right], \qquad x\in[0,l], \quad t>0. \tag{4.12}$$

In particular, from (4.12) we obtain

$$v_t(l,t-r/c)=\frac{ck\sqrt{A(l)}\,e^{ikc(t-r/c)}}{\sqrt{A(0)}\,F_\alpha(-k)}\left[f'(-k,l)-\frac{A'(l)}{2\,A(l)}\,f(-k,l)\right],\qquad t>r/c. \tag{4.13}$$

Using (2.2) in (4.13) we have

$$v_t(l,t-r/c)=-\frac{ck\sqrt{A(l)}\,e^{ikc(t-r/c-l/c)}}{\sqrt{A(0)}\,F_\alpha(-k)}\left[ik+\frac{A'(l)}{2\,A(l)}\right],\qquad t>r/c. \tag{4.14}$$

Finally, comparing (4.5) and (4.14) we get

$$p(l+r,t)=-\frac{ck\mu\sqrt{A(l)}\,e^{ikc(t-r/c-l/c)}}{4\pi r\,\sqrt{A(0)}\,F_\alpha(-k)}\left[ik+\frac{A'(l)}{2\,A(l)}\right],\qquad t>r/c, \tag{4.15}$$

or equivalently, with the help of (4.2), we have

$$F_\alpha(-k)=-\frac{ck\mu\sqrt{A(l)}\,e^{-ik(r+l)}}{4\pi r\,\sqrt{A(0)}\,P(k,l+r)}\left[ik+\frac{A'(l)}{2\,A(l)}\right]. \tag{4.16}$$

From (4.16) we can conclude that k appears as ik in $P(k,l+r)$ and hence

$$P(-k,l+r)=P(k,l+r)^*,\qquad k\in\mathbf{R}. \tag{4.17}$$

We emphasize that $P(k,x)$ given in (4.11) is valid only when $x\in[0,l]$, and hence $P(k,l+r)$ is not obtained from (4.11) by replacing x by $l+r$ there. Finally, we remark that, with the help of (4.1) and (4.12), we obtain

$$v(x,t)=-\frac{i\,\sqrt{A(x)}\,e^{ikct}}{\sqrt{A(0)}\,F_\alpha(-k)}\left[f'(-k,x)-\frac{A'(x)}{2\,A(x)}\,f(-k,x)\right],\qquad x\in[0,l],\quad t>0. \tag{4.18}$$

5. AREA AND RELATIVE AREA

Let us view (1.4) as the zero-energy Schrödinger equation, and consider the initial-value problem

$$\begin{cases} y''=Q(y)\,y, & x\in(0,l),\\ y(0)=\sqrt{A(0)}, & y'(0)=-\sqrt{A(0)}\,\cot\alpha, \end{cases} \tag{5.1}$$

where $\cot\alpha$ is the quantity in (4.9). Let $y_1(x)$ and $y_2(x)$ be any two linearly independent solutions to (5.1) on the interval $[0,l]$. Then, the unique solution to (5.1) can be written as

$$y(x) = \frac{\sqrt{A(0)}}{[y_1(x);y_2(x)]}\begin{vmatrix} 0 & y_1(x) & y_2(x) \\ -1 & y_1(0) & y_2(0) \\ \cot\alpha & y_1'(0) & y_2'(0) \end{vmatrix}, \tag{5.2}$$

where $[F;G] := FG' - F'G$ denotes the Wronskian. Let us define

$$\eta(x) := \frac{\sqrt{A(x)}}{\sqrt{A(0)}}. \tag{5.3}$$

We will refer to $[\eta(x)]^2$ as the relative area of the vocal tract. Let us remark that we can write (5.3) in the equivalent form

$$A(x) = A(0)\,[\eta(x)]^2, \qquad x \in [0,l]. \tag{5.4}$$

Recall that the Wronskian of any two solutions to the Schrödinger equation is independent of x, and $[y_1(x);y_2(x)] \neq 0$ if and only if y_1 and y_2 are linearly independent on $[0,l]$. For example, we can choose y_1 and y_2 as the zero-energy Jost solutions $g_{\rm l}(0,x)$ and $g_{\rm r}(0,x)$, respectively, for the full-line Schrödinger equation where the potential agrees with $Q(x)$ on the interval $(0,l)$, is zero when $x<0$, but is a real-valued, measurable, integrable function with a finite first moment when $x>l$. Let $\tau(k)$, $\ell(k)$, and $\rho(k)$ be the corresponding transmission coefficient, the left reflection coefficient, and the right reflection coefficient, respectively. Generically, we have $\tau(0)=0$ or equivalently $[g_{\rm l}(0,x);g_{\rm r}(0,x)] \neq 0$. In the exceptional case, we have $\tau(0)\neq 0$ or equivalently $[g_{\rm l}(0,x);g_{\rm r}(0,x)]=0$.

In the generic case, using [19,20,22,24]

$$[g_{\rm r}(k,x);g_{\rm l}(k,x)] = \frac{2ik}{\tau(k)}, \tag{5.5}$$

$$g_r(k,x) = \frac{g_{\rm l}(-k,x)+\rho(k)\,g_{\rm l}(k,x)}{\tau(k)},$$

$$g_{\rm r}(0,0)=1, \quad g_{\rm r}'(0,0)=0,$$

we can write (5.2) as

$$\eta(x)=\begin{vmatrix} 0 & -\frac{i}{2}\,\dot\tau(0)\,g_1(0,x) & i\,\dot g_1(0,x)-\frac{i}{2}\,\dot\rho(0)\,g_1(0,x) \\ 1 & g_1(0,0) & 1 \\ -\cot\alpha & g_1'(0,0) & 0 \end{vmatrix}, \tag{5.6}$$

where the overdot denotes the k-derivative.

