A NOTE ON HEAT KERNEL ESTIMATES FOR SECOND-ORDER ELLIPTIC OPERATORS

By

Seick Kim

IMA Preprint Series # 1995

(September 2004)

INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS

UNIVERSITY OF MINNESOTA

514 Vincent Hall

206 Church Street S.E.

Minneapolis, Minnesota 55455–0436

Phone: 612/624-6066 Fax: 612/626-7370

URL: http://www.ima.umn.edu

A NOTE ON HEAT KERNEL ESTIMATES FOR SECOND-ORDER ELLIPTIC OPERATORS.

SEICK KIM

Abstract. We study fundamental solutions to second order parabolic systems of divergence type with time independent coefficients, and give another proof of a result by Auscher, McIntosh and Tchamitchian on the Gaussian bounds for the heat kernels of second order elliptic operators in divergence form with complex bounded measurable coefficients.

1. Introduction

In [2] Auscher, McIntosh and Tchamitchian studied the heat kernels of second order elliptic operators in divergence form with complex bounded measurable coefficients on $\mathbb{R}^n$. In particular, in the case when $n = 2$, they obtained Gaussian bounds without further assumption on the coefficients.

In this article, we give another proof of their result when $n = 2$. In fact, we prove that fundamental solutions to second-order parabolic systems of divergence type on $\mathbb{R}^2$ with time independent coefficients have Gaussian bounds.

To be precise, we consider a system of equations defined on $\mathbb{R}^n$:

$$D_t u^i - \sum_{j=1}^{N} \sum_{\alpha,\beta=1}^{n} D_{x_\alpha}(A_{ij}^{\alpha\beta}(x) D_{x_\beta} u^j) = 0 \qquad (i = 1, \ldots, N). \tag{1}$$

Here t is a real number and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. For each $\alpha, \beta = 1, \ldots, n$, we shall denote by $\boldsymbol{A}^{\alpha\beta}(x)$ an $N \times N$ matrix with (i, j) entries of $A_{ij}^{\alpha\beta}(x)$.

It is convenient to write the system (1) in a vector form

$$\mathcal{L}\boldsymbol{u} := \boldsymbol{u}_t - \sum_{\alpha,\beta=1}^{n} D_\alpha(\boldsymbol{A}^{\alpha\beta}(x) D_\beta \boldsymbol{u}) = 0, \tag{2}$$

where $\boldsymbol{u} = (u^1, \ldots, u^N)^T$. We make use of a shorthand notation

$$\left\langle \boldsymbol{A}^{\alpha\beta}(x)\boldsymbol{\xi}_\beta, \boldsymbol{\eta}_\alpha \right\rangle := \sum_{\alpha,\beta=1}^{n} \sum_{i,j=1}^{N} A_{ij}^{\alpha\beta}(x) \xi_\beta^j \eta_\alpha^i, \tag{3}$$

where $\boldsymbol{\xi}_\beta = (\xi_\beta^1, \ldots, \xi_\beta^N)^T$ and $\boldsymbol{\eta}_\alpha = (\eta_\alpha^1, \ldots, \eta_\alpha^N)^T$ for $\alpha, \beta = 1, \ldots, n$.

We assume that the system (1) is strongly parabolic; i.e., there is a number $\nu > 0$ such that

$$\nu \left|\boldsymbol{\xi}\right|^2 \le \left\langle \boldsymbol{A}^{\alpha\beta}(x)\boldsymbol{\xi}_\beta, \boldsymbol{\xi}_\alpha \right\rangle. \tag{4}$$

2000 *Mathematics Subject Classification.* Primary 35A08; Secondary 35B45.

Here, we used the notation

$$|\boldsymbol{\xi}|^2 := \sum_{\alpha=1}^{n} |\boldsymbol{\xi}_\alpha|^2 = \sum_{\alpha=1}^{n}\sum_{i=1}^{N} (\xi_\alpha^i)^2.$$

We also assume that there is a number $M > 0$ such that

$$\left|\left\langle \boldsymbol{A}^{\alpha\beta}(x,t)\boldsymbol{\xi}_\beta, \boldsymbol{\eta}_\alpha \right\rangle\right| \le M\,|\boldsymbol{\xi}|\,|\boldsymbol{\eta}|\,. \tag{5}$$

Throughout this article, we make the qualitative assumption that the coefficients $A_{ij}^{\alpha\beta}(x)$ are smooth; however we emphasize that all qualitative estimates are only allowed to depend on the ellipticity constants ν, M.

