# Majorizing Kernels & Stochastic Cascades With Applications To Incompressible Navier-Stokes Equations*†

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**Abstract**

A general method is developed to obtain conditions on initial data and forcing terms for the global existence of unique regular solutions to incompressible 3d Navier-Stokes equations. The basic idea generalizes a probabilistic approach introduced by LeJan and Sznitman (1997) to obtain weak solutions whose Fourier transform may be represented by an expected value of a stochastic cascade. A functional analytic framework is also developed which partially connects stochastic iterations and certain Picard iterates. Some local existence and uniqueness results are also obtained by contractive mapping conditions on the Picard iteration.

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# 1 Introduction & Preliminaries

We develop two related approaches to obtain global and local existence, uniqueness and regularity, including spatial analyticity, of solutions to 3-dimensional incompressible Navier-Stokes (NS) equations governing fluid velocities

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = \nu \Delta u - \nabla p + g, \quad \nabla \cdot u = 0. \tag{1}$$

One approach is probabilistic and involves the construction of a multiplicative cascade solution to a related stochastic recursion in wave number Fourier space. The other approach is based on Picard iterations. Each of these approaches involves a notion of *Fourier multiplier* which we formalize as follows.

**Definition 1.1** Let $h : W_h \subseteq \mathbf{R}^n \backslash \{0\} \to (0, \infty)$ be a Lebesgue measurable function such that the closure of $W_h$ is a semigroup and $h = 0$ on $W_h^c$ with

$$0 < h * h(\xi) < \infty, \xi \in W_h. \tag{2}$$

The reciprocal function $1/h$ is referred to as a *Fourier multiplier.*

The probabilistic approach is based upon an interpretation of the integral equation governing Fourier transformed velocities scaled by a multiplier $1/h$. This is achieved in terms of expectation values of multiplicative cascade solutions to stochastic recursions generated by certain multi-type branching random walks in Fourier space. The transitions in wave-number are of the form $\xi \to (\xi_1, \xi_2), \xi_1 + \xi_2 = \xi$, with a transition probability kernel $h(\xi_1)h(\xi_2)/h * h(\xi)$. This generalizes branching random walks in the sense of LeJan and Sznitman (1997) for $n = 3$ dimensions where $h(\xi) = |\xi|^{-2}, \xi \in W_h = \mathbf{R}^3 \backslash \{0\}$. The essential requirement for this approach is that the above indicated expected values *exist.* Existence of these expected values is obtained in the present paper by constructions of a particular class of the Fourier multipliers, referred to as *majorizing kernels,* defined below.

The second approach is a purely analytic approach in which the Fourier multiplier $1/h$ is used to identify a Banach space norm for which iterations of the expected values may, under slightly more restrictive conditions, be interpreted as Picard iterates of successive approximations on a suitably identified function space defined via particular control of the Fourier transform by a majorizing Fourier multiplier, e.g. $u \in \mathcal{S}'$ such that $|\hat{u}(\xi, t)| \leq h(\xi)$. In particular, the Picard iteration may be expressed in terms of a contraction operator on such a space. It may be noted that a different function space for Picard iteration was identified by Kato (1984) in efforts to obtain existence and uniqueness for Navier-Stokes equations.

As noted above, the probabilistic approach gives a representation of the Fourier transform $\hat{u}(\xi, t)$ of the solution of the evolution equation in the LeJan-Sznitman form of an expected value

$$\hat{u}(\xi, t) = h(\xi) E_{\xi_\theta = \xi} \mathbb{X}(\theta, t). \tag{3}$$

Here $\mathbb{X}$ is a random multiplicative functional of scalar values $m(\cdot)$ and Fourier transformed initial data and/or forcing (vector) values over the vertices of a multi-type branching random walk tree $\tau_\theta(t)$ initiated in time $t$ from a single progenitor of type $\xi_\theta = \xi$. In general the scalar and vector value factors are evaluated at the wave-number (type) of the respective vertices appearing in the tree $\tau_\theta(t)$, with the initial and forcing terms appearing at the end-nodes. The holding times between branchings are determined from the principal part of the equation, while the branching probabilities depend on the lower order and forcing terms of the equation.

The framework developed here is also more generally applicable to diverse classes of evolution equations, including certain linear parabolic and fractional diffusion equations, semilinear reaction-diffusions, and some quasi-linear equations such as incompressible Navier-Stokes equations in dimension $n \geq 2$, as well as one-dimensional Burgers' equation. The following extremely simple example is selected to illustrate some of the most basic graph theoretic and probabilistic ideas involved in this approach. It is so simple however, that the notion of Fourier multiplier is not required. Consider

$$u_t = a\Delta u + b \cdot \nabla u \quad u(x, 0) = u_0(x), \tag{4}$$

in $n \geq 1$ dimensions, where $a > 0$, and $b \in \mathbf{R}^n$ are constants. To quickly get the flavor of the method, define the spatial Fourier transform of an integrable function $f$, or its distributional extension, by $\hat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R^n}} e^{-ix\cdot\xi} f(x)dx$, $\xi \in \mathbf{R}^n$. Then, from (4) one has

$$\hat{u}(\xi, t) = \hat{u}_0(\xi) e^{-a|\xi|^2 t} + \frac{ib \cdot \xi}{a|\xi|^2} \int_0^t a|\xi|^2 e^{-a|\xi|^2 s} \hat{u}(\xi, t - s) ds. \tag{5}$$

Now consider the random linear tree $\tau_\theta(t)$ rooted at a vertex $\theta$ of type $\xi_\theta = \xi$ which, after an exponential length of time is replaced by a single vertex $\langle 1 \rangle$

of the same type $\xi_{\langle 1 \rangle} = \xi$. Proceeding in this manner one may calculate that the solution $\exp(-a|\xi|^2 + ib \cdot \xi)\hat{u}_0(\xi)$ is the expectation of the random product $\mathbb{X}(\theta, t)$ initialized by $\xi_\theta = \xi$ and consisting of factors $m(\xi) = ib \cdot \xi / a|\xi|^2$ at each vertex until termination where one attaches the end factor $\hat{u}_0(\xi)$, i.e. $\mathbb{X}(\theta, t) = m(\xi)^{N(t)}\hat{u}_0(\xi)$, and

$$\hat{u}(\xi, t) = E_{\xi_\theta = \xi}\mathbb{X}(\theta, t) = E_{\xi_\theta = \xi} m(\xi)^{N(t)}\hat{u}_0(\xi), \tag{6}$$

where $N(t)$ is the Poisson process with parameter $\lambda(\xi) := a|\xi|^2$ which counts the number of times the exponential clocks ring before time $t$. In particular the Poisson process occupies a natural *dual* role to that played by the standard Brownian motion in the real space expectation formula. Similarly one may obtain a dual Feynman-Kac formula under the *complex measure condition* on coefficients given by Ito (1965); see Chen, Dobson, Guenther, Orum, Ossiander, Thomann, Waymire (2002). In particular this approach makes Ito's complex measure condition completely natural from a probabilistic point of view. One may also obtain a dual version of McKean's (1975) branching Brownian motion formula for KPP, and other interesting equations which will be treated in a forthcoming monograph by the authors; Bhattacharya et al. (2002). These also include, for example, the generalized fractional Burgers equation of the type considered by Woyczynski, Biler, and Funaki (1998), and the so-called "poor man's Navier-Stokes equation" discussed by Montgomery-Smith (2001) from the point of view of real-space iterative methods.

The primary focus of this paper is the 3d incompressible Navier-Stokes equation which may be expressed in the Fourier domain as follows:

$$\begin{aligned} &\hat{u}(\xi, t) = e^{-\nu|\xi|^2 t}\hat{u}_0(\xi) + \int_0^t e^{-\nu|\xi|^2 s}\{|\xi|(2\pi)^{-\frac{3}{2}} \\ &\int_{\mathbf{R}^3} \hat{u}(\eta, t-s) \otimes_\xi \hat{u}(\xi - \eta, t-s) d\eta + \hat{g}(\xi, t-s)\} ds, \end{aligned} \tag{FNS}$$

where for complex vectors $w, z$

$$w \otimes_\xi z = -i(e_\xi \cdot z)\pi_{\xi^\perp} w, \ e_\xi = \frac{\xi}{|\xi|}, \text{ and } \pi_{\xi^\perp} w = w - (e_\xi \cdot w)e_\xi \tag{7}$$

is the projection of $w$ onto the plane orthogonal to $\xi$, and $\nu > 0$ is the viscosity parameter. For $\xi \neq 0$, LeJan and Sznitman (1997) rescale the equation (FNS) to normalize the integrating factor $e^{-\nu|\xi|^2 s}$ to the exponential probability density $\nu|\xi|^2 e^{-\nu|\xi|^2 s}$. They then observe that the resulting equation is precisely the form for a branching random walk recursion for $\chi(\xi,t) := \nu|\xi|^2\hat{u}(\xi,t)$, for which the transition kernel $|\xi-\eta|^{-2}|\eta|^{-2}$ is naturally constrained by integrability to dimensions $d \geq 3$ for normalization to a probability.

Given a Fourier multiplier $1/h$ we consider the Fourier transformed equation (FNS) rescaled by factors $1/h(\xi)$, for $\xi \in W_h$. Namely, we consider the equation $(\mathrm{FNS})_h$ defined by

$$\begin{aligned}\chi(\xi,t) &= e^{-\nu t|\xi|^2}\chi_0(\xi) + \int_0^t \nu|\xi|^2 e^{-\nu|\xi|^2 s}\{\frac{1}{2}m(\xi)\int_{W_h\times W_h}\chi(\eta_1,t-s) \\ &\otimes_\xi \ \chi(\eta_2,t-s)H(\xi,d\eta_1\times d\eta_2) + \frac{1}{2}\varphi(\xi,t-s)\}ds, \quad \xi\in W_h \quad (\mathrm{FNS})_h\end{aligned}$$

where

$$m(\xi) = \frac{2h*h(\xi)}{\nu(2\pi)^{\frac{3}{2}}|\xi|h(\xi)}, \quad \chi_0(\xi) = \frac{\hat{u}_0(\xi)}{h(\xi)}, \quad \varphi(\xi,t) = \frac{2\hat{g}(\xi,t)}{\nu|\xi|^2 h(\xi)}, \tag{8}$$

and $H(\xi, d\eta_1 \times d\eta_2)$ is for $\xi \in W_h$ the transition probability kernel, with support contained in the set $\{(\eta_1,\eta_2)\in W_h\times W_h : \eta_1+\eta_2=\xi\}$, defined by

$$\int_{W_h\times W_h} f(\eta_1,\eta_2)H(\xi,d\eta_1\times d\eta_2) = \int_{W_h} f(\xi-\eta,\eta)\frac{h(\xi-\eta)h(\eta)}{h*h(\xi)}d\eta \tag{9}$$

for bounded, Borel measurable $f: W_h\times W_h \to \mathbf{R}$. Finally we include the following additional *exterior condition* in defining $(\mathrm{FNS})_h$ :

$$\chi(\xi,t) = 0, \xi\in W_h^c, t\geq 0. \tag{10}$$

**Remark 1.1** One may easily check using the semigroup requirement on $W_h$ that the exterior condition makes the equations (FNS) and $(\mathrm{FNS})_h$ equivalent if and only if $\hat{u}_0(\xi) = 0$ and $\varphi(\xi,t) = 0$ for a.e. $\xi \in W_h^c, t \geq 0$. In many examples of interest to the present paper one has $W_h = R^n\backslash\{0\}$. It should also be noted that the re-scaled functions $\chi(\xi,t), \varphi(\xi,t)$ provide a convenient notational device for presenting the essential calculations. However, in the end the conditions and results are stated in terms of the respective functions $\hat{u}(\xi,t) = h(\xi)\chi(\xi,t)$, and $\hat{g}(\xi,t) = \frac{\nu}{2}|\xi|^2 h(\xi)\varphi(\xi,t)$.

A first order approach to obtain finite expected values of the branching random walk cascade will be seen to result from the observation that the product $\otimes_\xi$ satisfies $|w \otimes_\xi z| \leq |w||z|, w, z \in C^n$, and the coefficients $m(\xi)$ may be controlled by selecting Fourier multipliers such that $m(\xi) \leq 1$. We refer to such a Fourier multiplier $h$ as a majorizing kernel (with exponent one and constant $B = \frac{\nu(2\pi)^{\frac{3}{2}}}{2}$). The following slightly more general definition is suitable for extensions to generalized Navier-Stokes equations with *fractional Laplacian* and, as will be seen more fully in Section 4, for considerations of local solutions.

**Definition 1.2** A positive locally integrable function $h$ on $W_h \subset \mathbf{R}^n \backslash \{0\}$ whose closure $\overline{W}_h$ is a semigroup and such that (i.) $h$ is continuous on $W_h$ (ii.) $h * h > 0$ a.e. on $W_h$ and (iii.) $h * h(\xi) \leq B|\xi|^\theta h(\xi)$, for $\xi \in W_h$ and some real exponent $\theta$ and some $B > 0$, will be referred to as an *FNS-admissible majorizing kernel with constant B and exponent $\theta$*. Majorizing kernels with a unit constant will be called *standard* kernels. We define $h = 0$ on $W_h^c$ and refer to $W_h$ as the *support* of $h$.

Since the focus of this paper is exclusively the Navier-Stokes equations, we will drop the prefix FNS-admissible in reference to majorizing kernels. Note that if $h$ is a majorizing kernel with constant $B$ then $\frac{h}{B}$ is a standard majorizing kernel. Alternatively, if $\overline{h}$ is a standard majorizing kernel then $h = B\overline{h}$ has constant $B$. If $h$ is a majorizing kernel then $h/B$, where $B = \sup\{h * h(\xi)/|\xi|^\theta h(\xi) : \xi \in W_h\}$, will be referred to as the *standardized* choice of $h$. Those majorizing kernels $h(\xi)$ which are defined and positive for all $\xi \neq 0$ are said to be *fully supported.* Some sense of the class of majorizing kernels may be derived by noting from Hölder's inequality that the set of fully supported majorizing kernels with a given exponent is a log-convex set. Also if $h(\xi)$ is a majorizing kernel then so is $ce^{a\cdot\xi}h(\xi)$ for arbitrary fixed vector $a$ and positive scalar $c$; note Theorems 2.1-2.4 in the next section in this general regard. Finally let us note that an exceptional role of $\xi = 0$ is linked to the use of the wave number $\xi$ in defining the exponential waiting time distribution with mean $1/\nu|\xi|^2$.

Formulated in these terms, the results of LeJan and Sznitman (1997) may be interpreted in terms of two exponent one, standardized majorizing kernels, $\pi^3/|\xi|^2$ and $\alpha e^{-\alpha|\xi|}/2\pi|\xi|$. These kernels are respectively non-integrable and integrable, with equality in (iii) of Definition 1.2. One may check that the only fully supported homogeneous majorizing kernels in $n \geq 3$ dimensions are those of degree $n - 1$. Development of majorizing kernels is somewhat generally treated in Section 2. As will be demonstrated in subsequent Sections 3 and 4, apart from their role in existence, uniqueness and expected value representations, the majorizing kernels also play a role in constraining such structure of the solutions as regularity, support size, complexification, etc.

Now let us define a Banach space $\mathcal{F}_{h,\gamma,T}$ with a norm that depends on a Fourier multiplier $1/h$ as the completion of the set

$$\{v \in \mathcal{S}' : \hat{v}(\xi,t) = 0, \xi \in W_h^c, \|v\|_{\mathcal{F}_{h,\gamma,T}} = \sup_{\substack{\xi \in W_h \\ 0 \leq t < T}} \frac{|\hat{v}(\xi,t)|}{e^{-\gamma\sqrt{t}|\xi|}h(\xi)} < \infty\} \tag{11}$$

under the indicated norm, where $\gamma \in \{0,1\}$ serves to conveniently index two different norms we wish to consider. Here $\mathcal{S}'$ is the space of tempered distributions on $\mathbf{R}^n$. Also, implicit to the definition of the Banach space $\mathcal{F}_{h,\gamma,T}$ is the requirement that tempered distributions belonging to this space have Fourier transforms which are functions. In the case $h(\xi) = |\xi|^{-2}$, $\mathcal{F}_{h,0,T}$ is the Besov type space introduced by Cannone and Planchon (2000). We will refer to such spaces $\mathcal{F}_{h,\gamma,T}$ as *majorizing spaces* in the case when $h$ is a majorizing kernel. The spaces $\mathcal{F}_{h,1,T}$ generalize those introduced by Lemarié-Rieusset (2000) to obtain conditions for spatial analyticity of solutions found by LeJan and Sznitman (1997).

Note that if $h$ is a majorizing kernel of exponent $\theta \leq 1$ and $u(x,t) \in \mathcal{F}_{h,\gamma,T} \cap \mathcal{C}^1([0,T], \mathcal{S}')$ is such that $\hat{u}(\xi,t)$ is a solution of the (FNS), $u = \check{\hat{u}}$ is a mild solution of the Navier-Stokes. Indeed, the definition of majorizing kernel and of the function spaces $\mathcal{F}_{h,\gamma,T}$ imply that the product of distributions in $\mathcal{F}_{h,\gamma,T}$ is itself a distribution. To see this, note that if $u$ and $v$ are elements of $\mathcal{F}_{h,\gamma,T}$ for a standard majorizing kernel $h$ of exponent $\theta$, $|\hat{u} * \hat{v}(\xi)| \leq Mh * h(\xi) \leq M|\xi|^{\theta}h(\xi)$, where $M = \|u\|_{\mathcal{F}_{h,\gamma,T}}\|v\|_{\mathcal{F}_{h,\gamma,T}}$. Using the definition of a majorizing kernel, it follows that $\hat{u} * \hat{v}(\xi)$ is locally integrable. Thus, in particular one has $\widehat{B(u,u)}(\xi,t) = \hat{B}(\hat{u},\hat{u})(\xi,t)$ as needed, where $B(u,v) = \int_0^t e^{\nu\Delta s}P(u \cdot \nabla v)ds$ for the Leray projection $P$ on divergence free vector fields and $\hat{B}(\hat{u},\hat{v})(\xi,t) := \int_0^t e^{-\nu|\xi|^2 s}|\xi|(2\pi)^{-\frac{3}{2}} \int \{\hat{u}(\xi-\eta,t-s) \otimes_\xi \hat{v}(\eta,t-s)\}d\eta ds$; see Galdi (1994), Temam (1995). Consequently, working in these function spaces, a direct relation between solutions obtained using the stochastic representation of Section 3 and the solutions obtained using Picard iteration methods can be seen. This is described in Section 4.

**Remark 1.2** In order to restrict the solutions to correspond to (real) vector-valued incompressible flows, one may simply replace the Banach space $\mathcal{F}_{h,\gamma,T}$

by the closed subset

$$\mathcal{G}_{h,\gamma,T} = \{v \in \mathcal{F}_{h,\gamma,T} : \xi \cdot \hat{v}(\xi,t) = 0, \hat{v}(-\xi,t) = \overline{\hat{v}(\xi,t)}, \xi \in W_h, 0 \le t \le T\}. \tag{12}$$

The main results of the paper use majorizing kernels of different exponents to establish existence, uniqueness and regularity properties of the solutions of the (FNS). Moreover these solutions have an expected value representation in terms of a suitably defined multiplicative stochastic functional $\chi(\theta,t)$ of a multitype branching random walk in Fourier wavenumber space. In the statements of these results, $(-\Delta)^\alpha$ denotes the fractional power of the Laplacian defined as the singular integral operator with symbol $|\xi|^{2\alpha}$. For example, using a majorizing kernel $h$ of exponent 1, and working on the space $\mathcal{F}_{h,0,T}$, existence of solutions can be obtained for small enough initial data and forcing on a time interval that is solely constrained by the length of time for which the forcing remains small. Specifically one has the following theorem.