In the exceptional case, we can choose

$$y_1(x)=g_1(0,x),\quad y_2(x)=g_1(0,x)\int_0^x \frac{dz}{g_1(0,z)^2}.$$

In this case, we have

$$y_1(0)=\frac{1+\ell(0)}{\tau(0)},\quad y_1'(0)=0,\quad y_2(0)=0,\quad y_2'(0)=\frac{1}{y_1(0)}=\frac{\tau(0)}{1+\ell(0)},$$

with $y_1(0)\neq 0$ because [19,20,22,24] we have $-1<\ell(0)<1$. Hence, from (5.2) we get

$$\eta(x)=g_1(0,x)\begin{vmatrix} 0 & 1 & \int_0^x \frac{dz}{g_1(0,z)^2} \\ 1 & g_1(0,0) & 0 \\ -\cot\alpha & 0 & \frac{1}{g_1(0,0)} \end{vmatrix}. \tag{5.7}$$

Theorem 5.1 *The relative area* $[\eta(x)]^2$ *for* $x\in[0,l]$ *is uniquely determined by the data* $\{|F_\alpha(k)|:\ k\in\mathbf{R}^+\}$*. Equivalently,* $\eta(x)$ *for* $x\in[0,l]$ *is uniquely determined from* $\{Q(x):x\in(0,l),\cot\alpha\}$*, where* $\cot\alpha$ *is the constant in (4.9).*

PROOF: From Theorem 3.1 we know that $\{|F_\alpha(k)|:\ k\in\mathbf{R}^+\}$ uniquely determines the potential $Q(x)$ for $x\in(0,l)$ and the constant $\cot\alpha$. From (4.8), (4.9), and (5.3) we see that $\eta(0)=1$ and $\eta'(0)=-\cot\alpha$. Thus, $\eta(x)$ is uniquely obtained by solving the initial value problem in (5.1) in the special case $A(0)=1$. ∎

It is possible to construct $\eta(x)$ from $\{|F_\alpha(k)|:\ k\in\mathbf{R}^+\}$ as follows. With the help of (3.4)-(3.12), we can construct the corresponding right reflection coefficient $R(k)$, the transmission coefficient $T(k)$, and the Jost solution $f(k,x)$. Hence, in the generic case every term appearing on the right hand side of (5.6) can be constructed from $\{|F_\alpha(k)|:\ k\in\mathbf{R}^+\}$, and we get

$$\eta(x)=\begin{vmatrix} 0 & -\dfrac{i}{2}\,\dot T(0)\,f(0,x) & i\,\dot f(0,x)-\dfrac{i}{2}\,\dot R(0)\,f(0,x)\\ 1 & f(0,0) & 1\\ -\cot\alpha & f'(0,0) & 0\end{vmatrix}. \tag{5.8}$$

In the exceptional case, from (5.7) we get

$$\eta(x)=f(0,x)\begin{vmatrix} 0 & 1 & \displaystyle\int_0^x \frac{dz}{f(0,z)^2}\\ 1 & f(0,0) & 0\\ -\cot\alpha & 0 & \dfrac{1}{f(0,0)}\end{vmatrix}.$$

Note that we can write the Jost solution $f(k,x)$ as a linear combination of $g_1(k,x)$ and $g_1(-k,x)$, where $g_1(k,x)$ is the quantity appearing in (5.5). With the help of (2.2) we obtain

$$f(k,x)=\frac{e^{ikl}}{2ik}\begin{vmatrix} 0 & g_1(k,x) & g_1(-k,x)\\ 1 & g_1(k,l) & g_1(-k,l)\\ ik & g_1'(k,l) & g_1'(-k,l)\end{vmatrix}. \tag{5.9}$$

We can also express $F_\alpha(k)$ with the help of $g_1(k,x)$. To do so, we can obtain $f(k,0)$ and $f'(k,0)$ from (5.9) and use (2.3) to get $F_\alpha(k)$. Alternatively, by using [cf. (2.5)]

$$g_1(k,0)=\frac{1+\ell(k)}{\tau(k)},\quad g_1'(k,0)=ik\,\frac{1-\ell(k)}{\tau(k)},$$

we can write $F_\alpha(k)$ with the help of the transmission and left reflection coefficients associated with $g_1(k,x)$.

6. RECOVERY FROM THE IMPEDANCE AT THE LIPS

The impedance at the lips is defined as

$$Z(k,l):=\frac{p(l,t)}{v(l,t)}. \tag{6.1}$$

Using (4.2), (4.11), and (4.18) in (6.1), we get

$$Z(k,l) = \frac{2ick\mu}{2ik\,A(l) + A'(l)}. \tag{6.2}$$

Thus, we can only hope to get $A(l)$ and $A'(l)$ from $Z(k,l)$. We can refer to $Z(k,l)$ as the output impedance because the volume velocity is from the glottis to the lips. In Section 11 we will analyze the impedance at the glottis, which we can identify as the input impedance.

