Global regular and singular solutions for a model of gravitating particles

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Abstract

The existence of solutions of a nonlinear parabolic equation describing the gravitational interaction of particles is studied for the initial data in spaces of (generalized) pseudomeasures. This approach permits us to relax regularity assumptions on the initial conditions and to prove asymptotic stability results for the above problem.

1. Introduction. We are concerned in this paper with a construction of solutions of the Cauchy problem for the nonlinear equation

$$\begin{aligned} u_t &= \Delta u + \nabla\cdot(u\nabla\varphi), \\ \nabla\varphi &= \nabla E_d * u, \end{aligned} \tag{1.1}$$

where the drift coefficient $\nabla\varphi$ is determined by u in a nonlocal way, $E_d(z) = -((d-2)\sigma_d)^{-1}|z|^{2-d}$ being the fundamental solution of the Laplacian in $I\!R^d$,

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$\sigma_d = 2\pi^{d/2}/\Gamma(d/2)$ — the area of the unit sphere in $I\!R^d$, $d \geq 3$. The initial condition

$$u(0)(x) = u(x,0) \tag{1.2}$$

supplements the above parabolic-elliptic system considered for $x \in I\!R^d$ and $t > 0$.

Physical interpretations of (1.1) are connected with astrophysical models of gravitational self-interaction of massive particles in a cloud or a nebula. Indeed, $u = u(x,t) \geq 0$ is the density of particles and φ is the (self-consistent) gravitational potential generated by u. We do not restrict, however, our attention in this paper to positive functions u, and thus we will also consider sign changing functions u. For $d \geq 3$

$$\tilde{u}(x) = 2(d-2)|x|^{-2} \tag{1.3}$$

is a stationary singular solution of (1.1) called the Chandrasekhar solution. Note that $\tilde{u}$ is a solution of (1.1) pointwise for all $x \neq 0$, and also in the sense of distributions $\mathcal{D}'(I\!R^d)$. Moreover, $\tilde{u} \in W^{1,p}_{\rm loc}(I\!R^d)$ with $p < d/3$ is a weak solution of (1.1) for $d \geq 4$. It is expected that $|x|^{-2}$ is the critical singularity of the initial data in the sense that for $|u(x,0)| \leq \varepsilon|x|^{-2}$ for some small $\varepsilon > 0$ there exists a solution $u = u(x,t)$ which is smooth for $t > 0$, and if $u(x,0) > 2(d-2)|x|^{-2}$ there are no (even local in time) solutions. Results in this direction have been proved in [2], [4], [16], where Kato's technique with two norms from [17] and various functional spaces of Besov and Morrey type have been employed besides the usual L^p framework for proving the existence of solutions of (1.1). All these results can be loosely summarized by saying that if possible singularities of $u(x,0)$ do not exceed those of $\tilde{u}$, then there exists a *smooth* solution of (1.1). Thus, keeping the size of singularities small, one obtains a strong parabolic regularization effect of the Laplacian operator in (1.1). Our goal here is to construct solutions that may have singularities even for $t > 0$, i.e. we will try to include the Chandrasekhar solution into the consideration of evolutionary solutions of (1.1), cf. Theorem 2.1 below. Our approach here is simple and use only one functional norm to prove the existence of solutions via the contraction mapping principle, as opposed to Kato's scheme involving two norms in [2], [9], etc. Moreover, our construction will permit us to obtain self-similar solutions of (1.1) as solutions of the Cauchy problem (1.1)–(1.2) with the initial data homogeneous of degree -2, cf. Corollary 2.1. Finally, the scheme of the proof of existence of solutions can be generalized to a proof of a kind of general asymptotic stability of solutions (Theorem 3.1), a novel result for the problem (1.1).