**Theorem 1.1** *Let $h(\xi)$ be a standard majorizing kernel with exponent $\theta = 1$. Fix $0 < T \le +\infty$. Suppose that $\|u_0\|_{\mathcal{F}_{h,0,T}} \le (\sqrt{2\pi})^3\nu/2$ and $\|(-\Delta)^{-1}g\|_{\mathcal{F}_{h,0,T}} \le (\sqrt{2\pi})^3\nu^2/4$. Then there is a unique solution $u$ in the ball $\mathcal{B}_0(0,R)$ centered at 0 of radius $R = (\sqrt{2\pi})^3\nu/2$ in the space $\mathcal{F}_{h,0,T}$. Moreover the Fourier transform of the solution is given by $\hat{u}(\xi,t) = h(\xi)E_{\xi_\theta=\xi}\chi(\theta,t), \xi \in W_h$.*

It should be remarked that regularity properties of the solutions can be inferred from the particular majorizing kernel being used. For example, note that the majorizing kernel $h_0(\xi) = \pi^3/|\xi|^2$ gives existence and uniqueness, but no control over regularity of the solution. However, solutions obtained using the majorizing kernels $h_\beta^{(\alpha)} = |\xi|^{\beta-2}e^{-\alpha|\xi|^\beta}, 0 < \beta \le 1, \alpha > 0$, maintain the same $C^\infty-$ regularity of the initial data. as can be seen from the bound on the Fourier transform of the solution. Moreover $\beta < 1$ permits smooth compactly supported initial data.

On the other hand, working in the function spaces $\mathcal{F}_{h,1,T}$ it is possible to use majorizing kernels to obtain spatial analyticity of the solution. However, it should be remarked that the size constraints imposed on the initial data and forcing are substantially more severe that those required in Theorem 1.1. Specifically one has

**Theorem 1.2** *Let $h(\xi)$ be a standard majorizing kernel with exponent $\theta = 1$. Fix $0 < T \le +\infty$. Assume $\|e^{\nu t\Delta}u_0(x)\|_{\mathcal{F}_{h,1,T}} \le \frac{(\sqrt{2\pi})^3}{2}\rho\nu e^{-1/2\nu}$ and that*

*$\|(-\Delta)^{-1}g(x,t)\|_{\mathcal{F}_{h,1,T}} \leq \frac{(\sqrt{2\pi})^3}{4}\rho\nu^2 e^{-1/2\nu}$ for some $0 \leq \rho < 1$. Then there is a unique solution $u$ in the ball $\mathcal{B}_1(0,R)$ centered at 0 of radius $R = (\rho/2)(\sqrt{2\pi})^3\nu e^{-\frac{1}{2\nu}}$ in the space $\mathcal{F}_{h,1,T}$.*

Under the conditions of Theorem 1.2 the asserted solution satisfies the following decay condition

$$\sup_{0\leq t<T}\sup_{\xi\in\mathbf{R}^3}\frac{e^{\sqrt{t}|\xi|}|\hat{u}(\xi,t)|}{h(\xi)} < \infty \tag{13}$$

Thus Theorem 1.2 provides another approach generalizing that of Lemarié-Rieusset (2000) to obtain conditions for regularity in the stronger form of spatial analyticity. More specifically, for example, if $\exp(-d|\xi|)h(\xi) \in L^1$ for some $d \in \mathbf{R}$, then one may conclude that $u(x+iy,t)$ is complex analytic for $|y| < \sqrt{t} - d$. Thus the generalized Lemarié-Rieusset estimate (13) may be applied to obtain spatial analyticity for suitable majorizing kernels with exponent 1. In particular, Theorem 1.2 extends the results of Lemarié-Rieusset since there are majorizing kernels that are larger than $h_0(\xi) = \pi^3/|\xi|^2$ as equation (26) shows.

One may also obtain local existence and uniqueness from more relaxed conditions on the majorizing kernels as illustrated by the following.

**Theorem 1.3** *Let $h(\xi)$ be a standard majorizing kernel with exponent $\theta < 1$. Fix $0 < T \leq +\infty, \gamma \in \{0,1\}$. Assume $e^{\nu t\Delta}u_0(x) \in \mathcal{F}_{h,\gamma,T}$ and for some $1 \leq \beta \leq 2$, $(-\Delta)^{-\frac{\beta}{2}}g(x,t) \in \mathcal{F}_{h,\gamma,T}$ Then there is a $0 < T_* \leq T$ for which one has a unique solution $u \in \mathcal{F}_{h,\gamma,T_*}$.*

**Remark 1.3** Kato and Fujita (1964) obtain global smooth solutions for initial velocities in $L^2$ with sufficiently small norm. In particular these results require finite energy conditions. Majorizing kernels can permit infinite energy and provide global smooth solutions if the initial data is sufficiently small in the norm $|\cdot|_h$. Kato (1984) assumes initial velocity fields in $L^3$, and proves existence of smooth global solutions if the $L^3$ norm of the initial velocity is suitably small. While these results allow infinite energy, they do not cover the cases obtained under majorization by $h^{\alpha}_{\beta}, 0 \leq \beta \leq 1$.

**Remark 1.4** Another variation on the general approach presented here leads to conditions for a local existence and uniqueness theory in all dimensions. Here one can use a particular perturbation to obtain results as follows: For a given $\nu > 0$ there is a time $T_*$, depending on $\nu$, such that one has existence and uniqueness in a ball of $\mathcal{G}_{h,T_*}$ which does not otherwise depend on $\nu$; see Orum (2002).

The organization of this paper is as follows. In the next Section 2 we identify various majorizing kernels, including kernels applicable to Navier-Stokes in $n \geq 2$ dimensions. In Section 3 the stochastic recursion is defined and Theorem 1.1 is proved. In Section 4 the Picard iteration is defined and proofs of Theorems 1.2, 1.3 are given. Conclusions and final remarks are presented in Section 5.

# 2 FNS-Majorizing Kernels

The FNS-admissible majorizing kernels play an important role in the development of our results. Recall that $h : W_h \to (0, \infty)$ is a standardized majorizing kernel with support $W_h \subset \mathbf{R}^n$ of exponent $\theta \geq 0$ if

$$h * h(\xi) \leq |\xi|^\theta h(\xi) \quad \text{for all} \quad \xi \in W_h.$$

The family of standard majorizing kernels of exponent $\theta$ on $\mathbf{R}^n$ is denoted by

$$\mathcal{H}_{n,\theta} = \{h : W_h \to (0, \infty) : h * h(\xi) \leq |\xi|^\theta h(\xi) \quad \text{for all} \quad \xi \in W_h \subset \mathbf{R}^n\}.$$

The first part of this section gives some building block structure of the sets $\mathcal{H}_{n,\theta}$ of majorizing kernels. The second part provides constructions of useful sub-families of $\mathcal{H}_{n,\theta}$. The main emphasis is on examples in $\mathcal{H}_{3,\theta}$ for $\theta = 0, 1$, although some examples are given in a more general setting. The section will close with some classes of examples of divergence free vector fields which are majorized by specific kernels.

We begin by showing that the $\mathcal{H}_{n,\theta}$'s are logarithmically convex for fixed dimension $n$.

**Theorem 2.1** *Suppose that $\{q_j : 1 \leq j \leq m\}$ satisfies $q_j > 0$, and $\sum_1^m q_j = 1$. Then for $h_j \in \mathcal{H}_{n,\theta_j}$, $j = 1, ..., m$,*

$$h(\xi) = \prod_{j=1}^{m} h_j^{q_j}(\xi) \ \in \mathcal{H}_{n, \sum_{j=1}^m q_j\theta_j}$$

*with support $W_h = \cap_{j=1}^m W_{h_j}$.*

**Corollary 2.1** *Suppose that* $\{q_j : 1 \leq j \leq m\}$ *satisfies* $q_j > 0$, *and* $\sum_1^m q_j = 1$. *Then for* $h_j \in \mathcal{H}_{n,\theta}$, $j = 1, ..., m$,

$$h(\xi) = \prod_{j=1}^{m} h_j^{q_j}(\xi) \ \in \mathcal{H}_{n,\theta}$$

*with support* $W_h = \cap_{j=1}^m W_{h_j}$.

*Proof*: Take $q_1, q_2 > 0$ with $q_1 + q_2 = 1$. Take $h_1 \in \mathcal{H}_{n,\theta_1}$ and $h_2 \in \mathcal{H}_{n,\theta_2}$ and let $h(\xi) = h_1^{q_1}(\xi) h_2^{q_2}(\xi)$. Using Hölder's inequality,

$$\begin{aligned} h * h(\xi) &= \int_\eta (h_1(\eta)h_1(\xi - \eta))^{q_1} (h_2(\eta)h_2(\xi - \eta))^{q_2} \, d\eta \\ &\leq (h_1 * h_1)^{q_1}(\xi)(h_2 * h_2)^{q_2}(\xi) \\ &\leq |\xi|^{q_1\theta_1 + q_2\theta_2} h_1^{q_1}(\xi) h_2^{q_2}(\xi) = |\xi|^{q_1\theta_1 + q_2\theta_2} h(\xi). \end{aligned}$$

The complete result follows by induction. ■

In addition relationships between the $\mathcal{H}_{n,\theta}$'s as both $n$ and $\theta$ vary are governed by a similar logarithmic convexity.

**Theorem 2.2** *Fix* $n \geq 1$. *Suppose that* $k_1, \ldots k_m$ *is a partition of* $n$ *and for each* $j = 1, \ldots, m$, $h_j$ *is in* $\mathcal{H}_{k_j,\theta_j}$. *Then*

$$h(\xi) = \prod_{j=1}^{m} h_j(\xi_j), \ \xi = (\xi_1, \ldots \xi_m) \quad \text{for } \xi_j \in \mathbf{R}^{k_j}$$

*is in* $\mathcal{H}_{n, \sum_{j=1}^m \theta_j}$ *with* $W_h = W_{h_1} \times ... \times W_{h_m}$.

*Proof:* For $h$ as defined, taking $\eta = (\eta_1, ..., \eta_m)$ with $\eta_j \in \mathbf{R}^{k_j}$,

$$\begin{aligned} h * h(\xi) &= \int_{\eta \in \mathbf{R}^n} \prod_{j=1}^{m} h_j(\eta_j) h_j(\xi_j - \eta_j) \, d\eta \\ &= \prod_{j=1}^{m} \int_{\eta_j \in \mathbf{R}^{k_j}} h_j(\eta_j) h_j(\xi_j - \eta_j) \, d\eta_j \\ &\leq \prod_{j=1}^{m} |\xi_j|^{\theta_j} h_j(\xi_j) \\ &= \prod_{j=1}^{m} |\xi_j|^{\theta_j} h(\xi) \\ &\leq |\xi|^{\sum_{j=1}^m \theta_j} h(\xi). \end{aligned}$$

■

**Theorem 2.3** *If $A$ is a $n \times n$ invertible matrix and $h \in \mathcal{H}_{n,\theta}$, then defining* $||A|| = \sup\{|A\mathbf{x}| : |\mathbf{x}| = 1\}$,

$$h_A(\xi) := |detA| \cdot ||A||^{-\theta} h(A\xi) \in \mathcal{H}_{n,\theta}$$

*with support $W_{h_A} = \{A^{-1}\xi : \xi \in W_h\}$.*

*Proof:* Take $A$ and $h$ as given and define $h_A$ as above. Then $W_{h_A} = \{\xi : 0 < h(A\xi) < \infty\} = \{A^{-1}\xi : 0 < h(\xi) < \infty\}$ and

$$\begin{aligned} h_A * h_A(\xi) &= |detA|^2 ||A||^{-2\theta} \int_{\mathbf{R}^n} h(A\eta)h(A(\xi-\eta))d\eta \\ &= |detA| \cdot ||A||^{-2\theta} \int_{\mathbf{R}^n} h(\eta)h(A\xi-\eta)d\eta \\ &\leq |detA| \cdot ||A||^{-2\theta} |A\xi|^\theta h(A\xi) \\ &\leq |detA| \cdot ||A||^{-\theta} |\xi|^\theta h(A\xi) \\ &= |\xi|^\theta h_A(\xi). \end{aligned}$$

■

The $\mathcal{H}_{n,\theta}$'s are also closed under logarithmic translation both linearly and in norm.

**Theorem 2.4** *If $h \in \mathcal{H}_{n,\theta}$ and $\psi : \mathbf{R}^n \to [0,\infty)$ satisfies $\psi(\xi) \leq \psi(\eta) + \psi(\xi - \eta)$ for all $\eta, \xi \in W_h$, then*

$$h_\psi(\xi) = e^{-\psi(\xi)} h(\xi) \in \mathcal{H}_{n,\theta}.$$

*Proof:* $h_\psi * h_\psi(\xi) \leq e^{-\psi(\xi)} h * h(\xi) \leq |\xi|^\theta h_\psi(\xi)$.

■

**Corollary 2.2** *If $h \in \mathcal{H}_{n,\theta}$, then*
*(i.) $e^{a\cdot\xi}h(\xi) \in \mathcal{H}_{n,\theta}$ for any fixed $a \in \mathbf{R}^n$,*
*and, for any pseudo-metric $\rho$ on a subset of $\mathbf{R}^3$ containing $W_h$,*
*(ii.) $e^{-a\rho(\xi_0,\xi)}h(\xi) \in \mathcal{H}_{n,\theta}$ for any $a > 0$ and $\xi_0$ fixed.*

**Note.** The example $e^{-a|\xi|^\beta}h(\xi) \in \mathcal{H}_{n,\theta}$ for any $a > 0$ and $0 < \beta \leq 1$ is a noteworthy special case of part (ii) of Corollary 2.2.

The question of existence of majorizing kernels is non-trivial. For example, it can be shown that any piecewise continuous $h \in \mathcal{H}_{1,1}$ must have

$W_h = (0, \infty)$ or $W_h = (-\infty, 0)$. This illustrates the tradeoff between $n$ and $\theta$; if exponent $\theta > 0$, the existence of majorizing kernels with support $\mathbf{R}^n \backslash \{0\}$ is problematic for smaller values of $n$. There are however fully supported majorizing kernels of exponent $\theta = 0$ for all $n \geq 1$.

**Example 2.1:** Let

$$h_1(\xi) = \frac{1}{2\pi(1+\xi^2)} \text{ for } \xi \in \mathbf{R}.$$

Then

$$h_1 * h_1(\xi) = \frac{1}{2\pi(4+\xi^2)} \leq h_1(\xi)$$

for all $\xi \in \mathbf{R}$, so $h_1 \in \mathcal{H}_{1,0}$ with $W_{h_1} = \mathbf{R}$. Using Theorem 2.2, it is easy to see that for $n > 1$,

$$h_n(\xi) = (2\pi)^{-n} \prod_{j=1}^{n} (1+\xi_j^2)^{-1} \in \mathcal{H}_{n,0}$$

with $W_h = \mathbf{R}^n$. The following rotationally invariant extension of $h_1$ is often more attractive:

$$\widetilde{h}_n(\xi) = \frac{\Gamma(\frac{n+1}{2})}{2\pi^{\frac{n+1}{2}}(1+|\xi|^2)^{\frac{n+1}{2}}}.$$

Then again

$$\widetilde{h}_n * \widetilde{h}_n(\xi) \leq \widetilde{h}_n(\xi) \qquad \text{for all} \qquad \xi \in \mathbf{R}^n, n \geq 1.$$

(See Folland (1992) page 247 for an indication of the necessary computation.)

The following Propositions 2.1 and 2.2 provide some examples of majorizing kernels in $\mathcal{H}_{3,1}$.

**Proposition 2.1** *Suitably normalized, each of the following kernels $h_\beta^{(\alpha)}$ are in $\mathcal{H}_{3,1}$ with support $W = \mathbf{R}^3 \backslash \{0\}$:*

$$h_\beta^{(\alpha)}(\xi) = |\xi|^{\beta-2} e^{-\alpha|\xi|^\beta}, \ \xi \neq 0, \ 0 \leq \beta \leq 1, \ \alpha > 0.$$

Using Theorem 2.1 the following is immediate.

**Corollary 2.3** *Suitably normalized, for each $\theta \in (0,1)$, $0 \leq \beta \leq 1$, and $\alpha > 0$,*

$$h^{(\alpha)}_{\theta,\beta}(\xi) = \frac{|\xi|^{\theta(\beta-2)} e^{-\alpha\theta|\xi|^\beta}}{(2\pi)^{3(1-\theta)} \prod_{j=1}^3 (1+\xi_j^2)^{(1-\theta)}}, \xi \neq 0$$

*and*

$$\widetilde{h}^{(\alpha)}_{\theta,\beta}(\xi) = \frac{|\xi|^{\theta(\beta-2)} e^{-\alpha\theta|\xi|^\beta}}{(1+|\xi|^2)^{2(1-\theta)}}, \xi \neq 0$$

*are both in $\mathcal{H}_{3,\theta}$ with support $W = \mathbf{R}^3 \backslash \{0\}$.*

The following lemma is sometimes useful for computing the convolution of two radially symmetric (rotationally invariant) functions, especially in dimension 3, due to the simplification of the integrand. It will be used in the proof of Proposition 2.1 below. Let $\sigma_n = 2\pi^{(n+1)/2}/\Gamma(\frac{n+1}{2})$ be the $n$-dimensional surface volume of a unit sphere $S^n$, and let

$$k(x,y,|\xi|) = \sqrt{(x+y+|\xi|)(-x+y+|\xi|)(x-y+|\xi|)(x+y-|\xi|)}$$

be 4 times the area of a triangle with side lengths $x$, $y$, and $|\xi|$.