Note that from (6.2) we get

$$|Z(k,l)|^2 = \frac{4c^2k^2\mu^2}{4k^2A(l)^2 + A'(l)^2}, \qquad k \in \mathbf{R}. \tag{6.3}$$

By using (6.2) at two distinct real k values, say k_1 and k_2, we can recover $A(l)$ and $A'(l)$ by solving a linear algebraic system and get

$$A(l) = \frac{c\mu}{k_1 - k_2}\left[\frac{k_1}{Z(k_1,l)} - \frac{k_2}{Z(k_2,l)}\right], \quad A'(l) = \frac{2ic\mu k_1 k_2}{k_1 - k_2}\left[\frac{1}{Z(k_2,l)} - \frac{1}{Z(k_1,l)}\right]. \tag{6.4}$$

On the other hand, if we only know $|Z(k,l)|$ without knowing its phase, then from (6.3) we get $A(l)$ and $|A'(l)|$ as

$$A(l) = \sqrt{\frac{c^2\mu^2}{k_1^2 - k_2^2}\left[\frac{k_1^2}{|Z(k_1,l)|^2} - \frac{k_2^2}{|Z(k_2,l)|^2}\right]}, \tag{6.5}$$

$$A'(l)^2 = \frac{4c^2\mu^2k_1^2k_2^2}{k_1^2 - k_2^2}\left[\frac{1}{|Z(k_2,l)|^2} - \frac{1}{|Z(k_1,l)|^2}\right]. \tag{6.6}$$

As seen from (6.2), $Z(k,l)$ by itself contains no other information related to Q, η, or A.

7. RECOVERY FROM PRESSURE AT A MICROPHONE

Let us place a microphone at a radial distance r from the lips and measure at that microphone the absolute value of the pressure resulting from the glottal volume velocity given in (4.1). With the help of (4.2) it follows that this is equivalent to having $|P(k,l+r)|$ at hand for $k \in \mathbf{R}^+$. From (4.17) we see that $|P(k,l+r)|$ is an even function of $k \in \mathbf{R}$, and from (4.16) we get

$$|P(k,l+r)|^2 = \left(\frac{ck\mu}{4\pi r}\right)^2 \frac{A(l)}{A(0)\,|F_\alpha(k)|^2}\left[k^2 + \frac{A'(l)^2}{4\,A(l)^2}\right], \qquad k \in \mathbf{R}. \tag{7.1}$$

In this section we show that the data set $\{|P(k, l+r)| : \ k \in \mathbf{R}^+\}$ by itself does not uniquely determine any of $Q(x)$, $\eta(x)$, or $A(x)$, and how additional data may be used for the unique determination.

Theorem 7.1 *The data set* $\{|P(k, l+r)| : \ k \in \mathbf{R}^+, A(l), |A'(l)|\}$ *uniquely determines each of* $Q(x)$, $\eta(x)$, *and* $A(x)$ *for* $x \in (0, l)$.

PROOF: It is known (cf. (3.9) of [21]) that for any fixed $\alpha \in (0, \pi)$ we have

$$|F_\alpha(k)| = k + O(1), \qquad k \to +\infty. \tag{7.2}$$

Thus, from (7.1) we obtain

$$|P(k, l+r)| = \frac{ck\mu\sqrt{A(l)}}{4\pi r\,\sqrt{A(0)}}\left[1 + O(1/k)\right], \qquad k \to +\infty,$$

or equivalently

$$\sqrt{A(0)} = \frac{c\mu\sqrt{A(l)}}{4\pi r}\lim_{k\to+\infty}\left[\frac{k}{|P(k, l+r)|}\right]. \tag{7.3}$$

Using (7.3) in (7.1) we get

$$\frac{k^2}{|F_\alpha(k)|^2} = \frac{4\,|P(k, l+r)|^2}{4k^2 + A'(l)^2/A(l)^2}\left[\lim_{k\to+\infty}\frac{k^2}{|P(k, l+r)|^2}\right], \qquad k \in \mathbf{R}. \tag{7.4}$$

From (4.17) and (7.4) we see that our data set uniquely determines $|F_\alpha(k)|$ for $k \in \mathbf{R}$, and hence, as indicated in Theorem 3.1, $Q(x)$ is uniquely determined for $x \in (0, l)$. Next, from Theorem 5.1 it follows that $\eta(x)$ is also uniquely determined for $x \in [0, l]$. Finally, from (7.3) we see that $A(0)$ is also determined by our data set, and thus we recover $A(x)$ for $x \in [0, l]$ uniquely by using (5.4). ∎

Note that we assume that $A(l)$ and $|A'(l)|$ do not change with k, and hence they are constants. As indicated in Section 6 they can be obtained via (6.5) and (6.6) by measuring the absolute value of the impedance at the lips at two different frequencies.