The results in this paper can be extended to a version of the Chavanis–Sommeria–Robert model introduced in [14] and studied recently in, e.g,

[13], [24], [15], [8], where (1.1) takes the form $u_t = \Delta u + \nabla \cdot (u\vartheta^{-1}\nabla\varphi)$ or $u_t = \vartheta\Delta u + \nabla \cdot (u\nabla\varphi)$ with the temperature $\vartheta = \vartheta(t) > 0$. The function ϑ is either a given continuous function or satisfies a constraint of energy $\beta\vartheta + \int_{I\!R^d} u\varphi\, dx = \text{const}$. In the latter case, we need to put everywhere an additional constraint $\int_{I\!R^d} u(x,t)\, dx = M = \text{const}$, and a restriction on $u\varphi$ which is necessary to make sense of the energy relation above.

Moreover, our methods apply to the, so called, (simplified) Debye system with (1.1) replaced by $u_t = \Delta u - \nabla \cdot (u\nabla\varphi)$ describing the electric repulsion of particles, as well as to multicomponent systems describing several species of particles, either attracting or repulsing themselves (e.g. the full Debye system, cf. [7]).

The ideas used in this paper have been first applied to the Navier–Stokes system, see [20] for an early approach involving pseudomeasures, [12], [10] and [11]. Clearly, the most important features of the both problems are suitable scaling properties of the nonlinear term, see the analysis in [16]. If one tries to compare the Navier–Stokes system and the problem studied here, the following remarks should be noted.

- The nonlinear terms in the Navier–Stokes equations have much more complicated structure while expected singularities (of degree -1 for space homogeneous solutions, cf. [18] or [26]) are weaker than $\tilde{u}(x) \sim |x|^{-2}$.
- There are intrinsic cancellations for solutions of the Navier–Stokes equations while the system (1.1)–(1.2) conserves the positivity of initial data so that formation of singularities in (1.1)–(1.2) is completely different from that in hydrodynamics equations, see [11] and [5].

Notations. In the sequel, all the integrals without the integration limits are meant $\int_{I\!R^d} \ldots\, dx$. The L^p-norms are denoted by $\|.\|_p$. The Fourier transform of a function v on $I\!R^d$ is $\widehat{v}(\xi) = \int v(x)e^{-ix\cdot\xi}\, dx$. Caution, the convention in [19] or [25] is different: $\mathcal{F}v(\xi) = \int v(x)e^{-2\pi ix\cdot\xi}\, dx$. All the inessential constants are denoted by C, even if they vary from line to line.

2. The Cauchy problem and self-similar solutions.

Let us introduce some Banach functional spaces relevant to the study of solutions of the Cauchy problem for the system (1.1):

$$\mathcal{PM}^a = \{v \in \mathcal{S}'(I\!R^d) : \widehat{v} \in L^1_{\text{loc}}(I\!R^d), \|v\|_{\mathcal{PM}^a} \equiv \operatorname*{ess\,sup}_{\xi\in I\!R^d} |\xi|^a |\widehat{v}(\xi)| < \infty\},$$

where $a \ge 0$. The notation $\mathcal{PM}$ stands for *pseudomeasure*, and the classical space of pseudomeasures considered in harmonic analysis corresponds to $a = 0$.

Remark. Given $f \in \mathcal{S}'(\mathbb{R}^d) \cap L^1_{loc}(\mathbb{R}^d)$ we denote the rescaling $f_\lambda(x) = f(\lambda x)$. In a standard way, we extend this definition to all tempered distributions. It follows from elementary calculations that $\widehat{f_\lambda}(\xi) = \lambda^{-d}\widehat{f}(\lambda^{-1}\xi)$. Hence, for every $\lambda > 0$, we obtain the scaling property of the norm in $\mathcal{PM}^a$

$$\|f(\lambda\,\cdot)\|_{\mathcal{PM}^a} = \lambda^{a-d}\|f\|_{\mathcal{PM}^a}. \tag{2.1}$$

In particular, the norm $\mathcal{PM}^{d-2}$ is invariant under rescaling $f \mapsto \lambda^2 f(\lambda\,\cdot)$.