**Lemma 2.1** *Suppose $n \geq 2$, and that $h_1, h_2 : \mathbf{R}^n \to \mathbf{C}$ are each rotationally invariant, i.e. $h_1(\xi) = g_1(|\xi|)$ and $h_2(\xi) = g_2(|\xi|)$. Then the convolution $h_1 * h_2(\xi)$, if it exists, may be computed for $|\xi| \neq 0$ as*

$$h_1 * h_2(\xi) = \frac{\sigma_{n-2}}{2^{n-3}|\xi|^{n-2}} \iint_{T_{|\xi|}} g_1(x) g_2(y)\, xy\, [k(x,y,|\xi|)]^{n-3}\, dxdy, \qquad (14)$$

*where $T_{|\xi|} = \{(x,y) \in \mathbf{R}^2 : y \geq -x + |\xi|, x - |\xi| \leq y \leq x + |\xi|\}$.*

*Proof:* The integrand in $h_1 * h_2(\xi) = \int h_1(\eta) h_2(\xi - \eta) d\eta$ is invariant under rotations around the axis defined by $\xi$ (or reflection if $n = 2$). Such rotations leave invariant the unit sphere $S^{n-2}$ centered at the origin in the hyperplane orthogonal to $\xi$. The following coordinates are therefore natural: $x = |\eta|$, $y = |\xi - \eta|$, $\omega \in S^{n-2}$. We transform to this coordinate system by first passing to ordinary spherical coordinates:

$$\begin{array}{lcll} \eta_1 & = & r\cos\theta_1 & 0 \leq \theta_1 \leq \pi \\ \eta_2 & = & r\sin\theta_1\cos\theta_2 & 0 \leq \theta_2 \leq \pi \\ \eta_3 & = & r\sin\theta_1\sin\theta_2\cos\theta_3 & 0 \leq \theta_3 \leq \pi \\ \vdots & & & \vdots \\ \eta_{n-1} & = & r\sin\theta_1 \cdots \sin\theta_{n-2}\sin\theta_{n-1} & 0 \leq \theta_{n-2} \leq \pi \\ \eta_n & = & r\sin\theta_1 \cdots \sin\theta_{n-2}\cos\theta_{n-1} & 0 \leq \theta_{n-1} < 2\pi. \end{array} \qquad (15)$$

Here $r = |\eta|$, and $\theta_1$ is the angle between $\eta$ and $\xi$. The $n$-dimensional volume element is

$$\begin{aligned} d\eta_1 d\eta_2 \cdots d\eta_n &= r^{n-1} dr \sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \cdots \sin\theta_{n-2} d\theta_1 \cdots d\theta_{n-1} \\ &= r^{n-1} dr \sin^{n-2}\theta_1 d\theta_1 d\omega \end{aligned}$$

where $d\omega$ is the surface element for the sphere $S^{n-2}$. Using spherical coordinates and performing the integration over $S^{n-2}$ gives, with $\theta = \theta_1$,

$$h_1 * h_2(\xi) = \sigma_{n-2} \int_0^\pi \int_0^\infty g_1(r) g_2(\sqrt{r^2 + |\xi|^2 - 2r|\xi|\cos\theta}\,) r^{n-1} \sin^{n-2}\theta dr d\theta.$$

Let $x = r = |\eta|$ and $y = \sqrt{r^2 + |\xi|^2 - 2r|\xi|\cos\theta} = |\xi - \eta|$. The new region of integration becomes the set $T_{|\xi|}$ of all possible ordered pairs of triangle side lengths when the third side of the triangle has length $|\xi|$. The Jacobian is

$$\left|\frac{\partial(r,\theta)}{\partial(x,y)}\right| = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|^{-1} = \left|\begin{matrix} \partial x/\partial r & 0 \\ * & \partial y/\partial\theta \end{matrix}\right|^{-1} = \frac{y}{x|\xi|\sin\theta};$$

hence,

$$h_1 * h_2(\xi) = \frac{\sigma_{n-2}}{|\xi|} \iint\limits_{T_{|\xi|}} g_1(x) g_2(y) xy \,[x\sin\theta]^{n-3}\, dxdy. \tag{16}$$

Expressed in terms of $x$ and $y$, $x\sin\theta = |2\xi|^{-1} k(x, y, |\xi|)$, giving equation (14). ■

*Proof of Proposition 2.1*: The cases $\beta = 0$ and $\beta = 1$ are treated by LeJan and Sznitman (1997). They are included for completeness here. The case $\beta = 0$ is treated first. Clearly $h_0^{(\alpha)} * h_0^{(\alpha)}(\xi)$ is finite for all $|\xi| \neq 0$. From Lemma 3.1,

$$\begin{aligned} h_0^{(\alpha)} * h_0^{(\alpha)}(\xi) &= 2\pi e^{-2\alpha} |\xi|^{-1} \iint\limits_{T_{|\xi|}} \frac{1}{xy}\, dx\, dy \\ &= 2\pi e^{-2\alpha} |\xi|^{-1} \iint\limits_{T_1} \frac{1}{xy}\, dx\, dy \\ &= 2\pi e^{-\alpha} \iint\limits_{T_1} \frac{1}{xy}\, dx\, dy\, |\xi|\; h_0^{(\alpha)}(\xi). \end{aligned}$$

For $\alpha > 0$ and $\beta \in (0,1]$ fixed we have $h_\beta^{(\alpha)}(\xi) = g(|\xi|)$ where $g(r) = r^{\beta-2}e^{-\alpha r^\beta}$. Note that for $r, x > 0,\ (x+r)^{\beta-1} - x^{\beta-1} \leq 0$ and for $0 \leq x \leq r$, $x^\beta + (r-x)^\beta \geq r^\beta$. For $|\xi| = r$ then

$$
\begin{aligned}
h * h(\xi) &= 2\pi r^{-1} \iint_{T_r} xg(x)yg(y)\,dx\,dy \\
&= \frac{2\pi r^{-1}}{\alpha\beta}(\int_{x=0}^{\infty} x^{\beta-1}e^{-\alpha x^\beta} \int_{y=|r-x|}^{r+x} \alpha\beta y^{\beta-1}e^{-\alpha y^\beta}\,dy\,dx \\
&= \frac{2\pi r^{-1}}{\alpha\beta} \int_{x=0}^{\infty} x^{\beta-1}e^{-\alpha x^\beta}(e^{-\alpha|r-x|^\beta} - e^{-\alpha(r+x)^\beta})\,dx \\
&\leq \frac{2\pi r^{-1}}{\alpha\beta}(\int_{x=0}^{r} x^{\beta-1}e^{-\alpha x^\beta - \alpha(r-x)^\beta}\,dx \\
&\quad + \int_{x=0}^{\infty}(x+r)^{\beta-1}e^{-\alpha(x+r)^\beta - \alpha x^\beta}\,dx - \int_{x=0}^{\infty} x^{\beta-1}e^{-\alpha x^\beta - \alpha(x+r)^\beta}\,dx) \\
&\leq \frac{2\pi r^{-1}}{\alpha\beta}e^{-\alpha r^\beta}\int_{x=0}^{r} x^{\beta-1}\,dx \\
&= \frac{2\pi r^{\beta-1}}{\alpha\beta^2}e^{-\alpha r^\beta}.
\end{aligned}
$$

■

One may also show that certain Bessel kernels and similar transforms provide further interesting examples of majorizing kernels in $\mathcal{H}_{n,1}$ for $n \geq 3$ as in the following proposition. These kernels are closely related to the Bessel kernels of Aronszajn and Smith (1961). They can also be combined with the kernels of Example 2.1 to construct kernels in $\mathcal{H}_{n,\theta}$ for $0 < \theta < 1$.

**Proposition 2.2** *For $n \geq 3$ and $(\beta, \gamma)$ with $0 \leq \beta \leq 1$ and $1 \leq \gamma \leq 1 + \beta$, suitably normalized, each of the following radially symmetric functions is in $\mathcal{H}_{n,1}$ with support $\mathbf{R}^n\backslash\{0\}$:*

$$h_{n,\beta,\gamma}(\xi) = \int_{t>0} t^{\frac{\gamma-n}{2}-1}e^{-t^\beta - |\xi|^2/t}\,dt,\ \xi \in \mathbf{R}^n.$$

**Remark 2.1** One may apply the Laplace method for estimating integrals to show that the Bessel type kernels $h = h_{n,\beta,\gamma}$ are also regularizing kernels in the sense that the distributions in the corresponding function space $\mathcal{F}_{h,0,T}$ are $C^\infty-$ functions.

The following lemma provides a comparison between the kernels of Proposition 2.1 and 2.2.

**Lemma 2.2** *(i.) For each $\alpha \in (0,1)$, there exists a constant $c^{(\alpha)}$ with*

$$h_{3,1,2}(\frac{\xi}{2}) \le c^{(\alpha)} h_1^{(\alpha)}(\xi).$$

*(ii.) For each $\alpha > 0$ and $\beta \in [0,1]$, there exists a constant $c_\beta^{(\alpha)}$ with*

$$h_{3,1,1+\beta}(\frac{\xi}{2}) \le c_\beta^{(\alpha)} h_\beta^{(\alpha)}(\xi).$$

*Proof*: Fix $\beta \in (0,1]$ and choose $\delta \in (0,1)$. Then

$$\begin{aligned} h_{3,1,1+\beta}(\frac{\xi}{2}) &= \int_{t>0} t^{\frac{\beta}{2}-2} e^{-\frac{|\xi|^2}{4t}-t}\, dt \\ &= e^{-\delta|\xi|} \int_{t>0} t^{\frac{\beta}{2}-2} e^{-\frac{(1-\delta^2)|\xi|^2}{4t}-(\sqrt{t}-\frac{\delta|\xi|}{2\sqrt{t}})^2}\, dt \\ &\le e^{-\delta|\xi|} \int_{t>0} t^{\frac{\beta}{2}-2} e^{-\frac{(1-\delta^2)|\xi|^2}{4t}}\, dt \\ &= (\frac{(1-\delta^2)|\xi|^2}{4})^{\frac{\beta}{2}-1} e^{-\delta|\xi|} \int_{s>0} s^{-\beta/2} e^{-s}\, ds \\ &= (\frac{(1-\delta^2)}{4})^{\frac{\beta}{2}-1} \Gamma(1-\beta/2) |\xi|^{\beta-2} e^{-\delta|\xi|}. \end{aligned}$$

For $|\xi| \ge 1$, trivially $e^{-\delta|\xi|} \le e^{-\delta|\xi|^\beta}$. For $|\xi| < 1$, $|\xi|^\beta - |\xi| \le (1-\beta)\beta^{\frac{\beta}{1-\beta}}$. Taking $\delta = \alpha$, this gives, for $0 < \beta \le 1$ and $0 < \alpha < 1$,

$$h_{3,1,1+\beta}(\frac{\xi}{2}) \le C_\beta^{(\alpha)} h_\beta^{(\alpha)}(\xi)$$

for $C_\beta^{(\alpha)} = 2^{2-\beta}\Gamma(1-\frac{\beta}{2})(1-\alpha^2)^{\frac{\beta}{2}-1} e^{\alpha(1-\beta)\beta^{\frac{\beta}{1-\beta}}}$.

For $0 < \beta < 1,\ 0 < \delta < 1$ and $\alpha \ge 1$,

$$e^{-\delta|\xi|} \le e^{-\alpha|\xi|^\beta} \quad \text{for } |\xi| \ge (\frac{\alpha}{\delta})^{\frac{1}{1-\beta}}.$$

For $|\xi| < (\frac{\alpha}{\delta})^{\frac{1}{1-\delta}}$, $-\delta|\xi| + \alpha|\xi|^\beta$ is maximized at $|\xi| = (\frac{\alpha\beta}{\delta})^{\frac{1}{1-\beta}}$ with a maximum of $(1-\beta)\alpha^{\frac{1}{1-\beta}}(\frac{\beta}{\delta})^{\frac{\beta}{1-\beta}}$. This gives

$$h_{3,1,1+\beta}(\frac{\xi}{2}) \le C_\beta^{(\alpha)} h_\beta^{(\alpha)}(\xi)$$

for $C_\beta^{(\alpha)} = 2^{2-\beta}\Gamma(1-\frac{\beta}{2})\inf_{0<\delta<1}(1-\delta^2)^{\frac{\beta}{2}-1}e^{(1-\beta)\alpha^{\frac{1}{1-\beta}}}(\frac{\beta}{\delta})^{\frac{\beta}{1-\beta}}$ ■

The majorizing kernels of Proposition 2.2 arise as weighted integrals of the function $t^{-\frac{n}{2}}e^{-\frac{|\xi|^2}{t}}$. The method of deriving these kernels can also be used to derive families of non-radial kernels as follows. Fix $\alpha \in (0,2]$ and define

$$f_\alpha(x) = \frac{1}{2\pi}\int_{\lambda\in\mathbf{R}} e^{-|\lambda|^\alpha + i\lambda x}\,d\lambda \qquad \text{for } x \in \mathbf{R}.$$

These $f_\alpha$'s correspond to the symmetric stable densities; for example $f_1(x) = (\pi(1+x^2))^{-1}$ and $f_2(x) = \frac{1}{2\sqrt{\pi}}e^{-x^2/4}$. The convolution and scaling properties of the $f_\alpha$'s give

$$(s+t)^{-\frac{1}{\alpha}}f_\alpha((s+t)^{-1/\alpha}x) = \int_{y\in\mathbf{R}}(st)^{-1/\alpha}f_\alpha(s^{-1/\alpha}(x-y))f_\alpha(t^{-1/\alpha}y)\,dy \quad (17)$$

for $s,t>0,\ x\in\mathbf{R}$.

For $n\ge 1,\ 0<\alpha\le 2$, and $g:\mathbf{R}^+\to\mathbf{R}$ define

$$T_{n,\alpha}g(x) = \int_{s>0} g(s)s^{-n/\alpha}\prod_{i=1}^n f_\alpha(s^{-1/\alpha}x_i)\,ds$$

for all $x\in\mathbf{R}^n$ such that this integral converges.

**Lemma 2.3** *Suppose $g_1, g_2 : \mathbf{R}^+\to\mathbf{R}$ such that $T_{n,\alpha}\,g_1,\ T_{n,\alpha}\,g_2,\ g_1 * g_2$ and $T_{n,\alpha}g_1 * T_{n,\alpha}g_2$ each exist a.e. with respect to Lebesgue measure. Then*

$$T_{n,\alpha}\,g_1 * T_{n,\alpha}\,g_2(x) = T_{n,\alpha}(g_1*g_2)(x) \text{ a.e.}$$

*Proof*: Use Fubini's Theorem and (17) above to check the result. ■

*Proof of Proposition 2.2* : For $\xi\in\mathbf{R}^n,\ h_{n,\beta,\gamma}(\xi) = 2^n\pi^{n/2}T_{n,2}h_{\beta,\gamma}(\xi)$ where

$$h_{\beta,\gamma}(t) = t^{\frac{\gamma}{2}-1}e^{-t^\beta}\cdot\mathbf{1}_{(0,\infty)}(t).$$

Note that since $u^\beta + (1-u)^\beta \ge 1$ for $0\le u\le 1$ and $0\le\beta\le 1$,

$$\begin{aligned} h_{\beta,\gamma} * h_{\beta,\gamma}(t) &= \int_{s=0}^t s^{\frac{\gamma}{2}-1}(t-s)^{\frac{\gamma}{2}-1}e^{-s^\beta-(t-s)^\beta}\,ds \\ &= t^{\gamma-1}\int_{u=0}^1 u^{\frac{\gamma}{2}-1}(1-u)^{\frac{\gamma}{2}-1}e^{-t^\beta(u^\beta+(1-u)^\beta)}\,du \\ &\le t^{\gamma-1}e^{-t^\beta}\int_{u=0}^1 u^{\frac{\gamma}{2}-1}(1-u)^{\frac{\gamma}{2}-1}\,du \\ &= B(\frac{\gamma}{2},\frac{\gamma}{2})h_{\beta,2\gamma}(t). \end{aligned}$$

This gives, for $\beta \in [0,1]$ and $\gamma > 0$,

$$h_{n,\beta,\gamma} * h_{n,\beta,\gamma}(\xi) \leq B(\frac{\gamma}{2}, \frac{\gamma}{2}) h_{n,\beta,2\gamma}(\xi)$$

for all $\xi$ such that $h_{n,\beta,\gamma}(\xi)$ exists.

We proceed by showing that, for $(\beta, \gamma)$ in the range given, the $h_{n,\beta,\gamma}$'s exist and the ratio

$$\frac{h_{n,\beta,2\gamma}(\xi)}{|\xi| h_{n,\beta,\gamma}(\xi)} \tag{18}$$

is bounded uniformly in $\xi \in \mathbf{R}^n$. For $z > 0$ and $\beta \in [0,1]$ define

$$g_{\beta,\alpha}(z) = \int_0^\infty t^{\alpha-1} e^{-\frac{z^2}{t} - t^\beta} \, dt.$$

For $|\xi| = z$ we have

$$h_{n,\beta,\gamma}(\xi) = g_{\beta,\frac{\gamma-n}{2}}(z)$$

and

$$h_{n,\beta,2\gamma}(\xi) = g_{\beta,\gamma-\frac{n}{2}}(z).$$

The following lemma is useful.

**Lemma 2.4** *For $\beta \in (0,1]$ and $z > 0$,*

*(i.) For $\alpha > 0$, $g_{\beta,\alpha}(z) \leq \frac{1}{\beta}\Gamma(\frac{\alpha}{\beta})$*

*(ii.) For $\alpha = 0$, $g_{\beta,0}(z) \leq \frac{1}{\beta}e^{-1} + \int_{s=z^2}^\infty s^{-1}e^{-s} \, ds$*

*(iii.) For $\alpha < 0$, $z^{-2\alpha} g_{\beta,\alpha}(z) \leq \Gamma(-\alpha)$; with $\lim_{z \downarrow 0} z^{-2\alpha} g_{\beta,\alpha}(z) = \Gamma(-\alpha)$.*

*Proof of Lemma*: Both $g_{\beta,\frac{\gamma-n}{2}}$ and $g_{\beta,\gamma-\frac{n}{2}}$ are continuous functions on $(0,\infty)$ for all $(\beta,\gamma)$ with $0 \leq \beta \leq 1$ and $1 \leq \gamma \leq 1+\beta$. For any $\alpha \in \mathbf{R}$ and $0 < \beta \leq 1$, the charge of variables $x = \frac{z^2}{t}$ gives

$$g_{\beta,\alpha}(z) = z^{2\alpha} \int_0^\infty s^{-\alpha-1} e^{-s-s^{-\beta}z^{2\beta}} \, ds. \tag{19}$$

In particular $z^{-2\alpha} g_{\beta,\alpha}(z)$ is continuous and strictly decreasing in $z > 0$. For $\alpha < 0$, $\lim_{z \searrow 0} z^{-2\alpha} g_{\beta,\alpha}(z) = \Gamma(-\alpha)$. Thus we see immediately that $g_{\beta,\frac{\gamma-n}{2}}$ is a continuous and decreasing function of $z$ specified by $\beta$ and $\gamma$, and $g_{\beta,\gamma-\frac{n}{2}}$ is a continuous and decreasing function of $z$ for $\gamma < \frac{n}{2}$. Next consider the case $n = 3,\ \beta \in [\frac{1}{2}, 1]$ and $\gamma = \frac{3}{2}$ :

$$\begin{aligned} g_{\beta,0}(z) &\leq \int_1^\infty t^{\beta-1} e^{-t^\beta}\, dt + \int_0^1 t^{-1} e^{-\frac{z^2}{t}}\, dt \\ &= \frac{1}{\beta} e^{-1} + \int_{z^2}^\infty s^{-1} e^{-s}\, ds. \end{aligned}$$

The case $n = 3,\ \gamma \in (\frac{3}{2}, 1+\beta)$ for $\beta \in (\frac{1}{2}, 1]$ is handled as follows. For $\alpha > 0$, the change of variables $s = t^\beta$ gives

$$\begin{aligned} g_{\beta,\alpha}(z) &= \frac{1}{\beta} \int_0^\infty s^{\frac{\alpha}{\beta}-1} e^{-s-z^2 s^{-\frac{1}{\beta}}}\, ds \\ &\leq \frac{1}{\beta} \Gamma(\frac{\alpha}{\beta}). \end{aligned}$$

∎

Returning to the proof of Proposition 2.2 we see that the key to bounding (18) uniformly in $\xi \in \mathbf{R}^n$ is showing that

$$\limsup_{z \searrow 0} \frac{g_{\beta,\gamma-\frac{n}{2}}(z)}{z g_{\beta,\frac{\gamma-n}{2}}(z)}$$

and

$$\limsup_{z \nearrow \infty} \frac{g_{\beta,\gamma-\frac{n}{2}}(z)}{z g_{\beta,\frac{\gamma-n}{2}}(z)}$$

are both finite.