Theorem 7.2 *The data set* $\{|P(k, l+r)| : \ k \in \mathbf{R}^+, |A'(l)|/A(l)\}$ *uniquely determines each of* $Q(x)$ *and* $\eta(x)$ *for* $x \in (0, l)$, *and it determines* $A(x)$ *for* $x \in [0, l]$ *up to a multiplicative constant.*

PROOF: From (2.4), (4.17), and (7.4) we see that $|F_\alpha(k)|$ for $k \in \mathbf{R}$ is uniquely determined by our data set, and hence $Q(x)$ via Theorem 3.1 and $\eta(x)$ via Theorem 5.1 are uniquely determined for $x \in (0, l)$. Furthermore, from (3.5) and (4.15) we see that if we multiply each of $A(0)$, $A(l)$, and $|A'(l)|$ by the same constant, we do not change $|p(l+r,t)|$ or equivalently we do not change $|P(k, l+r)|$ for $k \in \mathbf{R}$. Thus, our data set corresponds to the one-parameter family for $A(x)$, where the parameter $A(0)$ appears as a multiplicative parameter in (5.4). ∎

With the help of (7.4) and Theorem 7.2 we have the following conclusions.

Corollary 7.3 *Corresponding to the data set* $\{|P(k, l+r)| : \ k \in \mathbf{R}^+, A(l)\}$, *in general there exists a one-parameter family for each of* $Q(x)$, $\eta(x)$, *and* $A(x)$, *where* $|A'(l)|/A(l)$ *can be chosen as the parameter.*

Corollary 7.4 *Corresponding to the data set* $\{|P(k, l+r)| : \ k \in \mathbf{R}^+\}$, *in general there exists a two-parameter family for each of* $Q(x)$, $\eta(x)$, *and* $A(x)$, *where* $A(l)$ *and* $|A'(l)|$ *can be chosen as the parameters.*

8. RECOVERY FROM THE PRESSURE AT THE LIPS

Let us consider the recovery of $A(x)$ for $x \in [0, l]$ from the absolute value of the pressure at the lips resulting from the glottal volume velocity in (4.1). From (4.2) we see that our data set is equivalent to $\{|P(k, l)| : \ k \in \mathbf{R}^+\}$. With the help of (2.4), (2.5), (2.7), and (4.11), we notice that $|P(k, l)|$ is an even function of $k \in \mathbf{R}$, and hence we have our data actually available for $k \in \mathbf{R}$. In this section we show that this data set uniquely recovers each of $Q(x)$, $\eta(x)$, and $A(x)$ for $x \in (0, l)$, and we outline an explicit procedure to determine these quantities.

Theorem 8.1 *The data set* $\{|P(k, l)| : \ k \in \mathbf{R}^+\}$ *uniquely determines each of* $Q(x)$, $\eta(x)$, *and* $A(x)$ *for* $x \in (0, l)$.

PROOF: From (2.2), (2.4), and (4.11) we get

$$|P(k,l)| = \frac{c\mu\,|k|}{\sqrt{A(l)}\,\sqrt{A(0)}\,|F_\alpha(k)|}, \qquad k \in \mathbf{R}. \tag{8.1}$$

Using (7.2) in (8.1), we obtain

$$\sqrt{A(0)\,A(l)} = c\mu \left(\lim_{k\to+\infty} \left| \frac{1}{P(k,l)} \right| \right), \tag{8.2}$$

and hence

$$|F_\alpha(k)| = \frac{|k|}{|P(k,l)|} \left(\lim_{k\to+\infty} |P(k,l)| \right), \qquad k \in \mathbf{R}. \tag{8.3}$$

As seen from (2.4) and (8.3), by measuring the absolute value of the pressure at the lips for $k \in \mathbf{R}^+$, we get $|F_\alpha(k)|$ for $k \in \mathbf{R}$. Then, by proceeding as in Section 3, we can recover $Q(x)$ for $x \in (0,l)$ and the constant $\cot\alpha$ appearing in (4.9). Next, by proceeding as in Section 5, we determine $\eta(x)$ for $x \in (0,l)$. Note that $\sqrt{A(0)\,A(l)}$ is uniquely determined from our data via (8.2). Furthermore, as seen from (5.3), we have $\eta(l) = \sqrt{A(l)/A(0)}$. Thus, we obtain

$$A(0) = \frac{c\mu}{\eta(l)} \left(\lim_{k\to+\infty} \left| \frac{1}{P(k,l)} \right| \right),$$

and hence we get the area function uniquely via (5.4). ∎

9. RECOVERY FROM THE TRANSFER FUNCTION

The transfer function $\mathbf{T}(k,l)$ from the glottis to the lips is defined as

$$\mathbf{T}(k,l) := \frac{v(l,t)}{v(0,t)}, \tag{9.1}$$

and as we see from (4.1), (4.18), and (9.1), we have

$$\mathbf{T}(k,l) = \frac{\sqrt{A(l)}\,e^{-ikl}}{\sqrt{A(0)}\,F_\alpha(-k)} \left[-k + \frac{i}{2}\frac{A'(l)}{A(l)} \right], \qquad k \in \overline{\mathbf{C}^+}. \tag{9.2}$$

Hence, with the help of (2.4), we get

$$|\mathbf{T}(k,l)|^2 = \frac{A(l)}{A(0)\,|F_\alpha(k)|^2} \left[k^2 + \frac{A'(l)^2}{4A(l)^2} \right], \qquad k \in \mathbf{R}. \tag{9.3}$$

Using (7.2) in (9.2) we obtain

$$|\mathbf{T}(k,l)| = \frac{\sqrt{A(l)}}{\sqrt{A(0)}}\left[1+O(1/k)\right], \qquad k\to+\infty,$$

and as a result we can recover $A(0)$ as

$$A(0) = \frac{A(l)}{\lim\limits_{k\to+\infty}|\mathbf{T}(k,l)|^2}. \tag{9.4}$$

Thus, from (9.2) and (9.4), with the help of (2.4), we have

$$|F_\alpha(k)|^2 = \frac{\lim\limits_{k\to+\infty}|\mathbf{T}(k,l)|^2}{|\mathbf{T}(k,l)|^2}\left[k^2+\frac{1}{4}\frac{A'(l)^2}{A(l)^2}\right], \qquad k\in\mathbf{R}.$$

Comparing (9.3) with (7.1) we see that

$$|\mathbf{T}(k,l)|^2 = \left(\frac{4\pi r}{ck\mu}\right)^2|P(k,l+r)|^2, \qquad k\in\mathbf{R},$$

and hence we have the following conclusion.