By a fundamental solution (or a fundamental matrix) $\boldsymbol{\Gamma}_t(x,y)$ to the system (2) we mean an $N \times N$ matrix of functions defined for $t > 0$ which, as a function of x, each column is a solution of (2), and is such that

$$\lim_{t\downarrow 0} \int_{\mathbb{R}^2} \boldsymbol{\Gamma}_t(x,y)\boldsymbol{f}(y)\,dy = \boldsymbol{f}(x) \tag{6}$$

for any bounded continuous function $\boldsymbol{f} = (f^1, \ldots, f^N)^T$.

We close the introduction by stating our main result.

Theorem 1. *Let $n = 2$ and let $\boldsymbol{\Gamma}_t(x,y)$ be a fundamental solution to* (2)*. Then $\boldsymbol{\Gamma}_t(x,y)$ has an upper bound*

$$|\boldsymbol{\Gamma}_t(x,y)| \le \frac{K_0}{t} e^{-k_0|x-y|^2/t}, \tag{7}$$

where $|\boldsymbol{\Gamma}_t(x,y)|$ denotes the operator norm of fundamental matrix $\boldsymbol{\Gamma}_t(x,y)$. Here, $K_0 = K_0(\nu, M)$ and $k_0 = k_0(\nu, M)$.

2. L^2-estimates for the derivatives

In this section we derive a uniform a-priori bound for spatial L^2-norm of $\boldsymbol{u}_t$ at each time slice in terms of L^2-norm of $\boldsymbol{u}$, where $\boldsymbol{u}$ is a solution to (2).

First we will show that L^2-norm of $\boldsymbol{u}_t$ is controlled by L^2-norm of $D\boldsymbol{u}$. When the coefficients are symmetric, (i.e., $A_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha}$) this is well known (see, e.g., [7, pp. 172–181] and also [4, pp. 360–364]).

However, we must note that the coefficients satisfying (4) and (5) are not necessarily symmetric. The main task in this section lies in establishing the estimate in the case when the coefficients are allowed to be non-symmetric. This result on non-symmetric case seems to be new.

Let us fix some notations. For $X_0 = (x_0, t_0) \in \mathbb{R}^{n+1}$ and $r > 0$ we denote $Q_r(X_0) = B_r(x_0) \times (t_0 - r^2, t_0)$, where $B_r(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$.

Let $\boldsymbol{u}$ be a solution to (2) in $Q_R := Q_R(X_0)$. Fix positive numbers σ, τ such that $\sigma < \tau \le R$ and let ζ be a standard smooth cut-off function such that $\zeta \equiv 1$ in Q_σ and vanishes near the parabolic boundary of Q_τ (see e.g. [4, pp. 59]). In particular, ζ should satisfy

$$0 \le \zeta \le 1; \quad |\zeta_t| + |D\zeta|^2 \le 100(\tau - \sigma)^{-2}. \tag{8}$$

Note that

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} \Big[\boldsymbol{u}_t - D_\alpha(\boldsymbol{A}^{\alpha\beta}(x) D_\beta \boldsymbol{u})\Big] \cdot \zeta^2 \boldsymbol{u}_t \\ &= \int_{\mathbb{R}^n} \zeta^2 \left|\boldsymbol{u}_t\right|^2 + \int_{\mathbb{R}^n} \Big\langle \boldsymbol{A}^{\alpha\beta}(x) D_\beta \boldsymbol{u}, D_\alpha(\zeta^2 \boldsymbol{u}_t)\Big\rangle \\ &= \int_{\mathbb{R}^n} \zeta^2 \left|\boldsymbol{u}_t\right|^2 + \int_{\mathbb{R}^n} \zeta^2 \Big\langle \boldsymbol{A}^{\alpha\beta}(x) D_\beta \boldsymbol{u}, D_\alpha \boldsymbol{u}_t\Big\rangle + 2\zeta \Big\langle \boldsymbol{A}^{\alpha\beta}(x) D_\beta \boldsymbol{u}, D_\alpha \zeta \boldsymbol{u}_t\Big\rangle . \end{aligned}$$