By a solution of (1.1) we mean in this paper a function $u = u(t)$ belonging to the space of vector-valued functions $\mathcal{X} = \mathcal{C}_w([0,T];\mathcal{PM}^{d-2})$, $0 < T \le \infty$, such that

$$\widehat{u}(\xi,t) = e^{-t|\xi|^2}\widehat{u}(\xi,0) + \int_0^t e^{(t-s)|\xi|^2} i\xi\cdot\left(\widehat{u}(\xi,s) * \frac{i\xi}{|\xi|^2}\widehat{u}(\xi,s)\right)ds \tag{2.2}$$

for $\xi \in \mathbb{R}^d$ and $0 \le t < T$. The space $\mathcal{PM}^{d-2}$ is chosen because for $d \ge 3$ the function $|x|^{-2}$ belongs to this space (since its Fourier transform is a multiple of $|\xi|^{2-d}$) as well as $|x-x_0|^{-2}$ for each $x_0 \in \mathbb{R}^d$ does.

$\mathcal{C}_w$ denotes, as usual (cf. [2]), the space of vector-valued functions which are weakly continuous as distributions in t. The necessity of considering $\mathcal{C}_w$ instead of strongly continuous functions is an additional difficulty caused by the fact that the heat semigroup $(e^{t\Delta})_{t\ge0}$ is not strongly continuous on the spaces of pseudomeasures but only weakly continuous.

Usually, a *mild solution* of an evolution equation like (1.1) is defined as

$$u(t) = e^{t\Delta}u(0) + \int_0^t e^{(t-s)\Delta}\nabla\cdot(u(s)\nabla\varphi(s))\,ds, \tag{2.3}$$

where $\nabla\varphi(s) = \nabla E_d * u(s)$, and the integral is understood as the Bochner integral. However, such a meaning of a solution is not suitable for our construction of solutions of the Cauchy problem and, in particular, of self-similar solutions. Indeed, for stationary and homogeneous of degree -2 solutions u (given, e.g., by (1.3)), the nonlinear term $\nabla\cdot(u\nabla\varphi)$ corresponds to a tempered distribution which is homogeneous of degree -4, hence, there exists a distribution H such that

$$e^{(t-s)\Delta}\nabla\cdot(u\nabla\varphi) = t^{-2}H\left(\frac{\cdot}{\sqrt{t}}\right).$$

Now, computing the $\mathcal{PM}^{d-2}$ norm and using the scaling relation (2.1), we obtain

$$\|e^{t\Delta}\nabla\cdot(u\nabla\varphi)\|_{\mathcal{PM}^{d-2}} = t^{-1}\|H\|_{\mathcal{PM}^{d-2}}.$$

So, $e^{t\Delta}\nabla\cdot(u\nabla\varphi)$ is not Bochner integrable as a mapping on $[0,T)$ with values in $\mathcal{PM}^{d-2}$. On the other hand, the singularity at $t = 0$ does not appear in the Fourier transform of this quantity. Hence, the integral with respect to s in equations (2.2) and (2.3) should be defined in a weak sense like, e.g., it was done in [27, Def. 2]. For more explanations, we refer the reader to [21], because our spaces $\mathcal{PM}^a$ are the example of the shift-invariant Banach spaces of distributions systematically used in that book.

Nevertheless, a distributional solution of (1.1) is a solution in $\mathcal{X}$ of the integral equation (2.2) (or its equivalent version (2.3)), and vice versa. This equivalence can be proved along the lines of the computations for Navier–Stokes equations in [27, Th. 5.2].

To simplify the notation, the quadratic term in (2.2) or (2.3) will be denoted by $\widehat{B}(u,u)$ or $B(u,u)$ resp., with the bilinear form B defined by

$$B(u,v)(t) = \int_0^t e^{(t-s)\Delta}\nabla\cdot(u(s)\nabla\psi(s))\,ds, \tag{2.4}$$

where $\nabla\psi(s) = \nabla E_d * v(s)$, and $u = u(t)$, $v = v(t)$ are functions on $[0,T)$ with values in a vector space (here most frequently $\mathcal{PM}^{d-2}$).

The crucial tool used is the following technical estimate

Proposition 2.1 *If $d \geq 4$, then there exists a constant $K = K(d) > 0$ such that for all u, $v \in \mathcal{X} = \mathcal{C}_w([0,\infty);\mathcal{PM}^{d-2})$*

$$\|B(u,v)\|_{\mathcal{X}} \leq K\|u\|_{\mathcal{X}}\|v\|_{\mathcal{X}}$$

holds for the quadratic form B defined in (2.4).