Corollary 9.1 *For each fixed $r>0$, the information contained in the data set $\{|\mathbf{T}(k,l)|:\ k\in\mathbf{R}^+\}$ is equivalent to that in $\{|P(k,l+r)|:\ k\in\mathbf{R}^+\}$.*

In other words, measuring the absolute value of the pressure at a microphone placed at some distance from the lips is equivalent to measuring the absolute value of the transfer function from the glottis to the lips. Consequently, we have the following analogs of the results of Section 7.

Corollary 9.2 *The data set $\{|\mathbf{T}(k,l)|:\ k\in\mathbf{R}^+, A(l), |A'(l)|\}$ uniquely determines each of $Q(x)$, $\eta(x)$, and $A(x)$ for $x\in(0,l)$.*

Corollary 9.3 *The data set $\{|\mathbf{T}(k,l)|:\ k\in\mathbf{R}^+, |A'(l)|/A(l)\}$ uniquely determines each of $Q(x)$ and $\eta(x)$ for $x\in(0,l)$ and it determines $A(x)$ for $x\in[0,l]$ up to a multiplicative constant.*

Corollary 9.4 *Corresponding to the data set $\{|\mathbf{T}(k,l)|:\ k\in\mathbf{R}^+, A(l)\}$, in general there exists a one-parameter family for each of $Q(x)$, $\eta(x)$, and $A(x)$, where $|A'(l)|/A(l)$ can be chosen as the parameter.*

Corollary 9.5 *Corresponding to the data set* $\{|\mathbf{T}(k,l)| : \ k\in\mathbf{R}^+\}$, *in general there exists a two-parameter family for each of* $Q(x)$, $\eta(x)$, *and* $A(x)$, *where* $A(l)$ *and* $|A'(l)|$ *can be chosen as the parameters.*

10. RECOVERY FROM A GREEN'S FUNCTION AT THE LIPS

In this section we show that the absolute value of a Green's function for (1.3) at the lips measured for $k\in\mathbf{R}^+$ enables us to uniquely construct each of $Q(x)$, $\eta(x)$, and $A(x)$ for $x\in(0,l)$.

The Green's function at the lips can be defined [17] as the solution $\Phi(l,t)$ given in (1.2) when the glottal volume velocity is as in (4.1). Thus, from (1.2), (4.2), and (4.11), we get the Green's function at the lips as

$$\mathbf{G}(k,l;t)=\frac{-ck\mu\, e^{ik(ct-l)}}{\sqrt{A(0)}\,F_\alpha(-k)}.$$

Hence, with the help of (2.4) we obtain

$$|\mathbf{G}(k,l;t)|=\frac{c\,|k|\,\mu}{\sqrt{A(0)}\,|F_\alpha(k)|},\qquad k\in\mathbf{R}. \tag{10.1}$$

Theorem 10.1 *The data set* $\{|\mathbf{G}(k,l;t)| : \ k\in\mathbf{R}^+\}$ *uniquely determines each of* $Q(x)$, $\eta(x)$, *and* $A(x)$ *for* $x\in(0,l)$.

PROOF: From (2.4) and (10.1) it follows that $|\mathbf{G}(k,l;t)|$ is independent of t and is an even function of k on $\mathbf{R}$, and hence our data can be extended from $k\in\mathbf{R}^+$ to $k\in\mathbf{R}$. Using (7.2) in (10.1) we get

$$\sqrt{A(0)}=\frac{c\mu}{\lim\limits_{k\to+\infty}|\mathbf{G}(k,l;t)|}, \tag{10.2}$$

and hence

$$\frac{|k|}{|F_\alpha(k)|}=\frac{|\mathbf{G}(k,l;t)|}{\lim\limits_{k\to+\infty}|\mathbf{G}(k,l;t)|},\qquad k\in\mathbf{R}.$$

Thus, we get $|F_\alpha(k)|$ for $k\in\mathbf{R}$ whenever we have $|\mathbf{G}(k,l;t)|$ for $k\in\mathbf{R}^+$. Then, as in Section 3 we construct $Q(x)$ for $x\in(0,l)$ and $\cot\alpha$. Next, as in Section 5, we construct

$\eta(x)$ for $x \in [0, l]$. Finally, with the help of (5.4) and (10.2) we obtain

$$A(x) = \frac{c^2 \mu^2 \, [\eta(x)]^2}{\left(\lim_{k \to +\infty} |\mathbf{G}(k, l; t)| \right)^2}.$$