Therefore we estimate by using Cauchy's inequalities

$$\begin{aligned} \int_{\mathbb{R}^n} \zeta^2 \left|\boldsymbol{u}_t\right|^2 &\le M \int_{\mathbb{R}^n} \zeta^2 \left|D\boldsymbol{u}\right| \left|D\boldsymbol{u}_t\right| + 2M \int_{\mathbb{R}^n} \zeta \left|D\boldsymbol{u}\right| \left|D\zeta\right| \left|\boldsymbol{u}_t\right| \\ &\le \frac{\epsilon}{2} \int_{\mathbb{R}^n} \zeta^2 \left|D\boldsymbol{u}_t\right|^2 + \frac{M^2}{2\epsilon} \int_{\mathbb{R}^n} \zeta^2 \left|D\boldsymbol{u}\right|^2 + 2M^2 \int_{\mathbb{R}^n} \left|D\zeta\right|^2 \left|D\boldsymbol{u}\right|^2 \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^n} \zeta^2 \left|\boldsymbol{u}_t\right|^2 . \end{aligned}$$

Thus we have

$$\int_{Q_\tau} \zeta^2 \left|\boldsymbol{u}_t\right|^2 \le \epsilon \int_{Q_\tau} \zeta^2 \left|D\boldsymbol{u}_t\right|^2 + \frac{C}{\epsilon} \int_{Q_\tau} \zeta^2 \left|D\boldsymbol{u}\right|^2 + C \int_{Q_\tau} \left|D\zeta\right|^2 \left|D\boldsymbol{u}\right|^2 . \tag{9}$$

On the other hand, since $\boldsymbol{v} := \boldsymbol{u}_t$ also satisfies (2), we have the following Caccioppoli type estimates for $\boldsymbol{u}_t$:

$$\int_{Q_\tau} \zeta^2 \left|D\boldsymbol{u}_t\right|^2 \le C \int_{Q_\tau} \Big(\left|\zeta_t\right| + \left|D\zeta\right|^2\Big) \left|\boldsymbol{u}_t\right|^2 . \tag{10}$$

This is the part where we exploit the assumption that the coefficients are time independent. Combining (8), (9) and (10), we have

$$\int_{Q_\sigma} \left|\boldsymbol{u}_t\right|^2 \le \frac{C_0 \epsilon}{(\tau-\sigma)^2} \int_{Q_\tau} \left|\boldsymbol{u}_t\right|^2 + \frac{C}{\epsilon} \int_{Q_\tau} \left|D\boldsymbol{u}\right|^2 + \frac{C}{(\tau-\sigma)^2} \int_{Q_\tau} \left|D\boldsymbol{u}\right|^2 . \tag{11}$$

If we set $\epsilon = (\tau-\sigma)^2/2C_0$, we finally obtain

$$\int_{Q_\sigma} \left|\boldsymbol{u}_t\right|^2 \le \frac{1}{2} \int_{Q_\tau} \left|\boldsymbol{u}_t\right|^2 + \frac{C}{(\tau-\sigma)^2} \int_{Q_\tau} \left|D\boldsymbol{u}\right|^2 . \tag{12}$$

Here, we emphasize that $C = C(\nu, M)$ is a constant independent of σ, τ. Then by a standard iteration argument (see e.g. [5, Lemma 5.1, pp. 81]) we conclude that for $0 < r < R$ we have

$$\int_{Q_r} \left|\boldsymbol{u}_t\right|^2 \le \frac{C}{(R-r)^2} \int_{Q_R} \left|D\boldsymbol{u}\right|^2 , \tag{13}$$

where $C = C(\nu, M)$.

Now we are ready to state and prove our key lemma.

Lemma 1. *Let $\boldsymbol{u}$ be a solution of* (2) *in $Q_{2r}(X_0)$. Then*

$$\sup_{t_0 - r^2 < s < t_0} \int_{B_r(x_0)} \left|\boldsymbol{u}_t(\cdot, s)\right|^2 + \int_{Q_r(X_0)} \left|D\boldsymbol{u}_t\right|^2 \le \frac{C}{r^6} \int_{Q_{2r}(X_0)} \left|\boldsymbol{u}\right|^2 . \tag{14}$$

Proof. We may and do assume that $r = 1$ and $X_0 = (0, 0)$. By the energy estimate (see e.g. [7, pp. 139–144]) applied to $\boldsymbol{u}_t$ we obtain

$$\sup_{-1 < s < 0} \int_{B_1} \left|\boldsymbol{u}_t(\cdot, s)\right|^2 + \int_{Q_1} \left|D\boldsymbol{u}_t\right|^2 \le C \int_{Q_{3/2}} \left|\boldsymbol{u}_t\right|^2 .$$