Proof. First observe that $\nabla\psi(t) = \nabla E_d * v(t)$, so that

$$\widehat{\nabla\psi}(\xi,t) = i\xi|\xi|^{-2}\widehat{v}(\xi,t)$$

and $\nabla\psi(t) \in \mathcal{PM}^{d-1}$. Next we obtain

$$\begin{aligned} |(\widehat{u}(t) * \widehat{\nabla\psi}(t))(\xi)| &\leq \left(\int |\xi-\zeta|^{2-d}|\zeta|^{1-d}\,d\zeta\right)\|u(t)\|_{\mathcal{PM}^{d-2}}\|v(t)\|_{\mathcal{PM}^{d-2}} \\ &\equiv K|\xi|^{3-d}\|u(t)\|_{\mathcal{PM}^{d-2}}\|v(t)\|_{\mathcal{PM}^{d-2}} \end{aligned} \tag{2.5}$$

because $d \geq 4$ (for $d = 3$ the above estimate is not valid). Then we estimate the pseudomeasure $\mathcal{PM}^{d-2}$ norm of $B(u,v)$ by

$$\begin{aligned} &\sup_{\xi} K\left||\xi|^{d-2}\int_0^t i\xi e^{-(t-s)|\xi|^2}\,ds|\xi|^{3-d}\right|\|u\|_{\mathcal{X}}\|v\|_{\mathcal{X}} \\ = \ &K|\xi|^2\left(\int_0^t e^{-s|\xi|^2}\,ds\right)\|u\|_{\mathcal{X}}\|v\|_{\mathcal{X}} \leq K\|u\|_{\mathcal{X}}\|v\|_{\mathcal{X}} \end{aligned}$$

as stated. The proof of the weak continuity follows the lines of the argument in [22, Ch. 18, Lemma 24] or [27, Th. 3.1]. □

The value of the constant K can be calculated explicitly using the following

Lemma 2.1 *Let* $0 < \alpha,\ \beta < d$. *Then*

$$|x|^{-\alpha} * |x|^{-\beta} = C|x|^{d-\alpha-\beta},$$

where $C = C(\alpha,\beta,d) = \pi^{d/2}\frac{\Gamma\left(\frac{d-\alpha}{2}\right)\Gamma\left(\frac{d-\beta}{2}\right)\Gamma\left(\frac{\alpha+\beta-d}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\beta}{2}\right)\Gamma\left(d-\frac{\alpha+\beta}{2}\right)}$.

Proof. It follows from [19, Th. 5.9] or [25, Ch. V, Sec.1, (8)] that

$$\mathcal{F}(|x|^{-\alpha}) = \pi^{\alpha-d/2}\frac{\Gamma\left(\frac{d-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)}|\xi|^{\alpha-d}.$$

In this way we get

$$\mathcal{F}(|x|^{-\alpha} * |x|^{-\beta}) = \pi^{\alpha+\beta-d}\frac{\Gamma\left(\frac{d-\alpha}{2}\right)\Gamma\left(\frac{d-\beta}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\beta}{2}\right)}|\xi|^{\alpha+\beta-2d}$$

which is equivalent to [19, Cor. 5.10, (3)]

$$|x|^{-\alpha} * |x|^{-\beta} = \pi^{3d/2-\alpha-\beta}\frac{\Gamma\left(\frac{\alpha+\beta-d}{2}\right)}{\Gamma\left(d-\frac{\alpha+\beta}{2}\right)}\pi^{\alpha+\beta-d}\frac{\Gamma\left(\frac{d-\alpha}{2}\right)\Gamma\left(\frac{d-\beta}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\beta}{2}\right)}|x|^{d-\alpha-\beta},$$

so that we obtain the value of $C(\alpha,\beta,d)$. □

Therefore we have $K = \pi^{d/2}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{d-3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{d-1}{2}\right)\Gamma\left(\frac{d}{2}-1\right)} = \frac{4\pi^{d/2}}{(d-3)\Gamma\left(\frac{d}{2}-1\right)} = \frac{d-2}{d-3}\sigma_d$ in Proposition 2.1.