Thus, the proof is complete. ▮

11. RECOVERY FROM THE IMPEDANCE AT THE GLOTTIS

The impedance at the glottis is defined as

$$Z(k, 0) := \frac{p(0, t)}{v(0, t)}. \tag{11.1}$$

Using (4.1), (4.2), and (4.11) in (11.1) we get

$$Z(k, 0) = -\frac{ck\mu \, f(-k, 0)}{A(0) \, F_\alpha(-k)}. \tag{11.2}$$

From (2.4), (2.5), (2.7), and (11.2) we see that $|Z(k, 0)|$ is an even function of $k \in \mathbf{R}$, and hence $|Z(k, 0)|$ is known for $k \in \mathbf{R}$ if it is known for $k \in \mathbf{R}^+$. In this section we show that the information contained in $\{|Z(k, 0)| : \ k \in \mathbf{R}^+\}$ enables us to uniquely construct Q, η, and A.

Theorem 11.1 *The data set* $\{|Z(k, 0)| : \ k \in \mathbf{R}^+\}$ *uniquely determines each of* $Q(x)$, $\eta(x)$, *and* $A(x)$ *for* $x \in (0, l)$.

PROOF: With the help of (2.4) and (11.2) we get

$$|Z(k, 0)| = \frac{c \, |k| \, \mu \, |f(k, 0)|}{A(0) \, |F_\alpha(k)|}, \qquad k \in \mathbf{R}, \tag{11.3}$$

and hence, using (7.2) and the fact [18-20,22] that $f(k, 0) = 1 + O(1/k)$ as $k \to +\infty$, we obtain

$$|Z(k, 0)| = \frac{c\mu}{A(0)} \left[1 + O(1/k) \right], \qquad k \to +\infty,$$

which leads to

$$A(0) = \frac{c\mu}{\lim_{k \to +\infty} |Z(k, 0)|}. \tag{11.4}$$

Thus, we can write (11.3) as

$$\left|\frac{k\,f(k,0)}{F_\alpha(k)}\right| = \frac{|Z(k,0)|}{\lim\limits_{k\to+\infty}|Z(k,0)|}. \tag{11.5}$$

As seen from (11.5), the recovery from $|Z(k,0)|$ for $k\in\mathbf{R}^+$ is equivalent to the recovery from $|f(k,0)/F_\alpha(k)|$ for $k\in\mathbf{R}$.

Recall that we assume that the half-line Schrödinger equation with the boundary condition (2.1) does not have any bound states. In that case, it is known [21] that $kf(k,0)/F_\alpha(k)$ is analytic in $\mathbf{C}^+$, continuous in $\overline{\mathbf{C}^+}$, nonzero in $\overline{\mathbf{C}^+}\setminus\{0\}$, and it is either nonzero at $k=0$ or has a simple zero there, and

$$\frac{k\,f(k,0)}{F_\alpha(k)} = 1+O(1/k), \qquad k\to\infty \text{ in } \overline{\mathbf{C}^+}.$$

As a result, we can recover $kf(k,0)/F_\alpha(k)$ for $k\in\overline{\mathbf{C}^+}$ from its amplitude known for $k\in\mathbf{R}$ via

$$\frac{k\,f(k,0)}{F_\alpha(k)} = \exp\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} dt\,\frac{\log|t\,f(t,0)/F_\alpha(t)|}{t-k-i0^+}\right), \qquad k\in\overline{\mathbf{C}^+}.$$

Having constructed $kf(k,0)/F_\alpha(k)$ for $k\in\mathbf{R}$, we can use (3.2) and obtain $\Lambda_\alpha(k)$ for $k\in\mathbf{R}$. Next, using (3.4) we construct $|F_\alpha(k)|$ for $k\in\mathbf{R}$. Then, as in Section 3 we construct $Q(x)$ for $x\in(0,l)$, and as in Section 5 we construct $\eta(x)$ for $x\in[0,l]$. Finally, $A(x)$ is constructed via (5.4) after obtaining $A(0)$ from (11.4). ∎

12. RECOVERY FROM THE REFLECTANCE AT THE GLOTTIS

The reflectance at the glottis is defined [3,4] as the ratio of the right-moving (from the glottis towards the lips) pressure wave to the left-moving pressure wave. From (4.11) that reflectance is seen to be equal to $L(-k)$ because the Jost solution to the full-line Schrödinger equation has the extension

$$f(k,x) = \frac{1}{T(k)}\,e^{ikx} + \frac{L(k)}{T(k)}\,e^{-ikx}, \qquad x\le 0,$$

where we recall that $L(k)$ is the left reflection coefficient appearing in (2.5). In this section we show that either of the real and imaginary parts of the reflectance at the glottis known for $k \in \mathbf{R}^+$ enables us to construct $Q(x)$ uniquely for $x \in (0, l)$. On the other hand, it cannot uniquely determine either $\eta(x)$ or $A(x)$.