First consider the case $(\beta, \gamma) = (0, 1)$. From (19), for all $n \geq 3$,

$$\frac{g_{0,1-\frac{n}{2}}(z)}{z g_{0,\frac{1-n}{2}}(z)} = \frac{\Gamma(\frac{n}{2} - 1)}{\Gamma(\frac{n-1}{2})} < \infty.$$

Next consider $\beta \in (1, 2]$ and $\gamma \in [1, 1+\beta] \cap [1, \frac{n}{2})$. From (iii.) of Lemma 2.4

$$\begin{aligned} \limsup_{z \searrow 0} \frac{g_{\beta,\gamma-\frac{n}{2}}(z)}{z g_{\beta,\frac{\gamma-n}{2}}(z)} &= \limsup_{z \searrow 0} z^{\gamma-1} \frac{z^{n-2\gamma} g_{\beta,\gamma-\frac{n}{2}}(z)}{z^{n-\gamma} g_{\beta,\frac{\gamma-n}{2}}(z)} \\ &= \begin{cases} \frac{\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n-1}{2})} & \gamma = 1 \\ 0 & \gamma \in (1, 1+\beta] \cap (1, \frac{n}{2}). \end{cases} \end{aligned}$$

For $n = 3,\ \gamma = \frac{3}{2}$, and $\beta \in [\frac{1}{2}, 1]$, using (ii.) and (iii.) of Lemma 2.4,

$$\begin{aligned}
\limsup_{z \searrow 0} \frac{g_{\beta,0}(z)}{z g_{\beta,-\frac{3}{4}}(z)} &= \frac{1}{\Gamma(\frac{3}{4})} \lim_{z \searrow 0} z^{\frac{1}{2}} g_{\beta,0}(z) \\
&\leq \frac{1}{\Gamma(\frac{3}{4})} \limsup_{z \searrow 0} (z^{\frac{1}{2}} \int_{z^2}^{1} s^{-1}\, ds + z^{\frac{1}{2}} \int_{1}^{\infty} e^{-s}\, ds) \\
&\leq \frac{2}{\Gamma(\frac{3}{4})} \lim_{z \searrow 0} z^{\frac{1}{2}} \ln z = 0.
\end{aligned}$$

For $n = 4,\ \gamma = 2$ and $\beta = 1$, again using (ii.) and (iii.) of Lemma 2.4

$$\limsup_{z \searrow 0} \frac{g_{1,0}(z)}{z g_{1,-1}(z)} \leq \lim_{z \searrow 0} z g_{1,0}(z) = 0.$$

For $n = 3,\ \beta \in (\frac{1}{2}, 1]$ and $\gamma \in (\frac{3}{2}, 1 + \beta]$,

$$\begin{aligned}
\limsup_{z \searrow 0} \frac{g_{\beta,\gamma-\frac{3}{2}}(z)}{z g_{\beta,\frac{\gamma-3}{2}}(z)} &\leq \frac{\Gamma(\frac{\gamma-\frac{3}{2}}{\beta})}{\beta \Gamma(\frac{3-\gamma}{2})} \lim_{z \searrow 0} z^{2-\gamma} \\
&= \begin{cases} 1 & \text{for } \beta = 1,\ \gamma = 2 \\ 0 & \text{for } \beta \in (\frac{1}{2}, 1],\ \gamma \in (\frac{3}{2}, 1+\beta] \cap (\frac{3}{2}, 2). \end{cases}
\end{aligned}$$

Now consider the limit of the ratio as $z \nearrow \infty$. Fix $\beta \in (0, 1]$ and $\gamma \in [1, 1+\beta]$. For the minute fix $z \geq 1$ and consider $f(t) = \frac{z^2}{t} + t^\beta$. $f$ is minimized at $t_0 = (\frac{z^2}{\beta})^{\frac{1}{\beta+1}}$, decreases on $(0, t_0)$ and increases to $\infty$ on $(t_0, \infty)$. Fix $r \geq 2^{\frac{2}{n-\gamma}} \beta^{-\frac{1}{\beta+1}}$ sufficiently large to satisfy

$$\frac{1}{r} + r^\beta \leq \frac{1}{2}(\frac{1}{r^2} + r^{2\beta}).$$

In particular this gives

$$f(r z^{\frac{2}{\beta+1}}) \leq \frac{1}{2} f(r^2 z^{\frac{2}{\beta+1}}) \tag{20}$$

and $r z^{\frac{2}{\beta+1}} \geq t_0$. Then

$$g_{\beta,\frac{\gamma-n}{2}}(z) \geq \int_{t_0}^{r z^{\frac{2}{\beta+1}}} t^{\frac{\gamma-n}{2}-1} e^{-f(t)}\, dt$$

$$\begin{aligned}
&\geq e^{-f(rz^{\frac{2}{\beta+1}})}\cdot\frac{2}{n-\gamma}(t_0^{\frac{\gamma-n}{2}}-(rz^{\frac{2}{\beta+1}})^{\frac{\gamma-n}{2}}) \\
&= \frac{2z^{\frac{\gamma-n}{\beta+1}}}{n-\gamma}e^{-f(rz^{\frac{2}{\beta+1}})}(\beta^{\frac{n-\gamma}{2(\beta+1)}}-r^{\frac{\gamma-n}{2}}) \\
&\geq \frac{\beta^{\frac{n-\gamma}{2(\beta+1)}}}{n-\gamma}z^{\frac{\gamma-n}{\beta+1}}e^{-f(rz^{\frac{2}{\beta+1}})} \qquad (21)
\end{aligned}$$

and

$$\begin{aligned}
\int_{r^2z^{\frac{2}{\beta+1}}}^{\infty} t^{\gamma-\frac{n}{2}-1}e^{-f(t)}\,dt &\leq e^{-\frac{1}{2}f(r^2z^{\frac{2}{\beta+1}})}\int_{r^2z^{\frac{2}{\beta+1}}}^{\infty} t^{\gamma-\frac{n}{2}-1}e^{-\frac{1}{2}t^{\beta}}\,dt \\
&= e^{-\frac{1}{2}f(r^2z^{\frac{2}{\beta+1}})}z^{\frac{2\gamma-n}{\beta+1}}\int_{r^2}^{\infty} s^{\gamma-\frac{n}{2}-1}e^{-\frac{z^{\frac{2\beta}{\beta+1}}}{2}s^{\beta}}\,ds \qquad (22)
\end{aligned}$$

Combining (20), (21) and (22),

$$\frac{\int_{r^2z^{\frac{2}{\beta+1}}}^{\infty} t^{\gamma-\frac{n}{2}-1}e^{-f(t)}\,dt}{zg_{\beta,\frac{\gamma-n}{2}}(z)} \leq (n-\gamma)\beta^{\frac{\gamma-n}{2(\beta+1)}}z^{\frac{\gamma}{\beta+1}-1}\int_{r^2}^{\infty} s^{\gamma-\frac{n}{2}-1}e^{\frac{-z^{\frac{2\beta}{\beta+1}}}{2}s^{\beta}}\,ds \qquad (23)$$

For $\gamma \leq 1+\beta$, this goes to 0 as $z \to \infty$.

For $z \geq 1,\ t \leq r^2z^{\frac{2}{\beta+1}}$ gives

$$t^{\frac{\gamma}{2}} \leq r^{\gamma}z^{\frac{\gamma}{\beta+1}} \leq r^{\gamma}z,$$

so that

$$z^{-1}\int_0^{r^2z^{\frac{2}{\beta+1}}} t^{\gamma-\frac{n}{2}-1}e^{-f(t)}\,dt \leq r^{\gamma}\int_0^{r^2z^{\frac{2}{\beta+1}}} t^{\frac{\gamma-n}{2}-1}e^{-f(t)}\,dt \leq r^{\gamma}g_{\beta,\frac{\gamma-n}{2}}(z). \qquad (24)$$

Using (23) and (24), for $\beta \in (0,1]$ and $\gamma \in [1, 1+\beta]$,

$$\limsup_{z\nearrow\infty}\frac{g_{\beta,\gamma-\frac{n}{2}}(z)}{zg_{\beta,\frac{\gamma-n}{2}}(z)} \leq r^{\gamma} < \infty.$$

■

The same general technique that gave the kernels of Proposition 2.2 gives families of non-radial kernels that are not fully supported. These are the *larger* kernels that permit broader existence and uniqueness results for given initial data $u_0^{\lambda}$ of (FNS); see Remark 2.2 below.

**Proposition 2.3** *For each* $\alpha \in (0,1]$ *and* $n \geq 3$,

$$H_{n,\alpha}(\xi) = \int_{t>0} t^{-\frac{n-1+\alpha}{\alpha}} \prod_{i=1}^{n} f_\alpha(t^{-\frac{1}{\alpha}}\xi_i)\,dt$$

*is, suitably normalized, in* $\mathcal{H}_{n,1}$, *with support* $W_{n,\alpha} = \{\xi \in \mathbf{R}^n : \sum_1^n \mathbf{1}_{[\xi_i=0]} < \frac{n\alpha+1}{\alpha+1}\}$.

*Proof*: Fix $\alpha \in (0,1]$ and $n \geq 3$. Let $g_\gamma(t) = t^{\gamma-1}$ for $\gamma, t > 0$ and set

$$H_{n,\alpha}(\xi) = T_{n,\alpha}g_{\frac{1}{\alpha}}(\xi)$$

for all $\xi \in \mathbf{R}^n$ for which $T_{n,\alpha}g_{\frac{1}{\alpha}}(\xi)$ converges. The convolution $g_\gamma * g_\gamma(t) = B(\gamma,\gamma)g_{2\gamma}(t)$, so from Lemma 2.3,

$$H_{n,\alpha} * H_{n,\alpha}(\xi) = B(\frac{1}{\gamma},\frac{1}{\gamma})T_{n,\alpha}g_{\frac{2}{\alpha}}(\xi).$$

In order to check convergence of $T_{n,\alpha}g_\gamma(\xi)$ for $\gamma = \frac{1}{\alpha},\ \frac{2}{\alpha},\ \alpha \in (0,1]$ we rely on a series expansion of $f_\alpha(x)$ for $\alpha \in (0,1]$ and $|x|$ large given by Feller (1971) p. 583;

$$f_\alpha(x) = \frac{1}{\pi|x|}\sum_{k\geq 1}\frac{\Gamma(k\alpha+1)}{k!}(-1)^{k+1}|x|^{-\alpha k}\sin(\frac{k\alpha\pi}{2})$$

In particular, using this expansion it is straightforward to show that for $\alpha \in (0,1]$ and $|x| > 2^{\frac{1}{\alpha}},\ f_\alpha(x) < c_\alpha|x|^{-1-\alpha}$ where $c_\alpha$ is a constant depending on $\alpha$. In addition, it is easy to see that $f_\alpha(x)$ is maximized at $x = 0$.

Fix $n \geq 3$ and $x \in \mathbf{R}^n$ with $|x| > 0$. The change of variables $s = t|x|^{-\alpha}$ gives

$$T_{n,\alpha}g_\gamma(x) = |x|^{\gamma\alpha-n}T_{n,\alpha}g_\gamma(\frac{x}{|x|}). \tag{25}$$

Let $J(x) = \{i : x_i \neq 0\},\ j = j(x) = \sum_1^n \mathbf{1}_{[x_i=0]}$ and $r(x) = \min\{\frac{1}{2}(\frac{|x_j|}{|x|})^\alpha : j \in J(x)\}$. Then

$$\begin{aligned}\int_{s=0}^{r(x)} s^{\gamma-\frac{n}{\alpha}-1}\prod_1^n f_\alpha(\frac{x_i}{|x|}s^{-\frac{1}{\alpha}})\,ds &\leq f_\alpha^j(0)\int_{s=0}^{r(x)} s^{\gamma-\frac{n}{\alpha}-1}\prod_{i\in J(x)} c_\alpha(\frac{|x_i|}{|x|}s^{-\frac{1}{\alpha}})^{-1-\alpha}\,ds\\ &= f_\alpha^j(0)c_\alpha^{n-j}\prod_{i\in J(x)}(\frac{|x_i|}{|x|})^{-1-\alpha}\\ &\quad\cdot\int_{s=0}^{r(x)} s^{\gamma-\frac{n-(n-j)(1+\alpha)}{\alpha}-1}\,ds.\end{aligned}$$

For $\gamma = \frac{1}{\alpha}, \frac{2}{\alpha}$ this integral converges for $j < \frac{n\alpha+1}{\alpha+1}$. Also for $\gamma < \frac{n}{\alpha}$,

$$\int_{r(x)}^{\infty} s^{\gamma-\frac{n}{\alpha}-1} \prod_1^n f_\alpha(\frac{x_i}{|x|} s^{-\frac{1}{\alpha}})\, ds < f_\alpha^n(0) \int_{r(x)}^{\infty} s^{\gamma-\frac{n}{\alpha}-1}\, ds < \infty.$$

Together these give

$$T_{n,\alpha} g_\gamma(\frac{x}{|x|}) < \infty$$

for $\gamma = \frac{1}{\alpha}, \frac{2}{\alpha}$ and $\sum_1^n \mathbf{1}_{[x_i=0]} < \frac{n\alpha+1}{\alpha+1}$. From (25) we see that to verify $H_{n,\alpha}$ is a majorizing kernel, we need to show that for a constant $c_{n,\alpha} \in (0,\infty)$,

$$T_{n,\alpha} g_{\frac{2}{\alpha}}(\frac{x}{|x|}) \le c_{n,\alpha} T_{n,\alpha} g_{\frac{1}{\alpha}}(\frac{x}{|x|})$$

for all $x$ with $\sum_1^n \mathbf{1}_{[x_i=0]} < \frac{n\alpha+1}{\alpha+1}$. Fix $n \ge 3, \alpha \in (0,1]$, and $y \in \mathbf{R}^n$ with $|y| = 1$ and $\sum_1^n \mathbf{1}_{[y_i=0]} < \frac{n\alpha+1}{\alpha+1}$. For $\gamma = \frac{1}{\alpha}, \frac{2}{\alpha}$ let

$$I_\gamma^{(1)}(y) = \int_{s=0}^{1} s^{\gamma-\frac{n}{\alpha}-1} \prod_1^n f_\alpha(y_i s^{-\frac{1}{\alpha}})\, ds$$

and

$$I_\gamma^{(2)}(y) = \int_{s=1}^{\infty} s^{\gamma-\frac{n}{\alpha}-1} \prod_1^n f_\alpha(y_i s^{-\frac{1}{\alpha}})\, ds.$$

Immediately

$$I_{\frac{2}{\alpha}}^{(1)}(y) \le I_{\frac{1}{\alpha}}^{(1)}(y).$$

Using Yamazato (1978) we see that $f_\alpha(x)$ is uni-modal and strictly decreasing on $(0,\infty)$. This gives

$$I_{\frac{1}{\alpha}}^{(2)}(y) \ge \int_{s>1} s^{\frac{1-n}{\alpha}-1} \prod_1^n f_\alpha(1)\, ds = \frac{\alpha}{n-1} f_\alpha^n(1)$$

and

$$T_{n,\alpha} g_{\frac{2}{\alpha}}(x) \le c_{n,\alpha} |x| T_{n,\alpha} g_{\frac{1}{\alpha}}(x) \quad \text{for} \quad c_{n,\alpha} = \frac{n-1}{n-2} (\frac{f_\alpha(0)}{f_\alpha(1)})^n B(\frac{1}{\gamma}, \frac{1}{\gamma}).$$

∎

**Remark 2.2** In the case $n = 3, \alpha = 1$ the kernel $H_{3,1}$ can be written as

$$H_{3,1}(\xi) = \frac{1}{|\xi|^2} G(\frac{\xi}{|\xi|}), \tag{26}$$

where $G$ is defined a.e. on the unit sphere with $G(\theta) \to \infty$ as $\theta$ approaches the points $(0, 0, \pm 1), (0, \pm 1, 0), (\pm 1, 0, 0)$, respectively. In particular the growth of $H_{3,1}$ along particular directions is much larger than $h_0(\xi) = 1/|\xi|^2$. Transforming $H_{3,1}$ via a rotation as suggested in Theorem 2.3 permits such growth in any direction.

In view of the role of majorizing kernels in providing bounds on the Fourier transformed forcing and/or initial data, the theory contains a dual problem which is to identify classes of divergence free vector fields in physical space which are so dominated.

The first example is a class of divergence free vector fields on $\mathbf{R}^3$ whose Fourier transforms are dominated by $h_{3,\beta,\gamma}(\xi)$.

**Example 2.2:** Fix $0 \le \beta \le 1$ and $1 \le \gamma \le 1 + \beta$. For $1 \le j \le 3$ let $m_j(t)$ be measurable functions on $[0, \infty)$ such that $|m_j(t)| \le t^{\frac{\gamma}{2}-1} e^{-t^\beta}$ and $\int_{t>0} t^{-3/2} |m_j(t)| dt < \infty$. Let $v(x)$ be the vector field whose components $v_j(x)$ are the Laplace transforms of $m_j(t)$ evaluated at $|x|^2/4$; that is

$$v_j(x) = \int_0^\infty e^{-t|x|^2/4} m_j(t) dt$$

Let $u(x)$ be the divergence free projection of $v(x)$. Then the following calculation shows that

$$|\hat{u}(\xi)| \le c h_{3,\beta,\gamma}(\xi).$$

After using Tonelli's Theorem to check integrability, Fubini's Theorem gives

$$\begin{aligned}
|\hat{v}_j(\xi)| &= c \, | \int_{t>0} \int_{\mathbf{R}^3} e^{-i\xi\cdot x} e^{\frac{-t|x|^2}{4}} m_j(t) dx \, dt | \\
&\le c \int_{t>0} t^{-3/2} |m_j(t)| e^{-|\xi|^2/t} dt \\
&\le c \, h_{3,\beta,\gamma}(\xi).
\end{aligned}$$

The projection of the vector field $v(x)$ onto the divergence free component $u(x)$ becomes, on the Fourier side, $\hat{u}(\xi) = \hat{v}(\xi) - \frac{\xi}{|\xi|}(\hat{v}(\xi) \cdot \frac{\xi}{|\xi|}) = \pi_{\xi^\perp} \hat{v}(\xi)$. This contraction gives

$$|\hat{u}(\xi)| \le |\hat{v}(\xi)| \le c h_{3,\beta,\gamma}(\xi) \qquad \text{for all} \qquad \xi \in \mathbf{R}^3 \backslash \{0\}.$$

For the next example we consider majorization by the kernels $h_\beta^{(\alpha)}$.

**Example 2.3:** Let $\mathcal{M}$ denote the space of finite signed measures on $\mathbf{R}^3$ with total variation norm $\|\ \|$. Let $0 < \beta \le 1$ and denote the "Fourier transformed Bessel kernel" of order $\beta$ by $G_\beta(x) = (1+|x|^2)^{-\frac{1+\beta}{2}}$. Then for each $g = G_\beta * \mu, \mu \in \mathcal{M}$, one has for $\beta = 1,\ \alpha \in (0,1)$ and for $\beta \in (0,1),\ \alpha > 0$,

$$|\hat{g}(\xi)| \le C_\beta^{(\alpha)} h_\beta^{(\alpha)}(\xi)||\mu||, \quad \xi \neq 0,$$

for a constant $C_\beta^{(\alpha)} > 0$. In particular, if $v \in L^1$ is a divergence free vector field then $g = G_\beta * v$ is also a divergence free vector field whose Fourier transform is dominated by $h_\beta^{(\alpha)}$. To verify this class of examples it suffices to check that

$$|\hat{G}_\beta(\xi)| \le C_\beta^{(\alpha)} h_\beta^{(\alpha)}(\xi), \tag{27}$$

for some constant $C_\beta^{(\alpha)}$. For this we take the Fourier transform of $(1+|x|^2)^{-\frac{1+\beta}{2}}$ and then use Lemma 2.2. First notice that for any $a > 0$,

$$\Gamma(\frac{\beta+1}{2}) = a^{\frac{\beta+1}{2}} \int_0^\infty t^{\frac{\beta-1}{2}} e^{-at}\, dt.$$

Solving for $a^{-\frac{\beta+1}{2}}$ and then taking $a = 1+|x|^2$,

$$G_\beta(x) = \frac{1}{\Gamma(\frac{\beta+1}{2})} \int_0^\infty t^{\frac{\beta-1}{2}} e^{-(1+|x|^2)t}\, dt$$

and

$$\begin{aligned} \hat{G}_\beta(\xi) &= \frac{(2\pi)^{-\frac{3}{2}}}{\Gamma(\frac{\beta+1}{2})} \int_0^\infty t^{\frac{\beta-1}{2}} e^{-t} \int_{x\in\mathbf{R}^3} e^{-ix\cdot\xi-|x|^2 t}\, dx\, dt \\ &= \frac{2^{-\frac{3}{2}}}{\Gamma(\frac{\beta+1}{2})} \int_0^\infty t^{\frac{\beta-2}{2}-1} e^{-t-\frac{|\xi|^2}{4t}}\, dt \\ &= \frac{2^{-\frac{3}{2}}}{\Gamma(\frac{\beta+1}{2})} h_{3,1,\beta+1}(\frac{\xi}{2}). \end{aligned}$$

Then (27) follows from Lemma 2.2 with $C_\beta^{(\alpha)} = \frac{2^{-\frac{3}{2}}}{\Gamma(\frac{\beta+1}{2})} c_\beta^{(\alpha)}$.