On the other hand, the estimate (13) and then the energy estimate (this time applied to $\boldsymbol{u}$ itself) yields

$$\int_{Q_{3/2}} |\boldsymbol{u}_t|^2 \leq C \int_{Q_{7/4}} |D\boldsymbol{u}|^2 \leq C \int_{Q_2} |\boldsymbol{u}|^2 .$$

Combining together, we have the desired estimate (14). □

3. Proof of Theorem 1

Our proof is based on an approach appears in a paper [6] by Hofmann and the author. It relies on standard PDE methods and thus differs from the one appears in [1], [2] which uses a contour integral method.

First, we note that (4) and (5) remain unchanged for $\tilde{A}^{\alpha\beta}_{ij}(x) := A^{\beta\alpha}_{ji}(x)$. As it was indicated in [6, Remarks 2.1], if a local $L^2 \to L^\infty$ estimate called local boundedness property holds for solutions to the system (2), then the same property should hold for solutions to its adjoint system.

Then, it is shown in [6, Theorem 1.1] that the Gaussian bound follows from an argument based on a technique of Davies [3].

In order to get a local boundedness property for solutions to (2), we exploit the estimate (14) of the previous section. In Lemma 2 below, we rewrite (2) as $\mathcal{L}_0\boldsymbol{u} = \boldsymbol{u}_t$ where $\mathcal{L}_0$ is an elliptic operator

$$\mathcal{L}_0\boldsymbol{u} := \sum_{\alpha,\beta=1}^{n} D_\alpha(\boldsymbol{A}^{\alpha\beta}(x)D_\beta\boldsymbol{u}), \tag{15}$$

and apply the standard theory on elliptic operators defined on $\mathbb{R}^2$ to establish an a-priori Hölder estimate (16) for solutions $\boldsymbol{u}$ to (2), which particularly implies the desired local boundedness property for $\boldsymbol{u}$ (see the proof of [6, Proposition 2.1]).

We would like to mention that Auscher also makes use of elliptic regularity theory in his work [1] on Gaussian bounds in a somewhat different context.

It only remains to establish the following lemma.

Lemma 2. *Let $n = 2$ and $\boldsymbol{u}$ be a solution of (2) in $Q_{6R} = Q_{6R}(X_0)$. Then, the following a priori Hölder estimate holds:*

$$[\boldsymbol{u}]_{C^{\alpha,\alpha/2}(Q_R)} \leq \frac{C}{R^{2+\alpha}} \|\boldsymbol{u}\|_{L^2(Q_{6R})} , \tag{16}$$

where $\alpha = \alpha(\nu, M) > 0$ and $C = C(\nu, M)$.

Proof. We assume that $R = 1$ and $X_0 = (0, 0)$. The general case is recovered by a simple coordinate change $(x, t) \mapsto ((x - x_0)/R, (t - t_0)/R^2)$.

Let us denote $Q_{r,t}(X_0) = \{(x, t) \in Q_r(X_0)\}$. We may rewrite (2) as $\mathcal{L}_0\boldsymbol{u} = \boldsymbol{u}_t$ where $\mathcal{L}_0$ is an elliptic operator defined as in (15), and apply a well known theory on two dimensional linear elliptic systems (see e.g. [8, pp. 143–148]) to see

$$[\boldsymbol{u}(\cdot, \tau)]_{C^\alpha(Q_{4,\tau})} \leq C \left(\|\boldsymbol{u}(\cdot, \tau)\|_{L^2(Q_{5,\tau})} + \|\boldsymbol{u}_t(\cdot, \tau)\|_{L^2(Q_{5,\tau})} \right) \tag{17}$$

for some $\alpha = \alpha(\nu, M) > 0$. This is where the assumption $n = 2$ plays a crucial role.

By Lemma 1 we see that the right hand side of (17) is uniformly bounded for all $\tau \in (-4^2, 0)$ and thus

$$[\boldsymbol{u}(\cdot, \tau)]_{C^\alpha(Q_{4,\tau})} \leq C \|\boldsymbol{u}\|_{L^2(Q_6)} \quad \forall \tau \in (-4^2, 0). \tag{18}$$

Fix $X=(x,t)\in Q_1$ and $r\le 1$. By [6, Lemma 2.3], we have

$$\int_{Q_r(X)} |\boldsymbol{u}-\boldsymbol{u}_{X,r}|^2 \le P_0 r^2 \int_{Q_{2r}(X)} |D\boldsymbol{u}|^2, \tag{19}$$

where $\boldsymbol{u}_{X,r}$ denotes the average of $\boldsymbol{u}$ over $Q_r(X)$.