Now we recall well-known simple result on solving quadratic equations like (2.3) via contraction mapping (or successive approximations) argument.

Lemma 2.2 *Let* $\mathcal{X}$ *be a Banach space and* $B : \mathcal{X}\times\mathcal{X} \to \mathcal{X}$ *is a bilinear continuous form satisfying* $\|B(u,v)\|_{\mathcal{X}} \le K\|u\|_{\mathcal{X}}\|v\|_{\mathcal{X}}$ *for some* $K > 0$ *and all* $u,\ v \in \mathcal{X}$. *Then for every* $y \in \mathcal{X}$ *such that* $\|y\|_{\mathcal{X}} < 1/(4K)$, *the equation* $u = y + B(u,u)$ *has a solution* $u \in \mathcal{X}$. *Moreover, this solution is such that* $\|u\|_{\mathcal{X}} \le \left(1-\sqrt{1-4K\|y\|_{\mathcal{X}}}\right)/(2K) \le 2\|y\|_{\mathcal{X}}$, *and unique in the open ball in* $\mathcal{X}$ *of radius* $1/(2K)$. *The solution continuously depends on* y: *if* $\|z\|_{\mathcal{X}} \le \varepsilon < 1/(4K)$, $v = z + B(v,v)$ *and* $\|v\|_{\mathcal{X}} \le 2\varepsilon$, *then* $\|u-v\|_{\mathcal{X}} \le (1-4K\varepsilon)^{-1}\|y-z\|_{\mathcal{X}}$.

For the proof (its idea is clear in the simplest case $\mathcal{X} = I\!R$) we refer the reader to either [9], or [22], or [2, Lemma 2], see also [21, Th. 13.2].

Now we are ready to prove the basic existence result.

Theorem 2.1 *Let* $d \geq 4$*. Assume that the initial condition (1.2) satisfy* $u(0) \in \mathcal{PM}^{d-2}$ *and* $\|u(0)\|_{\mathcal{PM}^{d-2}} < 1/(4K)$*, where* K *was determined using Lemma 2.1. Then there exists a global in time solution* u *of the Cauchy problem in the space* $\mathcal{X} = \mathcal{C}_w([0,\infty); \mathcal{PM}^{d-2})$*. This solution is unique among those satisfying the condition* $\|u\|_{\mathcal{X}} < 1/(2K)$*.*

Proof. It suffices to apply Lemma 2.2 to the integral equation $u = e^{t\Delta}u(0) + B(u,u)$. Remember that

$$e^{t\Delta}u(0) \in \mathcal{PM}^{d-2} \tag{2.6}$$

if $u(0) \in \mathcal{PM}^{d-2}$, and the required estimate for B is provided by Proposition 2.1. □

Note that, using Lemma 2.1, one gets $\|\tilde{u}\|_{\mathcal{PM}^{d-2}} = 2^{d-1}(d-2)\pi^{d/2}\Gamma\left(\frac{d}{2}-1\right)$ which is $8\frac{d-2}{d-3}(2\pi)^d$ larger than the quantity $1/(4K)$ critical in the solvability condition in Theorem 2.1.

Remark. As was noticed in the introduction, this proof extends to the case of the Debye equation for electrically repulsing particles. However, the asymptotic behavior of solutions of arbitrary size of that problem (they are global in time) is expected to be different (and much simpler) than in our case. Namely, for L^1 initial data, the convergence to either steady states or to self-similar profiles holds, see, e.g., [7] and [6]. On the other hand, solutions of (1.1)–(1.2) with smooth, integrable but large initial data develop finite time blow-up, see [4], [5].