The following example uses the $h_\beta^{(\alpha)}$ majorizing kernels to give smooth divergence free vector fields, including some with compact support.

**Example 2.4:** Let $m_j(t), t > 0, j = 1,2,3$, be measurable functions such that $\int_0^\infty e^{-|x|^2 t}|m_j(t)|dt < \infty, x \in \mathbf{R}^3, j = 1,2,3$. Define a vector field with components $v_j, j = 1,2,3$, by

$$v_j(x) = \int_0^\infty e^{-|x|^2 t} m_j(t) dt, x \in \mathbf{R}^3.$$

Let $u$ be the divergence free projection of $v$. Then,

(i.) If $|m_j(t)| \le ct^{-\frac{1}{2}}$ then $|\hat{u}_j(\xi)| \le c' h_0^{(\alpha)}(\xi)$ for some $c' > 0, j = 1, 2, 3$.

(ii.) If $|m_j(t)| \le ce^{-2\alpha^2 t}$ then $|\hat{u}_j(\xi)| \le c' h_1^{(\alpha)}(\xi)$ for some $c' > 0, j = 1, 2, 3$.

(iii.) For arbitrary $\epsilon > 0$ there is a smooth probability density function $k_\epsilon$ supported on $[-\epsilon, \epsilon]^3$ such that

$$|\hat{k}_\epsilon(\xi)| \le c(\beta, \epsilon) \exp\{-|\epsilon\xi|^\beta\}, \xi \in \mathbf{R}^3, c(\beta, \epsilon) > 0.$$

Let $v$ be any divergence-free integrable vector field such that $|\hat{v}(\xi)| \le c|\xi|^{-2}$, $\xi \ne 0$. Then the componentwise perturbation $u = k_\epsilon * v$ is a divergence-free infinitely differentiable vector field such that $|\hat{u}_j(\xi)| \le c' h_\beta^{(\alpha)}(\xi)$, for $\alpha = \epsilon^\beta$ and some $c' > 0, j = 1, 2, 3$.

To verify (i.) and (ii.) first recall that

$$(2\pi)^{-\frac{3}{2}} \int_{\mathbf{R}^3} e^{-ix\cdot\xi - |x|^2 t}\, dx = (2t)^{-\frac{3}{2}} e^{-\frac{|\xi|^2}{4t}},$$

and therefore

$$|\hat{u}_j(\xi)| \le 2^{-\frac{3}{2}} \int_{t>0} t^{-\frac{3}{2}} |m_j(t)| e^{-\frac{|\xi|^2}{4t}}\, dt.$$

For $|m_j(t)| \le ct^{-\frac{1}{2}}$,

$$\begin{aligned} |\hat{u}_j(\xi)| &\le c2^{-\frac{3}{2}} \int_{t>0} t^{-2} e^{-\frac{|\xi|^2}{4t}}\, dt \\ &= c2^{\frac{1}{2}} |\xi|^{-2} \\ &= c2^{\frac{1}{2}} e^{\alpha} h_0^{(\alpha)}(\xi) \end{aligned}$$

for $j = 1, 2, 3$. For $|m_j(t)| \le ce^{-2\alpha^2 t}$, using the change of variables $s = \frac{|\xi|^2}{8t}$,

$$\begin{aligned} |\hat{u}_j(\xi)| &\le c2^{-\frac{3}{2}} \int_{t>0} t^{-\frac{3}{2}} e^{-2\alpha^2 t - \frac{|\xi|^2}{4t}}\, dt \\ &= c|\xi|^{-1} \int_{s>0} s^{-\frac{1}{2}} e^{-2s - \frac{\alpha^2|\xi|^2}{4s}}\, ds \\ &= c|\xi|^{-1} e^{-\alpha|\xi|} \int_{s>0} s^{-\frac{1}{2}} e^{-s-(\sqrt{s} - \frac{\alpha|\xi|}{2\sqrt{s}})^2}\, ds \\ &\le c\Gamma(\frac{1}{2}) h_1^{(\alpha)}(\xi). \end{aligned}$$

To check (iii) one may apply Theorem 10.2 of Bhattacharya and Rao (1976) to see that for any fixed $\beta \in (0,1)$ there exists a probability measure on $(\mathbf{R}, \mathbf{B})$ with density $k$ whose support is contained in $[-1,1]$ and

$$|\hat{k}(\xi)| \leq c(\beta) \exp\{-3^{\frac{\beta}{2}-1}|\xi|^{\beta}\} \quad \text{for } \xi \in \mathbf{R}.$$

Without loss of generality we can assume that $k$ is symmetric and infinitely differentiable. Fix $\epsilon > 0$ and take $k_\epsilon$ to be the density of the probability measure on $\mathbf{R^3}$ given by

$$K_\epsilon(A) = \iiint_{A_\epsilon} k(x_1)k(x_2)k(x_3)\, dx_1\, dx_2\, dx_3$$

where $A_\epsilon = \{\frac{x}{\epsilon} : x \in A\}$. Then $k_\epsilon$ has support contained in $[-\epsilon, \epsilon]^3$ and

$$\begin{aligned} |\hat{k}_\epsilon(\xi)| &\leq c^3(\beta) e^{-3^{\frac{\beta}{2}-1}\epsilon^\beta \sum_{i=1}^3 |\xi_i|^\beta} \\ &\leq c^3(\beta) e^{-\epsilon^\beta |\xi|^\beta} \end{aligned}$$

using Jensen's inequality in the exponent. If $v$ is an integrable divergence free vector field on $\mathbf{R}^3$ with $|\hat{v}(\xi)| \leq c|\xi|^{-2}$ then $u = k_\epsilon * v$ is both divergence free and infinitely differentiable with

$$\begin{aligned} |\hat{u}_j(\xi)| &= |\hat{k}_\epsilon|\,|\hat{v}_j(\xi)| \\ &\leq c^3(\beta) e^{-\epsilon^\beta|\xi|^\beta} \min\{c|\xi|^{-2},\ |\hat{v}_j(\xi)|\}. \end{aligned}$$

For $|\xi| \geq 1$ then

$$|\hat{u}_j(\xi)| \leq c'(\beta) h_\beta^{(\epsilon^\beta)}(\xi)$$

with $c'(\beta) = c \cdot c^3(\beta)$. For $|\xi| \leq 1$,

$$|\hat{u}_j(\xi)| \leq c''(\beta) h_\beta^{(\epsilon^\beta)}(\xi)$$

for $c''(\beta) = c^3(\beta)||v_j||$, where $||v_j||$ denotes the $L^1$−norm of $v_j$.

# 3 Stochastic Recursion

The vertex set $\mathcal{V}$ of a complete binary tree rooted at $\theta$ may be coded as (see Figure 1)

$$\mathcal{V} = \cup_{j=0}^{\infty}\{1,2\}^j = \{\theta, <1>, <2>, <11>, \ldots\}, \tag{28}$$

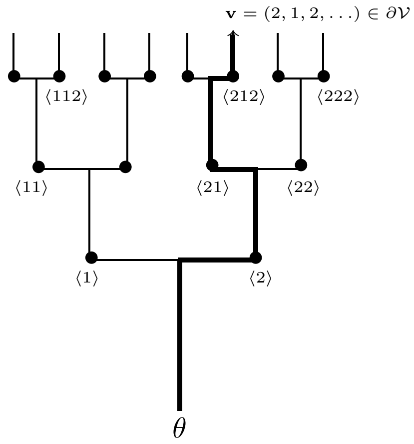



Figure 1: Full binary tree with index set $\mathcal{V}$ and boundary $\partial\mathcal{V}$. The path $\mathbf{v} = (2,1,2,\ldots) \in \partial\mathcal{V}$ is indicated in bold, with $\mathbf{v}|0 = \theta$, $\mathbf{v}|1 = \langle 2\rangle$, $\mathbf{v}|2 = \langle 21\rangle$, and $\mathbf{v}|3 = \langle 212\rangle$.

where $\{1,2\}^0 = \{\theta\}$. Also let $\partial\mathcal{V} = \prod_{j=0}^{\infty}\{1,2\} = \{1,2\}^{\mathbf{N}}$.

A stochastic model consistent with $(\mathrm{FNS})_h$ is obtained by consideration of a multitype branching random walk of nonzero Fourier wavenumbers $\xi$, thought of as particle *types*, as follows: A particle of type $\xi \neq 0$ initially at the root $\theta$ holds for an exponentially distributed length of time $S_\theta$ with holding time parameter $\lambda(\xi) = \nu|\xi|^2$; i.e. $ES_\theta = \frac{1}{\nu|\xi|^2}$. When this exponential clock rings, a coin $\kappa_\theta$ is tossed and either with probability $\frac{1}{2}$ the event $[\kappa_\theta = 0]$ occurs and the particle is terminated, or with probability $\frac{1}{2}$ one has $[\kappa_\theta = 1]$ and the particle is replaced by two offspring particles of types $\eta_1, \eta_2$ selected from the set $\eta_1 + \eta_2 = \xi$ according to the probability kernel $H(\xi, d\eta_1 \times d\eta_2)$ defined by (9). This process is repeated independently for the particle types $\eta_1, \eta_2$ rooted at the vertices $< 1 >, < 2 >$, respectively.

A more precise mathematical description of the stochastic model requires a bit more notation. For $\mathbf{v} = (v_1, v_2, \ldots, v_k) \in \mathcal{V}$, let $\mathbf{v}|j = (v_1, \ldots, v_j), j \leq k$. Also let $|\mathbf{v}| = k$, $|\theta| = 0$, denote the geneological length of the vertex $\mathbf{v} \in \mathcal{V}$. For $\mathbf{v} = (v_1, v_2, \ldots) \in \partial\mathcal{V}$, and $j = 0, 1, 2, \ldots$ let $\mathbf{v}|j = (v_1, \ldots v_j)$, $\mathbf{v}|0 = \theta$.

That is, for $\mathbf{v} \in \partial\mathcal{V},\ \mathbf{v}|0, \mathbf{v}|1, \mathbf{v}|2, \ldots$ may be viewed as a nonterminating *path* through vertices of the tree starting from the root $\mathbf{v}|0 = \theta$. For $\mathbf{u}, \mathbf{v} \in \partial\mathcal{V}$, or in $\mathcal{V}$, let $|\mathbf{u} \wedge \mathbf{v}| = \inf\{m \geq 1 : \mathbf{u}|m \neq \mathbf{v}|m\}$.

The following requirements provide the defining properties of the underlying stochastic model. The model depends on the initial frequency (wave number) $\xi$ and the choice of majorizing kernel $h$. Fix $h$ and let $W_h \subseteq \mathbf{R}^3\backslash\{0\}$ denote the support of $h$. Let $\mathcal{B}_h$ denote the Borel subsets of $W_h$. For fixed $\xi \in W_h$, let $\{(\xi_\mathbf{v}, \kappa_\mathbf{v}) : \mathbf{v} \in \mathcal{V}\}$ be the tree-indexed stochastic process starting at $(\xi_\theta, \kappa_\theta)$ with $\xi_\theta = \xi \in W_h,\ \kappa_\theta \in \{0, 1\}$, taking values in the state space $W_h \times \{0, 1\}$, and defined on a probability space $(\Omega, \mathcal{F}, P_\xi)$ by the following properties:

1. $P_\xi(\xi_\theta \in B,\ \kappa_\theta = \kappa) = \frac{1}{2}\delta_\xi(B),\ B \in \mathcal{B}_h,\ \kappa \in \{0, 1\}$.

2. For any fixed $\mathbf{v} \in \partial\mathcal{V}$, the sequence $(\xi_{\mathbf{v}|0},\ \kappa_{\mathbf{v}|0}),\ (\xi_{\mathbf{v}|1},\ \kappa_{\mathbf{v}|1}),\ (\xi_{\mathbf{v}|2},\ \kappa_{\mathbf{v}|2}), \ldots$ is a Markov chain with transition probabilities
$$\begin{aligned} &P_\xi(\xi_{\mathbf{v}|n+1} \in B,\ \kappa_{\mathbf{v}|n+1} = \kappa|\sigma(\{(\xi_\mathbf{u}, \kappa_\mathbf{u}) : |\mathbf{u}| \leq n\})) \\ &\quad = \frac{1}{2}\int_B \frac{h(\xi_{\mathbf{v}|n} - \eta)h(\eta)}{h * h(\xi_{\mathbf{v}|n})} d\eta \end{aligned} \tag{29}$$
for $B \in \mathcal{B}_h, \kappa \in \{0, 1\}$. In particular, for $\mathbf{v} \in \mathcal{V},\ \xi_{\mathbf{v}1} + \xi_{\mathbf{v}2} = \xi_\mathbf{v} \quad P_\xi -$ *a.s.*, where $\mathbf{v}j = (v_1 \ldots v_n)j := (v_1 \cdots v_n, j),\ j = 1, 2, \ldots$ is the concatenation operation.

3. For any $\mathbf{u}, \mathbf{v}, \in \partial\mathcal{V},\ \{(\xi_{\mathbf{u}|m}, \kappa_{\mathbf{u}|m})\}_{m=0}^\infty$ and $\{(\xi_{\mathbf{v}|m}, \kappa_{\mathbf{v}|m})\}_{m=0}^\infty$ are conditionally independent given $\sigma(\{(\xi_\mathbf{w}, \kappa_\mathbf{w}) : |\mathbf{w}| \leq |\mathbf{u} \wedge \mathbf{v}|\})$.

4. Let $\{\overline{S}_\mathbf{v} : \mathbf{v} \in \mathcal{V}\}$ be a sequence of *iid* mean one exponentially distributed random variables defined on $(\Omega, \mathcal{F}, P_\xi)$ and independent of $\{(\xi_\mathbf{v}, \kappa_\mathbf{v}) : \mathbf{v} \in \mathcal{V}\}$. Define $\lambda(\eta) = \nu|\eta|^2$ for $\eta \in W_h$ and
$$S_\mathbf{v} = \lambda(\xi_\mathbf{v})^{-1} \cdot \overline{S}_\mathbf{v},\ \mathbf{v} \in \mathcal{V}.$$

Conditionally given $\{\xi_\mathbf{v} : \mathbf{v} \in \mathcal{V}\}$, the collection $\{S_\mathbf{v} : \mathbf{v} \in \mathcal{V}\}$ consists of independent exponentially distributed random variables having respective conditional means $\{\lambda(\xi_\mathbf{v})^{-1} : \mathbf{v} \in \mathcal{V}\}$.

**Remark 3.1.** The above properties, although not an explicit construction, define the stochastic model; see Harris (1989) for an approach to construction of the underlying probability space.

Our objective now is to use the stochastic branching model represented by the collection of random variables $\{\xi_{\mathbf{v}}, \kappa_{\mathbf{v}}, S_{\mathbf{v}} : \mathbf{v} \in \mathcal{V}\}$ to recursively define a random functional related to (FNS) through its expected value. Namely, for measurable functions $\chi_0 : W_h \to \mathbf{C}^3$ and $\varphi : W_h \times [0,\infty) \to \mathbf{C}^3$, and for $\xi_\theta = \xi \in W_h, t \geq 0$, the stochastic functional $\chi(\theta, t)$ is recursively defined by

$$\chi(\theta,t) \quad = \quad \begin{cases} \chi_0(\xi_\theta), & \text{if } S_\theta > t \\ \varphi(\xi_\theta, t - S_\theta) & \text{if } S_\theta \leq t, \kappa_\theta = 0 \\ m(\xi_\theta)\chi(<1>,\ t - S_\theta) \otimes_{\xi_\theta} \chi(<2>, t - S_\theta), & else \end{cases} \tag{30}$$

where the product $\otimes_\xi$ and factors $m(\xi)$ are defined in (7), (8), respectively, and where $<1>, <2>$ are root vertices of the shifted full binary trees

$$\mathcal{V}_{<i>} := \{<i>, <i,1>, <i,2>, <i,1,1>, <i,1,2>, <i,2,1>, \ldots\}, \tag{31}$$

types $\xi_{<i>}, i = 1, 2$, respectively.