From $\mathcal{L}_0\boldsymbol{u}=\boldsymbol{u}_t$, we derive the Caccioppoli inequality (see e.g. [5, pp. 24])

$$\int_{Q_{2r,\tau}(X)} |D\boldsymbol{u}|^2 \le C\left(\frac{1}{r^2}\int_{Q_{3r,\tau}(X)} |\boldsymbol{u}-\boldsymbol{\lambda}|^2 + r^2 \|\boldsymbol{u}_t(\cdot,\tau)\|^2_{L^2(Q_{3r,\tau}(X))}\right).$$

If we set $\boldsymbol{\lambda}$ to be the average of $\boldsymbol{u}(\cdot,\tau)$ over the slice $Q_{3r,\tau}(X)$, then by (18) we have

$$\frac{1}{r^2}\int_{Q_{3r,\tau}(X)} |\boldsymbol{u}-\boldsymbol{\lambda}|^2 \le Cr^{2\alpha}[\boldsymbol{u}(\cdot,\tau)]^2_{C^\alpha(Q_{4,\tau})} \le Cr^{2\alpha}\|\boldsymbol{u}\|^2_{L^2(Q_6)}.$$

Also, by Lemma 1 we have

$$\|\boldsymbol{u}_t(\cdot,\tau)\|^2_{L^2(Q_{3r,\tau}(X))} \le \|\boldsymbol{u}_t(\cdot,\tau)\|^2_{L^2(Q_{4,\tau})} \le C\|\boldsymbol{u}\|^2_{L^2(Q_6)}.$$

Therefore, we have

$$\begin{aligned}\int_{Q_{2r}(X)} |D\boldsymbol{u}|^2 &= \int_{t-4r^2}^{t}\int_{Q_{2r,\tau}(X)} |D\boldsymbol{u}(y,\tau)|^2\,dyd\tau\\ &\le Cr^2\left(r^{2\alpha}+r^2\right)\|\boldsymbol{u}\|^2_{L^2(Q_6)} \le Nr^{2+2\alpha}\|\boldsymbol{u}\|^2_{L^2(Q_6)}.\end{aligned}$$

Hence, using (19) we conclude

$$\int_{Q_r(X)} |\boldsymbol{u}-\boldsymbol{u}_{X,r}|^2 \le Cr^{4+2\alpha}\|\boldsymbol{u}\|^2_{L^2(Q_6)}, \quad \forall r\le 1,\ \forall X\in Q_1.$$

Finally, from [6, Lemma 2.1], we get

$$[\boldsymbol{u}]_{\alpha,\alpha/2,Q_1} \le C\|\boldsymbol{u}\|_{L^2(Q_6)}.$$

The proof is complete. □

References

[1] Auscher, P. *Regularity theorems and heat kernel for elliptic operators.* J. London Math. Soc. (2) **54** (1996), no. 2, 284–296.

[2] Auscher, P.; McIntosh, A.; Tchamitchian, Ph. *Heat kernels of second order complex elliptic operators and applications.* J. Funct. Anal. 152 (1998), no. 1, 22–73.

[3] Davies, E. B. *Explicit constants for Gaussian upper bounds on heat kernels*, Amer. J. Math. **109** (1987), no. 2, 319–333.

[4] Evans, L. C. *Partial differential equations.* Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 1998.

[5] Giaquinta, M. *Introduction to regularity theory for nonlinear elliptic systems.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1993.

[6] Hofmann, S.; Kim, S. *Gaussian estimates for fundamental solutions to certain parabolic systems.* Publ. Mat. **48** (2004), 481-496.

[7] Ladyženskaja, O. A.; Solonnikov, V. A.; Ural'ceva, N. N. *Linear and quasilinear equations of parabolic type.* Translations of Mathematical Monographs; American Mathematical Society: Providence, RI, 1967; Vol. 23.

[8] Morrey, C. B., Jr. *Multiple integrals in the calculus of variations.* Springer-Verlag New York, Inc., New York 1966

Mathematics Department, University of Missouri, Columbia, Missouri 65211, USA
E-mail address: seick@math.missouri.edu
URL: http://www.math.missouri.edu/~seick/