Theorem 12.1 *Either of the data sets* $\{\text{Re}[L(k)] : \ k \in \mathbf{R}^+\}$ *and* $\{\text{Im}[L(k)] : \ k \in \mathbf{R}^+\}$ *uniquely determines* $Q(x)$ *for* $x \in (0, l)$*. On the other hand, to each of these data sets, there correspond a one-parameter family for* $\eta(x)$ *and a two-parameter family for* $A(x)$*.*

PROOF: From (2.7) we see that $\text{Re}[L(k)]$ is an even function of k on $\mathbf{R}$ and $\text{Im}[L(k)]$ is an odd function. Hence, both of our data sets can be extended from $k \in \mathbf{R}^+$ to $k \in \mathbf{R}$. In the absence of bound states, because $Q(x) \equiv 0$ for $x < 0$, it is known [22,24] that $L(k)$ is analytic in $\mathbf{C}^+$, continuous in $\overline{\mathbf{C}^+}$, and $o(1/k)$ as $k \to \infty$ in $\overline{\mathbf{C}^+}$. Thus, we can construct $L(k)$ from either data set via the Schwarz integral formula [cf. (3.4)] as

$$L(k) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{dt\, \text{Re}[L(t)]}{t - k - i0^+} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dt\, \text{Im}[L(t)]}{t - k - i0^+}, \qquad k \in \overline{\mathbf{C}^+}.$$

We can then obtain Q uniquely, for example, via [22,24] the Faddeev-Marchenko method, as

$$Q(x) = 2\, \frac{d}{dx} B_{\text{r}}(x, 0^+), \qquad x \in \mathbf{R},$$

where $B_{\text{r}}(x, y)$ is obtained by solving the (right) Faddeev-Marchenko integral equation

$$B_{\text{r}}(x, y) + \hat{L}(-2x + y) + \int_0^{\infty} dz\, \hat{L}(-2x + y + z)\, B_{\text{r}}(x, z) = 0, \qquad x \in \mathbf{R}, \quad y > 0.$$

It is already known [19,20,22,24] that $T(k)$ and $R(k)$ can be constructed from $L(k)$ with the help of (2.6) and

$$T(k) = \exp\left(\frac{1}{2\pi i} \int_{-\infty}^{\infty} ds\, \frac{\log(1 - |L(s)|^2)}{s - k - i0^+} \right), \qquad k \in \overline{\mathbf{C}^+},$$

and hence, as we see from (5.8), $\eta(x)$ is not uniquely determined and we have the corresponding one-parameter family for $\eta(x)$ with $\cot \alpha$ being the parameter. From (5.1) we see that we have the corresponding two-parameter family for $A(x)$, where the parameters can be chosen, for example, as $A(0)$ and $A'(0)$, or as $A(l)$ and $A'(l)$. ∎

Finally, let us remark that, as indicated in (6.4), we can obtain $A(l)$ and $A'(l)$ if we know the impedance at the lips at two distinct real k values.

13. EXAMPLES

In this section we illustrate the theoretical results presented in the previous sections with some examples.

Let us use $l = 17.5$ cm, $c = 3.43 \times 10^4$ cm/sec, $\mu = 1.2 \times 10^{-3}$ gm/cm^3, $A(0) = 5$ cm^2, and $A'(0) = -0.52$ cm, and

$$Q(x) = \frac{80(7 + 3\sqrt{5})\, e^{2\sqrt{5}x}}{\left[(7 + 3\sqrt{5})\, e^{2\sqrt{5}x} - 2\right]^2}.$$

When Q is viewed as a potential of the full-line Schrödinger equation with support on $\mathbf{R}^+$, the corresponding scattering coefficients $\tau(k)$, $\rho(k)$, $\ell(k)$ and the left Jost solution $g_l(k,x)$ introduced in Section 5 are rational functions of k, and it can be verified that

$$g_l(k,x) = e^{ikx}\left[1 + \frac{i}{k + i\sqrt{5}}\, \frac{4i\sqrt{5}}{(7 + 3\sqrt{5})\, e^{2\sqrt{5}x} - 2}\right], \qquad x \geq 0,$$

$$\tau(k) = \frac{k(k + i\sqrt{5})}{(k+i)(k+2i)}, \quad \ell(k) = \frac{2}{(k+i)(k+2i)},$$

$$\rho(k) = \frac{-2(k + i\sqrt{5})}{(k+i)(k+2i)(k - i\sqrt{5})}.$$

All the quantities related to (1.1), (1.3), and (1.6) can now be explicitly evaluated. For example, the left Jost solution $g_l(k,x)$ for $x \leq 0$ for the full-line Schrödinger equation can be obtained as

$$g_l(k,x) = \frac{e^{ikx}}{\tau(k)} + \frac{\ell(k)\, e^{-ikx}}{\tau(k)}, \qquad x \leq 0.$$

Via (4.9) we get $\cot \alpha = -0.052$, $\eta(x)$ can be obtained via (5.6), $A(x)$ via (5.4), $f(k,x)$ via (5.9), $F_\alpha(k)$ via (2.3), the scattering coefficients $T(k)$, $R(k)$, and $L(k)$ via (3.9) and (3.10),

$P(k,x)$ via (4.11), $v(x,t)$ via (4.18), $|P(k,l+r)|$ via (7.1), $|P(k,l)|$ via (8.1), $|Z(k,l)|$ via (6.3), $|\mathbf{T}(k,l)|$ via (9.3), $|\mathbf{G}(k,l;t)|$ via (10.1), $|Z(k,0)|$ via (11.3), $\Lambda_\alpha(k)$ via (3.2). We also compute $A(l) = 11.596$ cm^2 and $A'(l) = 0.681$ cm. Even though all these quantities can be explicitly written in terms of elementary functions in closed forms, the corresponding expressions are too long to display here, and instead we only show some of their graphs.