For evaluation of the stochastic functional $\chi(\theta, t)$, for a given $\xi_\theta = \xi$, it is useful to identify a particular tree structure intrinsic to the stochastic branching model by (see Figure 2).

$$\tau_\theta(t) = \{\mathbf{v} \in \mathcal{V} : \prod_{j=0}^{|\mathbf{v}|-1} k_{\mathbf{v}|j} = 1,\ B_{\mathbf{v}} \leq t\} \tag{32}$$

where

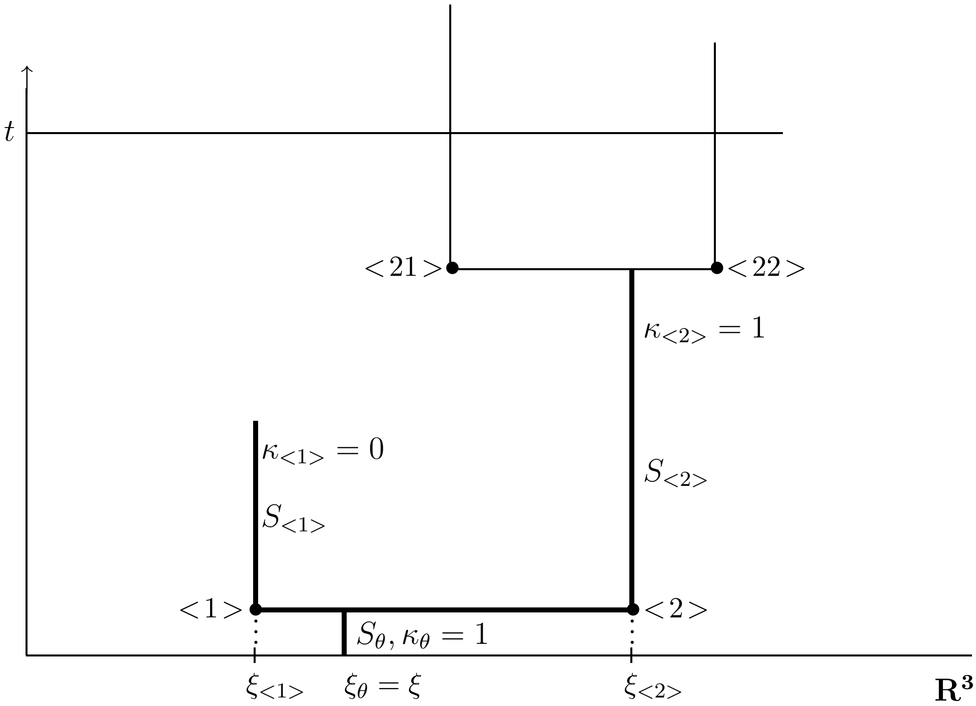
$$B_\theta = 0, \qquad B_{\mathbf{v}} = \sum_{j=0}^{|\mathbf{v}|-1} S_{\mathbf{v}|j}, \quad \theta \neq \mathbf{v} \in \mathcal{V}. \tag{33}$$

It is helpful to have a bit more notation and further decompose $\tau_\theta(t)$ into sets of vertices of two types. We say that $\mathbf{v} \in \mathcal{V}$, *born* at time $B_{\mathbf{v}}$, *survives* for a time $S_{\mathbf{v}}$ until the *clock ring* at time $R_{\mathbf{v}} := B_{\mathbf{v}} + S_{\mathbf{v}} = \sum_{j=0}^{|\mathbf{v}|} S_{\mathbf{v}|j}$. In this way we can partition $\tau_\theta(t)$ into the vertices born before time $t$ with clock rings before and after time $t$; see Figure 2. Namely,

$$\tau_\theta(t) = \tau_\theta(<t) \cup \tau_\theta(>t)$$

where

$$\begin{aligned} \tau_\theta(<t) &= \{\mathbf{v} \in \tau_\theta(t) : R_{\mathbf{v}} \leq t\} \\ \tau_\theta(>t) &= \{\mathbf{v} \in \tau_\theta(t) : B_{\mathbf{v}} \leq t < R_{\mathbf{v}}\} \end{aligned}$$




$$\tau_\theta(t)$$

Figure 2: Schematic of a tree indexed branching random walk with $\tau_\theta(< t)$ denoted in bold lines

Since the discrete branching process defined by $\{\mathbf{v} \in \mathcal{V} : \prod_{j=0}^{|\mathbf{v}|} \kappa_{\mathbf{v}|j} = 1\}$ is a critical binary Galton-Watson process, the recursion will terminate in a finite number of iterations with probability one. In particular, $\mathcal{X}(\theta, t)$ is simply a *finite* product of values of $\chi_0$ and/or $\varphi$. For example, the functional evaluation of the sample tree in Figure 2 is given by

$$\mathcal{X}(\theta, t) = m(\xi_\theta) m(\xi_{<2>}) \varphi(\xi_{<1>}, t - R_{<1>}) \otimes_{\xi_\theta} [\chi_0(\xi_{<21>}) \otimes_{\xi_{<2>}} \chi_0(\xi_{<22>})]$$

In particular, the product is over vertices $\mathbf{v} \in \tau_\theta(t)$ with evaluations of factors at the *leaves* $\mathbf{v}$ of $\tau_\theta(< t)$ as $\varphi(\xi_{\mathbf{v}},\ t - R_{\mathbf{v}})$ and at the *leaves* $\mathbf{v}$ of $\tau_\theta(> t)$ as $\chi_0(\xi_{\mathbf{v}})$; here a *leaf* refers to a terminal vertex, while a nonterminating vertex is referred to as a *branch point*. No essential use of graph theoretic notions is made beyond their descriptive role in this development.

**Remark 3.2.** The branching random walk constructed here differs from that introduced by LeJan and Sznitman (1998) in that by constructing the process forward in time we eliminate the dependence of the model $(\Omega, \mathcal{F}, P_\xi)$ on $t$. Secondly, a larger class of transition probabilities is furnished by the respective class of majorizing kernels. In order to relate the stochastic framework to (FNS) and/or $(\text{FNS})_h$, we require a notion of *solution.* The first is a variant on that formulated by LeJan and Sznitman (1997) for solutions to $(\text{FNS})_h$ in the special case $h = h_0^{(\alpha)}$. Since we do not wish to exclude the analysis of complex valued solutions, we do not include their condition $\overline{h(\xi)\chi(\xi,t)} = h(-\xi)\chi(-\xi,t)$ in the definition of solution, but choose to consider it as a possible subsequent property of solutions.

**Definition 3.1.** A function $\chi : W_h \times [0,T] \to C^3$ which is

1. continuous in $t \in [0,T]$ for each fixed $\xi \in W_h$,
2. measurable in $\xi \in W_h$ for each fixed $t \in [0,T]$,

   and satisfies

3. $\int_0^T \int_{W_h \times W_h} |\chi(\xi_1, s) \cdot e_\xi| \cdot |\pi_{\xi\perp}\chi(\xi_2, s)| H(\xi, d\xi_1 \times d\xi_2) < \infty$ for a.e. $\xi \in W_h$, and
4. $\chi(\xi,t) \cdot \xi = 0,\ 0 \le t \le T$,

will be called a *solution* to $(\text{FNS})_h$ for initial data $\chi_0 : W_h \to \mathbf{C}^3,\ \chi_0(\xi) \cdot \xi = 0$, and forcing $\varphi : W_h \times [0,T] \to \mathbf{C}^3,\ \int_0^T |\varphi(\xi,t)| dt < \infty,\ \varphi(\xi,t) \cdot \xi = 0$, provided $(\text{FNS})_h$ holds for *a.e.* $\xi \in W_h$.

**Remark 3.3.** Global solutions are defined by requiring the conditions of the definition for all $T > 0$. In the case that a solution to $(\text{FNS})_h$ also satisfies

$$\overline{h(\xi)\chi(\xi,t)} = h(-\xi)\chi(-\xi,t)$$

we will say $\chi(\xi,t)$ is a *solution in the sense of LeJan-Sznitman.*

Although our focus is on *majorizing kernels,* the stochastic model may be constructed for any measurable $h : W_h \to (0,\infty)$ such that $h * h(\xi) < \infty$. With this in mind we make the following definition.

**Definition 3.2.** Let $1/h$ be a Fourier multiplier on $W_h$. We say that the pair $(u_0, g)$ is $(\text{FNS})_h-$admissible if

1. $\hat{u}_0(\xi) = \hat{g}(\xi,t) = 0$ for a.e. $\xi \in W_h^c, t \geq 0$.
2. $\mathbf{E}_{\xi_\theta=\xi}|\mathbb{X}(\theta,t)| < \infty$ for a.e. $\xi \in W_h,\ t \geq 0$,

where $\chi_0(\xi) = \hat{u}_0(\xi)/h(\xi)$, and $\varphi(\xi,t) = 2\hat{g}(\xi,t)/(\nu|\xi|^2 h(\xi)), t \geq 0$, as in (8).

**Theorem 3.1** *(Existence) If $(u_0, g)$ is (FNS)$_h$-admissible for a given Fourier multiplier $1/h$, then*

$$\hat{u}(\xi,t) \quad = \quad \begin{cases} h(\xi)\mathbf{E}_{\xi_\theta=\xi}\mathbb{X}(\theta,t), & \text{if } \xi \in W_h,\ t \geq 0, \\ 0 \quad \text{if } \xi \in W_h^c, t \geq 0, \end{cases}$$

*is a solution to (FNS).*

*Proof.* As noted in Remark 1.1, it suffices to consider (FNS)$_h$. To verify that (FNS)$_h$ is satisfied, decompose $\mathbb{X}(\theta,t)$ as

$$\begin{aligned} \mathbb{X}(\theta,t) \quad &= \quad \mathbb{X}(\theta,t)\mathbf{1}[S_\theta > t] + \mathbb{X}(\theta,t)\mathbf{1}[S_\theta \leq t, \kappa_\theta = 0] \\ &\qquad + \mathbb{X}(\theta,t)\mathbf{1}[S_\theta \leq t, \kappa_\theta = 1], \end{aligned}$$

take expectation starting at $\xi$, and use the strong Markov property and conditional independence in the recursive definition of $\mathbb{X}(\theta,t)$ on $[S_\theta \leq t, \kappa_\theta = 1]$. Specifically,

$$\begin{aligned} & \mathbf{E}_{\xi_\theta=\xi}\{m(\xi_\theta)\mathbb{X}(<1>, t-S_\theta) \otimes_{\xi_\theta} \mathbb{X}(<2>, t-S_\theta)\mathbf{1}[S_\theta \leq t, \kappa = 1]\} \\ = \; & m(\xi)\mathbf{E}_{\xi_\theta=\xi}\{\mathbf{1}[S_\theta \leq t, \kappa_\theta = 1]\mathbf{E}\{\mathbb{X}(<1>, t-S_\theta) \\ & \quad \otimes_{\xi_\theta}\mathbb{X}(<2>, t-S_\theta)|\xi_{<1>}, \xi_{<2>}, S_\theta, \kappa_\theta\}\} \\ = \; & m(\xi)\mathbf{E}_{\xi_\theta=\xi}\{\mathbf{1}[S_\theta \leq t, \kappa_\theta = 1]\chi(\xi_{<1>}, t-S_\theta) \otimes_{\xi_\theta} \chi(\xi_{<2>}, t-S_\theta)\} \\ = \; & \frac{1}{2}m(\xi)\int_0^t \lambda(\xi)e^{-\lambda(\xi)s}\int_{W_h\times W_h}\chi(\eta_1, t-s)\otimes_\xi \chi(\eta_2, t-s)H(\xi, d\eta_1 \times d\eta_2)ds. \end{aligned}$$

The continuity requirement in (1) of Definition 3.1 is evident in the representation of $\chi(\xi,t)$ by (FNS)$_h$. The measurability (2) may be obtained from the measure theoretic construction of the stochastic branching model. The condition (3) is contained in the (FNS)$_h$-admissibility definition. To check the incompressibility condition (4) simply observe that samplepointwise one has

$$\mathbb{X}(\theta,t)\cdot\xi_\theta = 0$$

by the definition of $\mathcal{X}(\theta, t)$, orthogonality of $\pi_{\xi\perp}$, and corresponding hypothesis on $\chi_0(\xi)$ and $\varphi(\xi, t)$. ∎

The proof of the existence part of Theorem 1.1 stated in the introduction now follows as a corollary to Theorem 3.1 as follows:
*Proof of Existence in Theorem 1.1.* Defining $c_\nu = \nu(2\pi)^{\frac{3}{2}}/2$, the conditions of Theorem 1.1 state that

$$(i.) h * h(\xi) \leq |\xi| h(\xi), \quad (ii.) |\hat{u}_0(\xi)| \leq c_\nu h(\xi), \quad (iii.) |\hat{g}(\xi, t)| \leq \nu c_\nu |\xi|^2 h(\xi)/2.$$

Thus one may define a majorizing kernel $h_\nu$ with constant $c_\nu$ by

$$h_\nu(\xi) = c_\nu h(\xi), \xi \in W_{h_\nu} = W_h.$$

The conditions (i)-(iii) may then be expressed with respect to the majorizing kernel $h_\nu$ as

$$(i.) m(\xi) = h_\nu * h_\nu(\xi)/(c_\nu |\xi| h_\nu(\xi)) \leq 1, \quad (ii.) |\chi_0(\xi)| \leq 1, \quad (iii.) |\varphi(\xi, t)| \leq 1,$$

where $\chi_0(\xi) = \hat{u}_0(\xi)/h_\nu(\xi)$, and $\varphi(\xi, t) = 2\hat{g}(\xi, t)/\nu|\xi|^2 h_\nu(\xi)$. In particular it follows that $|\mathcal{X}(\theta, t)| \leq 1$ for this choice of majorizing kernel, and hence Theorem 3.1 applies. Now one may check that cancellations make the formula defining the solution $\hat{u}(\xi, t)$ invariant under rescalings of $h$ by constants. Specifically, it follows from the defining stochastic recursion (30) that for any positive constant $c > 0$ one has

$$c\mathcal{X}_{ch}(\theta, t) = \mathcal{X}_h(\theta, t) \quad a.s., \tag{34}$$

where $\mathcal{X}_h$ denotes the functional corresponding to the Fourier multiplier $h$. Note that the stochastic functional is always a.s. finite since the stochastic recursion terminates in a finite number of steps a.s.. ∎

**Remark 3.4.** Note that for a Fourier multiplier $1/h$ induced by a majorizing kernel $h$ with constant $B > 0$, the corresponding factor $m(\xi)$ is bounded by one provided this constant is sufficiently small, i.e. $B \leq c_\nu = \nu(2\pi)^{\frac{3}{2}}/2$. In this case one sees that $(u_0, g)$ is (FNS)$_h$-admissible under the condition that $|\hat{u}_0(\xi)| \leq Bh(\xi)$, and $|\hat{g}(\xi, t)| \leq B\nu|\xi|^2 h(\xi)/2$ by virtue of the implied a.s. unit bound on the functional $\mathcal{X}$. In particular there is an implied competition over the size of the majorizing constant $B$ in this approach. Recently Chris

Orum (2002) has shown that one may further exploit incompressibility as reflected in the geometry of the product $\otimes_\xi$ to obtain $(\text{FNS})_h-$admissible majorizing kernels with constants which are twice as large as these.

Under the additional hypothesis that $h(\xi) = h(-\xi)$ one may check that

$$\overline{\mathbb{X}(\theta,t)}|_{\xi_\theta=\xi} \stackrel{dist}{=} \mathbb{X}(\theta,t)|_{\xi_\theta=-\xi}.$$

As a result it will follow that $\chi(\xi,t) = \mathbf{E}_{\xi_\theta=\xi}\mathbb{X}(\theta,t)$ is also a solution in the sense of LeJan-Sznitman under this additional condition. However, we shall also see in a later section that this assumption is not necessary for the expected value.

The above proof of the existence part of Theorem 1.1 provides a global solution in the ball $\mathcal{B}_0(0,R)$ in the space $\mathcal{F}_{h,0,T}, T > 0$, of radius $R = c_\nu = \nu(2\pi)^{\frac{3}{2}}/2$. For uniqueness of solutions within this ball an argument along the lines of that used by LeJan and Sznitman (1997) may be applied to obtain the following.

**Theorem 3.2** *(Uniqueness). Let $h(\xi)$ be a standard majorizing kernel with exponent $\theta = 1$. Fix $0 < T \leq +\infty$. Suppose that $\|u_0\|_{\mathcal{F}_{h,0,T}} \leq \nu(\sqrt{2\pi})^3/2$ and $\|\Delta^{-1}g\|_{\mathcal{F}_{h,0,T}} \leq \nu^2(\sqrt{2\pi})^3/4$. Then the solution*

$$\hat{u}(\xi,t) = \begin{cases} h(\xi)\mathbf{E}_{\xi_\theta=\xi}\mathbb{X}(\theta,t), & \textit{if } \xi \in W_h,\ t \geq 0, \\ 0 \quad \textit{if } \xi \in W_h^c, t \geq 0, \end{cases}$$

*is unique in the ball $\mathcal{B}_0(0,R)$ centered at 0 of radius $R = \nu(\sqrt{2\pi})^3/2$ in the space $\mathcal{F}_{h,0,T}$.*

*Proof.* Suppose that $\hat{w}(\xi,t)$ is another solution to (FNS) with $|\hat{w}(\xi,t)| \leq Rh(\xi)$. As in the proof of Theorem 1.1, without loss of generality one may replace $h$ by $h_\nu = c_\nu h$, where $c_\nu \equiv R = \nu(2\pi)^{\frac{3}{2}}/2$ and define

$$\gamma(\xi,t) = \hat{w}(\xi,t)/h_\nu(\xi).$$

Then

$$\sup_{\substack{\xi\in W_h \\ 0\leq t\leq T}} |\gamma(\xi,t)| \leq 1.$$

Define a truncation of $\tau_\theta(t)$ by

$$\tau_\theta^{(n)}(t) = \{\mathbf{v} \in \tau_\theta(t) : |\mathbf{v}| \le n\}, \quad n = 0, 1, 2, \ldots.$$

Let $Y(\tau_\theta^{(n)}(t))$ be the recursively defined random functional given by

$$Y(\tau_\theta^{(n)}(t)) \;=\; \begin{cases} \chi_0(\xi_\theta) \quad \text{if } S_\theta > t \\ \varphi(\xi_\theta, t - S_\theta) \quad \text{if } S_\theta \le t, \kappa_\theta = 0 \\ m(\xi_\theta) Y(\tau_{<1>}^{(n-1)}(t - S_\theta)) \otimes_{\xi_\theta} Y(\tau_{<2>}^{(n-1)}(t - S_\theta)), \text{else} \end{cases}$$

for $n = 1, 2, \ldots,$ where $\chi_0(\xi) = \hat{u}_0(\xi)/h_\nu(\xi)$, $\varphi(\xi, t) = 2\hat{g}(\xi, t)/(\nu|\xi|^2 h_\nu(\xi))$, $m(\xi) = 2h_\nu * h_\nu(\xi)/(\nu(2\pi)^{3/2}|\xi| h_\nu(\xi)) \le 1$, and

$$Y(\tau_\theta^{(0)}(t)) \;=\; \begin{cases} \chi_0(\xi_\theta) \quad \text{if } S_\theta > t \\ \varphi(\xi_\theta, t - S_\theta) \quad \text{if } S_\theta \le t, \kappa_\theta = 0 \\ m(\xi_\theta)\gamma(\xi_{<1>}, t - S_\theta) \otimes_{\xi_\theta} \gamma(\xi_{<2>}, t - S_\theta), \text{else.} \end{cases}$$

Observe that since $\hat{w}(\xi, t)$ is an assumed solution to (FNS) it follows directly from $(\text{FNS})_{h_\nu}$ that

$$\mathbf{E}_{\xi_\theta = \xi} Y(\tau_\theta^{(0)}(t)) = \gamma(\xi, t).$$

Moreover, using $(\text{FNS})_{h_\nu}$ and conditioning on

$$\mathcal{F}_n = \sigma(\{S_\mathbf{v}, \xi_\mathbf{v}, \kappa_\mathbf{v} : |\mathbf{v}| \le n\}),$$

this extends by induction to yield

$$\gamma(\xi, t) = \mathbf{E}_{\xi_\theta = \xi} Y(\tau_\theta^{(n)}(t)) \quad \text{for } n = 0, 1, 2, \ldots.$$

Specifically, one has

$$\begin{aligned} & \mathbf{E}_{\xi_\theta = \xi} Y(\tau_\theta^{(n+1)}(t)) \\ = \;& \chi_0(\xi) e^{-\lambda(\xi) t} + \frac{1}{2}\int_0^t \lambda(\xi) e^{-\lambda(\xi)s} \varphi(\xi, t - s) ds \\ & + m(\xi) \mathbf{E}_{\xi_\theta = \xi}\{Y(\tau_{<1>}^{(n)}(t - S_\theta)) \otimes_\xi Y(\tau_{<2>}^{(n)}(t - S_\theta)) \mathbf{1}[S_\theta \le t, \kappa_\theta = 1]\} \\ = \;& \chi_0(\xi) e^{-\lambda(\xi) t} + \frac{1}{2}\int_0^t \varphi(\xi, t - s)\lambda(\xi) e^{-\lambda(\xi)s} ds + m(\xi)\frac{1}{2}\int_0^t \lambda(\xi) e^{-\lambda(\xi)s} \\ & \cdot \int \mathbf{E}_{\xi_{<1>}} Y(\tau_{<1>}^{(n)}(t - s)) \otimes_\xi \mathbf{E}_{\xi_{<2>}}(Y(\tau_{<2>}^{(n)}(t - s)) H(\xi, d\xi_{<1>} \times d\xi_{<2>})\, ds. \end{aligned}$$

Now observe that

$$Y(\tau_\theta^{(0)}(t)) = \mathbb{X}(\theta, t) \quad \text{on } [\tau_\theta^{(0)}(t) = \tau_\theta(t)],$$

and more generally, since the terms $\gamma(\xi_{\mathbf{v}}, t - R_{\mathbf{v}})$ appear in $Y$ only at truncated vertices,

$$Y(\tau_\theta^{(n)}(t)) = \mathbb{X}(\theta, t) \quad \text{on } [\tau_\theta^{(n)}(t) = \tau_\theta(t)].$$

Thus, since

$$\mathbf{E}|Y(\tau_\theta^{(n)}(t))| \le 1 \quad \text{for all } n$$

and

$$\mathbf{E}|\mathbb{X}(\theta, t)| \le 1 \quad \text{for all } n$$

we have

$$\begin{aligned} |\gamma(\xi, t) - \mathbf{E}\mathbb{X}(\theta, t)| &= |\mathbf{E}\{Y(\tau_\theta^{(n)}(t)) - \mathbb{X}(\theta, t)\mathbf{1}[\tau_\theta^{(n)}(t) \neq \tau_\theta(t)]\}| \\ &\le 2P(\tau_\theta^{(n)}(t) \neq \tau_\theta(t)) \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

■

**Corollary 3.1** *Under the conditions of the theorem one has*

$$Y(\tau_\theta^{(n)}(t)) = \mathbf{E}_{\xi_\theta = \xi}\{\mathbb{X}(\theta, t) | \mathcal{F}_n\}, \quad n = 0, 1, 2, \ldots$$

*where*

1. $\mathcal{F}_n = \sigma(\{S_{\mathbf{v}}, \xi_{\mathbf{v}1}, \xi_{\mathbf{v}2}, \kappa_{\mathbf{v}} : |\mathbf{v}| \le n\})$
2. $$Y(\tau_\theta^{(0)}(t)) = \begin{cases} \chi_0(\xi_\theta) & \text{if } S_\theta > t \\ \varphi(\xi_\theta, t - S_\theta) & \text{if } S_\theta \le t, \kappa_\theta = 0 \\ m(\xi_\theta)\mathbf{E}_{\xi_{<1>}}\mathbb{X}(<1>, t - S_\theta) & \\ \quad \otimes_{\xi_\theta}\mathbf{E}_{\xi_{<2>}}\mathbb{X}(<2>, t - S_\theta), & \text{else} \end{cases}$$

*In particular,* $\{Y(\tau_\theta^{(n)}(t)) : n = 0, 1, 2, \ldots\}$ *is a martingale with respect to the filtration* $\{\mathcal{F}_n : n \ge 0\}$.

*Proof.* First note from the recursive definition of the functional $Y(\tau_\theta^{(n)}(t))$ that for any $N \geq n$,

$$\mathbf{E}(Y(\tau_\theta^{(N)}(t))|\mathcal{F}_n) = Y(\tau_\theta^{(n)}(t)),\ \ N \geq n.$$

Let $G = G(S_{\mathbf{v}}, \xi_{\mathbf{v}1}, \xi_{\mathbf{v}2}, \kappa_{\mathbf{v}} : |\mathbf{v}| \leq n))$ be a bounded $\mathcal{F}_n$-measurable function. Then, for $N \geq n$,

$$\begin{aligned}
\mathbf{E}\{G \cdot Y(\tau_\theta^{(n)}(t))\} &= \mathbf{E}\{G \cdot \mathbf{E}\{Y(\tau_\theta^{(N)}(t))|\mathcal{F}_n\}\} \\
&= \mathbf{E}\{\mathbf{E}\{G \cdot Y(\tau_\theta^{(N)}(t))|\mathcal{F}_n\} \\
&= \mathbf{E}\{G \cdot Y(\tau_\theta^{(N)}(t))\} \\
&= \lim_{N\to\infty} \mathbf{E}\{GY(\tau_\theta^{(N)}(t))\} \\
&= \mathbf{E}\{\lim_{N\to\infty} GY(\tau_\theta^{(N)}(t))\mathbf{1}[\tau_\theta^{(N)}(t) = \tau_\theta(t)]\} \\
&= \mathbf{E}\{G\chi(\theta,t)\}.
\end{aligned}$$

■

# 4 Picard Iterations of a Contraction Map

In this section we show how majorizing kernels can be used to obtain local or global solutions of the Navier-Stokes equations following a contraction mapping argument. At the same time, relations of the stochastic cascade theory with a Picard iteration scheme are established.

Recall that the (FNS) equations are

$$\begin{aligned}
\hat{u}(\xi,t) = e^{-\nu|\xi|^2 t}\hat{u}_0(\xi) + \hat{B}(\hat{u},\hat{u})(\xi,t) \\
+ \int_0^t e^{-\nu|\xi|^2 s}\hat{g}(\xi,t-s)ds := \hat{\mathcal{Q}}[\hat{u};\hat{u}_0,\hat{g}](\xi,t)
\end{aligned} \tag{35}$$

where

$$\begin{aligned}
\hat{B}(\hat{u},\hat{v})(\xi,t) := \int_0^t e^{-\nu|\xi|^2 s}|\xi|(2\pi)^{-\frac{3}{2}} \\
\int \{\hat{u}(\xi-\eta,t-s) \otimes_\xi \hat{v}(\eta,t-s)\}d\eta ds.
\end{aligned}$$

Consider the Picard iteration scheme naturally associated with the (projected) Navier-Stokes equation

$$u_{n+1}(x,t) = F(x,t) + B(u_n, u_n)(x,t) \tag{36}$$

where $F(x,t) = e^{t\nu\Delta}u_0(x) + \int_0^t e^{s\nu\Delta}g(x,t-s)ds$, $u^{(0)}(x,t) = e^{t\nu\Delta}u_0(x)$ and $u_1(x,t) = F(x,t) + B(u^{(0)}, u^{(0)})(x,t)$. The convergence of the iterates follows from showing that that $\mathcal{Q}$ is a contraction in an appropriate ball in $\mathcal{F}_{h,\gamma,T}$.

**Remark 4.1** In the case $\gamma = 1$ the smaller ball for existence and uniqueness is related to the increased regularity, namely spatial analyticity, implied by the decay on the Fourier transform in this case. Existence and uniqueness results in the larger balls obtained with $\gamma = 0$ are aimed at $C^\infty$-smoothness.

The following lemmas summarize some of the technical details.

**Lemma 4.1** *Let* $0 \le \beta \le 2$, $\mu > 0$ *and* $M(\beta) = \sup_{\lambda>0} \frac{1-e^{-\lambda}}{\lambda^{(2-\beta)/2}}$. *Then*

$$\int_0^t |\xi|^\beta e^{-\mu|\xi|^2 s} ds \le t^{(2-\beta)/2}\mu^{-\beta/2}M(\beta)$$

*Proof:* A direct calculation gives for $0 \le \beta \le 2$,

$$\int_0^t |\xi|^\beta e^{-\mu|\xi|^2 s} ds = \frac{1-e^{-\mu|\xi|^2 t}}{\mu|\xi|^{2-\beta}} = \frac{t^{(2-\beta)/2}}{\mu^{\beta/2}} \frac{1-e^{-\lambda}}{\lambda^{(2-\beta)/2}}$$

where $\lambda = \mu|\xi|^2 t$ and the result follows immediately. ∎

In the spirit of Foias and Temam (1989) and Lemarié-Rieusset (2000), one has the following estimate.

**Lemma 4.2** *Let* $\xi, \eta \in R^n$, $0 \le s \le t$. *Then*

$$e^{-\nu s|\xi|^2 - \sqrt{t-s}|\xi-\eta| - \sqrt{t-s}|\eta|} \le e^{1/(2\nu)} e^{-\sqrt{t}|\xi|} e^{-\nu s|\xi|^2/2}$$

*Proof*: Using the triangle inequality, it suffices to show that

$$f(|\xi|) := \frac{1}{2\nu} + \frac{1}{2}\nu|\xi|^2 s + \sqrt{t-s}|\xi| - \sqrt{t}|\xi| \ge 0.$$

A simple calculation shows that $f(r)$ achieves its minimum value at $r = (\sqrt{t} - \sqrt{t-s})/(\nu s) = 1/[\nu(\sqrt{t} + \sqrt{t-s})]$ of

$$\frac{1}{\nu} \frac{\sqrt{t-s}}{\sqrt{t} + \sqrt{t-s}}$$

which is non-negative for $0 \leq s \leq t$. ■

Using the above lemmas, it is possible to estimate the bilinear form $B(u,v)$. When considering majorizing kernel of exponent 1, it is the size of the data that is used to show that $\mathcal{Q}$ is a contraction on a small ball centered at the origin. For this pointwise estimates of $\hat{B}$ will be needed.

**Proposition 4.1** *Let $h$ be a standard majorizing kernel of exponent $\theta = 1$. For $\gamma = 0$ or $1$, let $C(1,\gamma) = (2\pi)^{-3/2}2^{\gamma}$. Then for $u(x,t), v(x,t) \in \mathcal{F}_{h,\gamma,T}$, and $0 \leq t \leq T$,*

$$|\hat{B}(\hat{u},\hat{v})(\xi,t)| \leq \|u\|_{\mathcal{F}_{h,\gamma,T}}\|v\|_{\mathcal{F}_{h,\gamma,T}} h(\xi)e^{-\gamma\sqrt{t}|\xi|}C(1,\gamma)\frac{1-e^{-\nu|\xi|^2t/2^{\gamma}}}{\nu}e^{\gamma/(2\nu)}$$

*Proof:* Considering the case $\gamma = 0$ first one has;

$$\begin{aligned}
|\hat{B}(\hat{u},\hat{v})(\xi,t)| &\leq \|u\|_{\mathcal{F}_{h,0,T}}\|v\|_{\mathcal{F}_{h,0,T}}\int_0^t [e^{-\nu|\xi|^2s}|\xi|(2\pi)^{-\frac{3}{2}}\int h(\xi-\eta)h(\eta)d\eta]ds \\
&\leq \|u\|_{\mathcal{F}_{h,0,T}}\|v\|_{\mathcal{F}_{h,0,T}} h(\xi)(2\pi)^{-\frac{3}{2}}\int_0^t e^{-\nu|\xi|^2s}|\xi|^2 ds \\
&\leq \|u\|_{\mathcal{F}_{h,0,T}}\|v\|_{\mathcal{F}_{h,0,T}} h(\xi)(2\pi)^{-\frac{3}{2}}\frac{1}{\nu}(1-e^{-\nu|\xi|^2t}).
\end{aligned}$$

Similarly, for $\gamma = 1$ and using Lemma 4.2,

$$\begin{aligned}
|\hat{B}(\hat{u},\hat{v})(\xi,t)| &\leq \|u\|_{\mathcal{F}_{h,1,T}}\|v\|_{\mathcal{F}_{h,1,T}}\int_0^t \Big[e^{-\nu|\xi|^2s}|\xi|(2\pi)^{-\frac{3}{2}} \\
&\qquad \int h(\xi-\eta)h(\eta)e^{-\sqrt{t-s}|\xi-\eta|}e^{-\sqrt{t-s}|\eta|}d\eta\Big] ds \\
&\leq \|u\|_{\mathcal{F}_{h,1,T}}\|v\|_{\mathcal{F}_{h,1,T}} h(\xi)e^{-\sqrt{t}|\xi|}e^{1/(2\nu)}(2\pi)^{-\frac{3}{2}}\int_0^t e^{-\nu|\xi|^2s/2}|\xi|^2 ds \\
&\leq \|u\|_{\mathcal{F}_{h,1,T}}\|v\|_{\mathcal{F}_{h,1,T}} h(\xi)e^{-\sqrt{t}|\xi|}e^{1/(2\nu)}(2\pi)^{-\frac{3}{2}}\frac{2}{\nu}(1-e^{-\nu|\xi|^2t/2}).
\end{aligned}$$

■

When using majorizing kernels of exponent $\theta < 1$, estimates on the norm of the bilinear form $B(u,v)$ are obtained using the time integral as follows.

**Proposition 4.2** *Let $h$ be a standard majorizing kernel of exponent $\theta < 1$ and let $C(\theta,\gamma) = M(\theta+1)(2\pi)^{-3/2}2^{\gamma(\theta+1)/2}$ where $M(\theta+1)$ is defined in Lemma 4.1. Then for $u, v \in \mathcal{F}_{h,\gamma,T}$,*

$$\|B(u,v)\|_{\mathcal{F}_{h,\gamma,T}} \le \|u\|_{\mathcal{F}_{h,\gamma,T}} \|v\|_{\mathcal{F}_{h,\gamma,T}} C(\theta,\gamma) T^{(1-\theta)/2} \left(\frac{1}{\nu}\right)^{(\theta+1)/2} e^{\gamma/(2\nu)}.$$

*Proof:* Considering $\gamma = 0$ one has

$$\begin{aligned}
|\hat{B}(\hat{u},\hat{v})|(\xi,t) &\le (2\pi)^{-3/2}\|u\|_{\mathcal{F}_{h,0,T}}\|v\|_{\mathcal{F}_{h,0,T}} \int_0^t |\xi| \left[e^{-\nu|\xi|^2 s}\int h(\xi-\eta)h(\eta)d\eta\right] ds \\
&\le (2\pi)^{-3/2}\|u\|_{\mathcal{F}_{h,0,T}}\|v\|_{\mathcal{F}_{h,0,T}} h(\xi) \int_0^T |\xi|^{1+\theta} e^{-\nu|\xi|^2 s} ds \\
&\le (2\pi)^{-3/2}\|u\|_{\mathcal{F}_{h,0,T}}\|v\|_{\mathcal{F}_{h,0,T}} h(\xi) M(1+\theta)\left(\frac{1}{\nu}\right)^{(\theta+1)/2} T^{(1-\theta)/2}.
\end{aligned}$$

Similarly, for $\gamma = 1$ use Lemma 4.1 and Lemma 4.2 to get

$$\begin{aligned}
|\hat{B}(\hat{u},\hat{v})|(\xi,t) &\le (2\pi)^{-3/2}\|u\|_{\mathcal{F}_{h,1,T}}\|v\|_{\mathcal{F}_{h,1,T}} \int_0^t |\xi| \left[e^{-\nu|\xi|^2 s}\int [e^{-\sqrt{t-s}|\xi-\eta|}e^{-\sqrt{t-s}|\eta|}\right. \\
&\quad h(\xi-\eta)h(\eta)d\eta]\, ds \\
&\le (2\pi)^{-3/2}e^{1/(2\nu)}\|u\|_{\mathcal{F}_{h,1,T}}\|v\|_{\mathcal{F}_{h,1,T}} h(\xi)e^{-\sqrt{t}|\xi|} \int_0^t |\xi|^{1+\theta} e^{-\nu|\xi|^2 s/2} ds \\
&\le (2\pi)^{-3/2}e^{1/(2\nu)}\|u\|_{\mathcal{F}_{h,1,T}}\|v\|_{\mathcal{F}_{h,1,T}} h(\xi)e^{-\sqrt{t}|\xi|} \\
&\quad \cdot M(1+\theta)\left(\frac{2}{\nu}\right)^{(\theta+1)/2} T^{(1-\theta)/2}.
\end{aligned}$$

■

The first result on global existence is an immediate consequence of these propositions assuming the initial data and forcing are small. As noted in the introduction, the solution determined by this theorem exists for the same time interval on which the forcing remains small.

**Theorem 4.1** *Let $h$ be a standard majorizing kernel of exponent $\theta = 1$. For $\gamma = 0$ or 1, let $\rho_\gamma = \rho_\gamma(\nu) < \min\{1, (\nu/2)(1/C(1,\gamma))\}$ where $C(1,\gamma)$ is defined in Proposition 4.1. Then, if $\|e^{\nu t\Delta}u_0(x)\|_{\mathcal{F}_{h,\gamma,T}} \le \rho_\gamma e^{-\gamma/(2\nu)}$ and $\|(\Delta)^{-1}g(x,t)\|_{\mathcal{F}_{h,\gamma,T}} \le \rho_\gamma(\nu/2)e^{-\gamma/(2\nu)}2^{-\gamma}$, the Navier-Stokes equation have a unique solution $u(x,t) \in \mathcal{F}_{h,\gamma,T}$ satisfying $\|u\|_{\mathcal{F}_{h,\gamma,T}} \le \rho_\gamma$.*

*Proof*: Let $\hat{F}(\xi,t) = e^{-\nu|\xi|^2 t}\hat{u}_0(\xi) + \int_0^t e^{-\nu|\xi|^2 s}\hat{g}(\xi,t-s)ds$.

Consider the case $\gamma = 0$ first. Then

$$|\hat{F}(\xi,t)| \le \rho_0 h(\xi)\left(e^{-\nu|\xi|^2 t} + \frac{1}{2}(1-e^{-\nu|\xi|^2 t})\right), \tag{37}$$

Also, if $\|u\|_{\mathcal{F}_{h,0,T}} \le \rho_0$ it follows from the choice of $\rho_0$ and Proposition 4.1 that

$$|\hat{B}(\hat{u},\hat{u})(\xi,t)| \le \rho_0 h(\xi)\frac{1}{2}(1-e^{-\nu|\xi|^2 t}). \tag{38}$$

Thus, using (37) and (38), one has

$$|\hat{\mathcal{Q}}[\hat{u};\hat{u}_0,\hat{g}](\xi,t)| \le |\hat{F}(\xi,t)| + |\hat{B}(\hat{u},\hat{u})(\xi,t)| \le \rho_0 h(\xi)$$

and so if also $\|v\|_{\mathcal{F}_{h,0,T}} \le \rho_0$, using Proposition 4.1 one has

$$\begin{aligned}\|B(u,u)-B(v,v)\|_{\mathcal{F}_{h,0,T}} &= \|B(u,u-v)+B(u-v,v)\|_{\mathcal{F}_{h,0,T}}\\ &\le \rho_0 C(1,0)(2/\nu)(\|u-v\|_{\mathcal{F}_{h,0,T}}).\end{aligned}$$

The result follows by the contraction mapping theorem since $\rho_0 C(1,0)(2/\nu) < 1$.

Considering $\gamma = 1$, note that $|\hat{u}_0(\xi)|/h(\xi) \le \rho_1 e^{-1/(2\nu)}$ so

$$e^{-\nu|\xi|^2 t}|\hat{u}_0(\xi)| \le \rho_1 h(\xi)e^{-1/(2\nu)}e^{-\nu|\xi|^2 t} \le \rho_1 h(\xi)e^{-\sqrt{t}|\xi|}e^{-\nu|\xi|^2 t/2} \tag{39}$$

where in the last step, Lemma 4.2 with $s = t$ was used. Similarly,

$$\begin{aligned}\left|\int_0^t e^{-\nu|\xi|^2 s}\hat{g}(\xi,t-s)ds\right| &\le \rho_1 h(\xi)\frac{1}{2}\frac{\nu}{2}e^{-1/(2\nu)}\int_0^t e^{-\nu|\xi|^2 s}|\xi|^2 e^{-\sqrt{t-s}|\xi|}ds\\ &\le \rho_1 h(\xi)e^{-\sqrt{t}|\xi|}\frac{1}{2}\frac{\nu}{2}\int_0^t e^{-\nu|\xi|^2 s/2}|\xi|^2 ds\\ &\le \rho_1 h(\xi)e^{-\sqrt{t}|\xi|}\frac{1}{2}(1-e^{-\nu|\xi|^2 t/2})\end{aligned} \tag{40}$$

where in the last step, Lemma 4.2 with $\eta = 0$ was used. Thus, from (39) and (40) it follows that

$$|\hat{F}(\xi,t)| \le \rho_1 h(\xi)e^{-\sqrt{t}|\xi|}\left[e^{-\nu|\xi|^2 t/2} + \frac{1}{2}(1-e^{-\nu|\xi|^2 t/2})\right]. \tag{41}$$

As before, if $\|u\|_{\mathcal{F}_{h,1,T}} \le \rho_1$, it follows from the choice of $\rho_1$ and Proposition 4.1 that

$$|\hat{B}(\hat{u},\hat{u})(\xi,t)| \le \rho_1 h(\xi) e^{-\sqrt{t}|\xi|}\frac{1}{2}(1-e^{-\nu|\xi|^2 t/2}). \tag{42}$$

Thus, using (41) and (42), one has for $\|u\|_{\mathcal{F}_{h,1,T}} \le \rho_1$,

$$|\hat{\mathcal{Q}}[\hat{u};\hat{u}_0,\hat{g}](\xi,t)| \le \rho_1 h(\xi) e^{-\sqrt{t}|\xi|}.$$

So if also $\|v\|_{\mathcal{F}_{h,1,T}} \le \rho_1$,

$$\begin{aligned}\|\hat{\mathcal{Q}}[\hat{u};\hat{u}_0,\hat{g}](\xi,t) - \hat{\mathcal{Q}}[\hat{v};\hat{u}_0,\hat{g}](\xi,t)\|_{\mathcal{F}_{h,1,T}} &= \|B(u,u)-B(v,v)\|_{\mathcal{F}_{h,1,T}} \\ &= \|B(u,u-v)+B(u-v,v)\|_{\mathcal{F}_{h,1,T}} \\ &\le \rho_1 C(1,1)\frac{2}{\nu}(\|u-v\|_{\mathcal{F}_{h,0,T}}),\end{aligned}$$

and as before the proposition follows by the contraction mapping theorem. ■

It is possible to show that solutions exist locally in time when the forcing satisfies a bound involving fractional powers of the Laplace operator. A result along this lines is given by the following theorem.