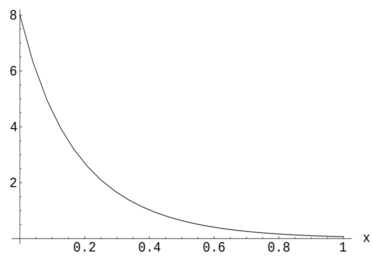

Fig. 13.1 $Q(x)$.

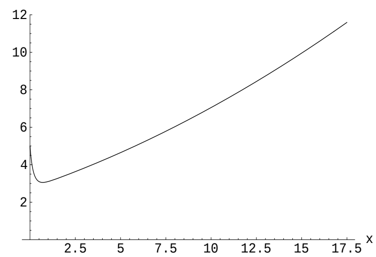

Fig. 13.2 $A(x)$.

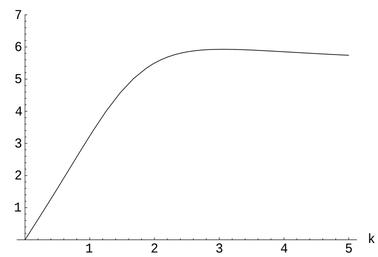

Fig. 13.3 $|P(k,l)|$.

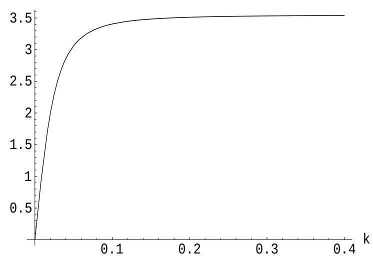

Fig. 13.4 $|Z(k,l)|$.

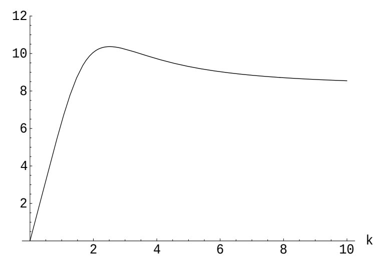

Fig. 13.5 $|Z(k,0)|$.

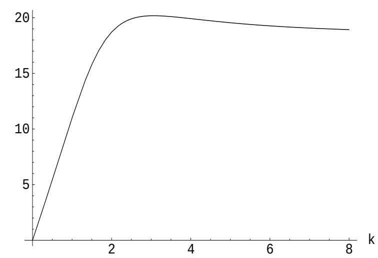

Fig. 13.6 $|\mathbf{G}(k,l;t)|$.

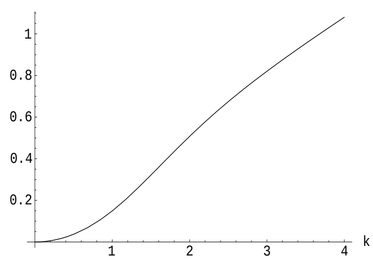

Fig. 13.7 $|P(k,l+20)|$.

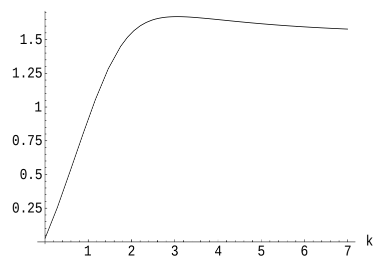

Fig. 13.8 $|\mathbf{T}(k,l)|$.

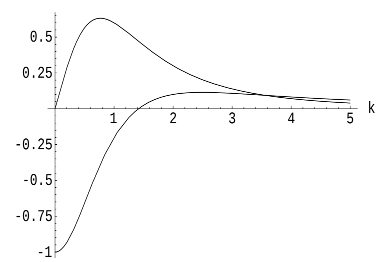

Fig. 13.9 $L(-k)$.

Let us remark that in Fig. 13.7 the absolute value of the pressure is measured at a distance of 20 cm from the lips. In Fig. 13.9 the real and imaginary parts of the reflectance are displayed, which can be told apart from $\text{Re}[L(0)] = -1$ and $\text{Im}[L(0)] = 0$.

It is known from Section 6 that the graph in Fig. 13.4 cannot determine either of

Figs. 13.1 and 13.2. We know from Section 8 that the graph in Fig 13.3 uniquely determines the graphs of Figs. 13.1 and 13.2. As we know from Section 11, the graph in Fig 13.5 uniquely determines the graphs of Figs. 13.1 and 13.2. From Section 10 we know that the graph in Fig 13.6 also uniquely determines the graphs of Figs. 13.1 and 13.2. We know from Section 9 that the graphs of Figs. 13.7 and 13.8 contain the same information, but neither is sufficient to determine uniquely either of the graphs in Figs. 13.1 and 13.2. On the other hand, as we have seen in Sections 7 and 9, the graphs in Figs. 13.4 and 13.7 together, or the graphs in Figs. 13.4 and 13.8 together, uniquely determine the graphs of Figs. 13.1 and 13.2. We know from Section 12 that either graph in Fig. 13.9 uniquely constructs the graph in Fig. 13.1, but not that in Fig. 13.2.

Acknowledgments. The author has benefited from discussions with Roy Pike, Barbara Forbes, and Paul Sacks. The research leading to this article was supported in part by the National Science Foundation under grant DMS-0204437 and the Department of Energy under grant DE-FG02-01ER45951.

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