**Theorem 4.2** *Let $h$ be a standard majorizing kernel of exponent $\theta = 1$ and let $\rho_\gamma$ be as in Theorem 4.1. Then if $\|e^{\nu t\Delta}u_0(x)\|_{\mathcal{F}_{h,\gamma,T}} \le \rho_\gamma$ and for some $0 \le \beta < 2$, $(-\Delta)^{-\beta/2} g(x,t) \in \mathcal{F}_{h,\gamma,T}$, then there exits $T_*$ and $u(x,t) \in \mathcal{F}_{h,\gamma,T_*}$ satisfying the Navier-Stokes equation and $\|u\|_{\mathcal{F}_{h,\gamma,T_*}} \le \rho_\gamma$.*

*Proof*: New estimates are required for the forcing term. Considering first $\gamma = 0$, note that $0 \le t \le T$

$$\begin{aligned}\left|\int_0^t e^{-\nu|\xi|^2 s}\hat{g}(\xi,t-s)ds\right| &\le \|g\|_{\mathcal{F}_{h,0,T}} h(\xi)\int_0^t e^{-\nu|\xi|^2 s}|\xi|^\beta ds \\ &\le \|g\|_{\mathcal{F}_{h,0,T}} h(\xi) M(\beta) t^{(2-\beta)/2}\nu^{-\beta/2}\end{aligned}$$

where in the last step, Lemma 4.1 was used.

As in the proof of Theorem 4.1 one has for $0 \leq t \leq T$

$$\begin{aligned} |\hat{\mathcal{Q}}[\hat{u};\hat{u}_0,\hat{g}](\xi,t)| &\leq h(\xi)\left[\|e^{\nu t\Delta}u_0\|_{\mathcal{F}_{h,0,T}}e^{-\nu|\xi|^2 t}\right. \\ +\|g\|_{\mathcal{F}_{h,0,T}}M(\beta)t^{(2-\beta)/2}\nu^{-\beta/2} &+ \left.\|u\|^2_{\mathcal{F}_{h,0,T}}(2\pi)^{-3/2}\frac{1}{\nu}(1-e^{-\nu|\xi|^2 t})\right] \end{aligned}$$

The result follows by choosing $T_*$ small enough so that $\hat{\mathcal{Q}}$ is a contraction of the ball of radius $\rho_\gamma$ centered at the origin into itself.

Similarly, for $\gamma = 1$, one has

$$\begin{aligned} \left|\int_0^t e^{-\nu|\xi|^2 s}\hat{g}(\xi,t-s)ds\right| &\leq \|g\|_{\mathcal{F}_{h,1,T}}h(\xi)\int_0^t e^{-\nu|\xi|^2 s}|\xi|^\beta e^{-\sqrt{t-s}|\xi|}ds \\ &\leq \|g\|_{\mathcal{F}_{h,1,T}}h(\xi)e^{-\sqrt{t}|\xi|}e^{1/(2\nu)}\int_0^t e^{-\nu|\xi|^2 s/2}|\xi|^\beta ds \\ &\leq \|g\|_{\mathcal{F}_{h,1,T}}h(\xi)e^{-\sqrt{t}|\xi|}M(\beta)t^{(2-\beta)/2}(\nu/2)^{-\beta/2} \end{aligned}$$

Thus,

$$\begin{aligned} |\hat{\mathcal{Q}}[\hat{u};\hat{u}_0,\hat{g}](\xi,t)| &\leq h(\xi)e^{-\sqrt{t}|\xi|}\left[\|e^{\nu t\Delta}u_0\|_{\mathcal{F}_{h,1,T}}e^{-\nu|\xi|^2 t}\right. \\ +\|g\|_{\mathcal{F}_{h,1,T}}M(\beta)t^{(2-\beta)/2}\nu^{-\beta/2} &+ \left.\|u\|^2_{\mathcal{F}_{h,1,T}}(2\pi)^{-3/2}\frac{1}{\nu}(1-e^{-\nu|\xi|^2 t})\right]. \end{aligned}$$

As before, the result follows by choosing $T_*$ small enough so that the contraction mapping theorem can be applied to $\hat{\mathcal{Q}}$ as a mapping on the ball or radius $\rho_\gamma$ centered at the origin. ∎

Finally, a further local existence result can be obtained if majorizing kernels of exponent $\theta < 1$ are considered.

**Theorem 4.3** *Let $h$ be a standard majorizing kernel of exponent $\theta < 1$. Assume that for some $1 \leq \beta \leq 2$*

$$(\Delta)^{-\beta/2}g(x,t) \in \mathcal{F}_{h,\gamma,T}. \tag{43}$$

*Then, for any initial data $e^{t\Delta}u_0(x) \in \mathcal{F}_{h,\gamma,T}$ and forcing satisfying (43) there exists $T_* \leq T$ and a unique $u(x,t) \in \mathcal{F}_{h,\gamma,T_*}$ satisfying the Navier-Stokes equation.*

*Proof:* A straightforward calculation shows that $\hat{F}(\xi,t) = e^{-\nu|\xi|^2 t}\hat{u}_0(\xi) + \int_0^t e^{-\nu|\xi|^2 s}\hat{g}(\xi,t-s)ds$ satisfies

$$\|F\|_{\mathcal{F}_{h,\gamma,T}} \leq M$$

for an appropriate $M$.

Using Lemma 4.1, it follows that

$$\begin{aligned}\|\mathcal{Q}[\hat{u};\hat{u}_0,\hat{g}] - F\|_{\mathcal{F}_{h,\gamma,T}} &\leq \|B(u,u)\|_{\mathcal{F}_{h,\gamma,T}} \\ &\leq \left[\|u-F\|_{\mathcal{F}_{h,\gamma,T}} + \|F\|_{\mathcal{F}_{h,\gamma,T}}\right]^2 M(\gamma,\theta)T^{(1-\theta)/2} \quad (44)\end{aligned}$$

Similarly,

$$\begin{aligned}\|\mathcal{Q}[\hat{u};\hat{u}_0,\hat{g}] - \mathcal{Q}[\hat{v};\hat{u}_0,\hat{g}]\|_{\mathcal{F}_{h,\gamma,T}} &\leq \|B(u,u) - B(v,v)\|_{\mathcal{F}_{h,\gamma,T}} \\ &\leq \|B(u,(u-v)) + B(u-v,v)\|_{\mathcal{F}_{h,\gamma,T}} \\ &\leq M(\gamma,\theta)(\|u\|_{\mathcal{F}_{h,\gamma,T}} + \|v\|_{\mathcal{F}_{h,\gamma,T}}) \quad (45) \\ &\quad \cdot\|u-v\|_{\mathcal{F}_{h,\gamma,T}} T^{(1-\theta)/2}. \quad (46)\end{aligned}$$

Now, use (44) and (45) to choose $T_* \leq T$ such that if for some $\rho > 0$,

$$\|u - F\|_{\mathcal{F}_{h,\gamma,T_*}} < \rho,$$

$\mathcal{Q}$ is a contraction in the ball centered at $F$ of radius $\rho$. ∎

**Remark 4.2.** Theorem 4.3 establishes uniqueness and regularity for solutions to (FNS) on a finite time interval $[0, T_*)$ for all initial $u_0 \in \mathcal{F}_{h,\gamma,T_*}$ without further restricting $\|u_0\|_{\mathcal{F}_{h,\gamma,T_*}}$. Here $T_* \to 0$ as $\nu \to 0$. This is consistent with other known local existence and uniqueness theorems, e.g. see Temam (1995), Kato (1984).

**Remark 4.3.** Recall that a Banach space $X$ is called a *limit space* for the Navier-Stokes equations iff $\|u\|_X = \|u_\lambda\|_X$ where $u_\lambda = \lambda u(\lambda x, \lambda^2 t)$. If $h$ is a majorizing kernel with exponent $\theta = 1$ then $h_\lambda(\xi) = \lambda^{-2}h(\xi/\lambda)$ is also a majorizing kernel of the same exponent. Moreover, if $u \in \mathcal{F}_{h,\gamma,T}$, $u_\lambda \in \mathcal{F}_{h_\lambda,\gamma,T}$ and $\|u\|_{\mathcal{F}_{h,\gamma,T}} = \|u_\lambda\|_{\mathcal{F}_{h_\lambda,\gamma,T}}$. Thus an exponent one majorizing kernel $h$ such that $h = h_\lambda$ defines a limit space $X = \mathcal{F}_{h,\gamma,T}$ in the usual sense, whereas the relation $\|u\|_{\mathcal{F}_{h,\gamma,T}} = \|u_\lambda\|_{\mathcal{F}_{h_\lambda,\gamma,T}}$ defines a slightly more general version of this notion. Nonetheless the global existence result of Theorem 4.1 is in agreement with the similar results known for the usual limit spaces; c.f. Cannone and Meyer (1995), Cannone and Planchon (2000), and Chen and Xin (2001).

Finally, the relation between the iteration scheme and the expected value representation of the solution obtained in Section 3 is established in the following proposition. For reference, recall that the replacement time of a vertex $\mathbf{v}$ is defined as

$$R_{\mathbf{v}} = \sum_{k=0}^{|\mathbf{v}|} S_{\mathbf{v}|k}$$

Introduce

$$A_n(\theta, t) = [|\mathbf{v}| \le n \ \forall \mathbf{v} \in \tau_\theta(t)] \cap [R_{\mathbf{v}} > t \ \forall \mathbf{v} \in \{\mathbf{u} \in \tau_\theta(t) : |\mathbf{u}| = n\}],$$

and let $\mathbf{1}[n; \theta, t]$ the indicator of the event $A_n(\theta, t)$. Observe that the definition of the event $A_n(\theta, t)$ and its indicator extends to $A_n(< i >, t - S_\theta), i = 1, 2,$ and inductively to $A_n(\mathbf{v}, t - B_{\mathbf{v}})$, using the shifted binary tree defined by (31) and the time shift $t - S_\theta$.

**Proposition 4.3** *Let*

$$\begin{aligned} v_k(\xi, t) &= h(\xi)\chi_k(\xi, t) \\ &= h(\xi)\mathbf{E}_\xi\{\mathbf{1}[k; \xi, t] \between (\theta, t)\} \end{aligned}$$

*and denote by $\hat{u}_k(\xi, t)$ the Fourier transform of the $k^{\text{th}}$ iterate of the iteration scheme defined in (36). Then $v_k(\xi, t) = \hat{u}_k(\xi, t)$.*

*Proof:* The proof is by induction on $k$. Note that

$$\begin{aligned} v_0(\xi, t) &= h(\xi)\mathbf{E}_{\xi_\theta = \xi}\{\mathbf{1}[0; \theta, t] \between (\theta, t)\} \\ &= h(\xi)\mathbf{E}_{\xi_\theta = \xi}\{\between (\theta, t) | S_\theta > t\}\mathbf{P}[S_\theta > t] \\ &= h(\xi)\chi_0(\xi)e^{-\nu|\xi|^2 t} \\ &= \hat{u}^{(0)}(\xi). \end{aligned}$$

The proof for the general case rests on the following identity,

$$\begin{aligned} h(\xi)\mathbf{E}_{\xi_\theta = \xi}\{\mathbf{E}_{\xi_\theta = \xi}\{\mathbf{1}[k+1; \theta, t]\mathbf{1}[\kappa_\theta = 1]\mathbf{1}[S_\theta < t]m(\xi_\theta) \\ \cdot \between (< 1 >, t - S_\theta) \otimes_{\xi_\theta} \between (< 2 >, t - S_\theta) | \xi_{<1>}, \xi_{<2>}, S_\theta\}\} = \\ \int_0^t e^{-\nu|\xi|^2 s}|\xi|(2\pi)^{-\frac{3}{2}} \int_{\mathbf{R}^3} \chi_k(\eta, t-s) \otimes_\xi \chi_k(\xi - \eta, t - s)h(\eta)h(\xi - \eta)d\eta. \end{aligned} \tag{47}$$

To see this, recall that $\mathbf{P}[\kappa_\theta = 1] = 1/2$ as well as both the recursive definition of the $\between$ functional together with the following factorization on the event $[\kappa_\theta = 1, S_\theta < t]$,

$$\mathbf{1}[k+1; \theta, t] = \mathbf{1}[k; < 1 >, t - S_\theta]\mathbf{1}[k; < 2 >, t - S_\theta]. \tag{48}$$

Also recall the exterior condition (10) and the definitions of $m$ given in (8) and of the transition probability kernel given in (9). With these in mind, the left hand side of (47) can be computed as

$$h(\xi)\frac{1}{2}\int_0^t e^{-\nu|\xi|^2 s}\nu|\xi|^2\frac{2h*h(\xi)}{\nu(2\pi)^{\frac{3}{2}}|\xi|h(\xi)}\int_{\mathbf{R}^3}\mathbf{E}_{\xi_\theta=\xi}\{\mathbf{1}[k;<1>,t-s]\chi(<1>,t-s)$$
$$\otimes_\xi\mathbf{1}[k;<2>,t-s]\chi(<2>,t-s)|\xi_{<1>}=\eta,\xi_{<2>}=\xi-\eta\}$$
$$\frac{h(\xi-\eta)h(\eta)}{h*h(\xi)}d\eta ds$$
$$=\int_0^t e^{-\nu|\xi|^2 s}|\xi|(2\pi)^{-\frac{3}{2}}\mathbf{E}_{\xi_\theta=\xi}\{\int_{\mathbf{R}^3}\mathbf{1}[k;<1>,t-s]\chi(<1>,t-s)$$
$$\otimes_\xi\mathbf{1}[k;<2>,t-s]\chi(<2>,t-s)|\xi_{<1>}=\eta,\xi_{<2>}=\xi-\eta\}h(\eta)h(\xi-\eta)d\eta\}\,ds.$$

Thus, using the conditional independence of the recursive functional it follows that the last equation can be written as

$$\int_0^t e^{-\nu|\xi|^2 s}|\xi|(2\pi)^{-\frac{3}{2}}\int_{\mathbf{R}^3}[h(\eta)\mathbf{E}_{\xi_{<1>}=\eta}\{\chi(<1>,t-s)\mathbf{1}[k;<1>,t-s]\}]$$
$$\otimes_\xi[h(\xi-\eta)\mathbf{E}_{\xi_{<2>}=\xi-\eta}\{\chi(<2>,t-s)\mathbf{1}[k;<2>,t-s]\}]\,d\eta ds$$

as needed to establish (47).

To complete the proof, condition on the value of the first clock ring $S_\theta$, recall the definitions of $m$ and $\varphi$ given in (8) and use (47) to get

$$\begin{aligned}
v_{k+1}(\xi,t) &= h(\xi)\mathbf{E}_{\xi_\theta=\xi}\{\mathbf{E}_{\xi_\theta=\xi}\{\mathbf{1}[k+1;\theta,t]\chi(\theta,t)|\xi_{<1>},\xi_{<2>},S_\theta\}\}\\
&= h(\xi)\left[\chi_0(\xi)e^{-\nu|\xi|^2 t}+\frac{1}{2}\int_0^t e^{-\nu|\xi|^2 s}\nu|\xi|^2\varphi(\xi,t-s)ds\right]\\
&+ h(\xi)\mathbf{E}_{\xi_\theta=\xi}\{\mathbf{E}_{\xi_\theta=\xi}\{\mathbf{1}[k+1;\theta,t]\mathbf{1}[\kappa_\theta=1]\mathbf{1}[S_\theta<t]m(\xi)\chi(<1>,t-S_\theta)\\
&\quad\otimes_{\xi_\theta}\chi(<2>,t-S_\theta)|\xi_{<1>},\xi_{<2>},S_\theta\}\}\\
&= \hat{u}^{(0)}(\xi)+\int_0^t e^{-\nu|\xi|^2 s}\hat{g}(\xi,t-s)ds\\
&+ \int_0^t e^{-\nu|\xi|^2 s}|\xi|(2\pi)^{-\frac{3}{2}}\int_{\mathbf{R}^3}v_k(\eta,t-s)\otimes_\xi v_k(\xi-\eta,t-s)d\eta ds\\
&= \hat{F}(\xi,t)+\int_0^t e^{-\nu|\xi|^2 s}|\xi|(2\pi)^{-\frac{3}{2}}\int_{\mathbf{R}^3}\hat{u}_k(\eta,t-s)\otimes_\xi\hat{u}_k(\xi-\eta,t-s)d\eta ds
\end{aligned}$$

by the induction hypothesis and the definition of $\hat{F}$. This last equation is $\hat{u}_{k+1}(\xi,t)$ as claimed. ∎

A consequence of the proposition is that the convergence of the iteration scheme (36) and the existence of the expected value in Theorem 3.1 are essentially equivalent.

# 5 Conclusions and Remarks

The introduction and identification of majorizing kernels provides a way to obtain existence and uniqueness of mild solutions of Navier-Stokes equations and track regularity of initial data to solutions. The same methods may be applied to the Fourier coefficients in the case of periodic initial data and forcing. In fact the identification of majorizing kernels is somewhat simpler here due to the fact that on the integer lattice the origin need not be a singularity of the majorizing kernel. One may use a lattice versions of the theory for constructions of majorizing kernels, e.g. Theorems 2.1-2.2 to construct fully supported majorizing kernels on the integer lattice in all dimensions $d \geq 2$. In the case $d = 1$ one also obtains cascade representations of solutions to Burgers' equation by these techniques. For example majorizing kernels supported on the positive half-line, $h(\xi) = \mathbf{1}[\xi > 0]$, also appear naturally and yield an existence/uniqueness theory for complex-valued solutions in Hardy spaces $H^p$.

As emphasized in the introduction, in principle the theory may be approached from the perspective of identifying Fourier multipliers for which $E_{\xi_\theta=\xi}|\mathbb{X}(\theta,t)| < \infty$. While majorizing kernels are sufficient for this purpose, this neither exploits the geometric structure of the product $\otimes_\xi$ nor the "size"(number of vertices) of the underlying stochastic tree structure beyond simple first order considerations.

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