Homogeneity properties of the problem (1.1)–(1.2) imply that if u solves the Cauchy problem, then the rescaled function $u_\lambda(x,t) = \lambda^2 u(\lambda x, \lambda^2 t)$ is also a solution for each $\lambda > 0$. Thus, it is natural to consider solutions which satisfy the scaling invariance property $u_\lambda \equiv u$ for all $\lambda > 0$, i.e. *forward self-similar* solutions. By the very definition, they are global in time, and one may expect that they describe the large time behavior of general solutions of (1.1)–(1.2). Indeed, if $\lim_{\lambda\to\infty} \lambda^2 u(\lambda x, \lambda^2 t) = U(x,t)$ in an appropriate sense, then $tu(xt^{1/2}, t) \to U(x,1)$ as $t \to \infty$ (take $t = 1$, $\lambda = t^{1/2}$), and $U \equiv U_\lambda$ is scale invariant. Hence U is a self-similar solution, and

$$U(x,t) = t^{-1}U(x/t^{1/2}, 1) \tag{2.7}$$

is thus determined by a function of d variables $U(y) \equiv U(y,1)$, $y = x/t^{1/2}$ being the Boltzmann substitution.

A discussion of self-similar radially symmetric solutions can be found in [3], see also [8]. They have been analyzed using ordinary differential equations techniques.

If $u_\lambda \equiv u$ for all $\lambda > 0$, then from the self-similar form (2.7), the initial condition (1.2) $\lim_{t\searrow 0} u(x,t)$ is a distribution homogeneous of degree -2 at the origin. Of course, the Chandrasekhar solution $\tilde{u}$ is a self-similar solution which is time independent.

Self-similar solutions can be obtained directly from Theorem 2.1 by taking $u(0)$ homogeneous of degree -2 of small $\mathcal{PM}^{d-2}$ norm. By the uniqueness property of solutions of the Cauchy problem constructed in Theorem 2.1, they have the form (2.7). An alternative way to prove the existence of self-similar solutions is to convert (1.1)–(1.2) into the integral formulation (2.3) and check that the form B reproduces the scaling property (2.7). Thus, the equation (2.3) can be solved in a subspace of $\mathcal{X}$ formed by self-similar functions, as was done in [9], [22] for the Navier–Stokes system, or in [2].

Proceeding in this way we arrive at

Corollary 2.1 *If $d \geq 4$, $u(0) \in \mathcal{PM}^{d-2}$ is homogeneous of degree -2 and $\|u(0)\|_{\mathcal{PM}^{d-2}} < 1/(4K)$, then there exists a self-similar solution U of the problem (1.1)–(1.2). This solution is unique in the class of pseudomeasures satisfying $\|U\|_{\mathcal{PM}^{d-2}} < 1/(2K)$.* □

The remaining three-dimensional case of the Cauchy problem whose analysis is slightly more complicated will be treated at the end of Section 4.

Remark. In principle, stationary solutions of the system (1.1) can also be studied using the ideas developed here in Section 2, and further in Sections 3 and 4. Namely, for a stationary solution $u \in \mathcal{PM}^{d-2}$ relations

$$u = e^{t\Delta}u + \int_0^t e^{(t-s)\Delta}\nabla\cdot(u\nabla\varphi)\,ds \tag{2.8}$$

and

$$u = \int_0^\infty e^{s\Delta}\nabla\cdot(u\nabla\varphi)\,ds \tag{2.9}$$

are equivalent. The equation (2.8) means that u is a time independent solution of (1.1) in the sense of our definition; the equality (2.9) should be understood in the weak sense, in the Fourier variables as was for (2.8), cf. (2.2).

Indeed, since u is independent of t, the equation (2.8) can be written as

$$\begin{aligned} \widehat{u}(\xi) &= e^{-t|\xi|^2}\widehat{u}(\xi) + \left(\int_0^t e^{-(t-s)|\xi|^2}\,ds\right)\nabla\cdot\widehat{(u\nabla\varphi)} \\ &= e^{-t|\xi|^2}\widehat{u}(\xi) - |\xi|^{-2}\left(1-e^{-t|\xi|^2}\right)\nabla\cdot\widehat{(u\nabla\varphi)}. \end{aligned}$$

Taking the limit $t\to\infty$ we get for $\xi \neq 0$ $\quad \widehat{u}(\xi) = |\xi|^{-2}\nabla\cdot\widehat{(u\nabla\varphi)}$. Since $|\xi|^{-2} = \int_0^\infty e^{-s|\xi|^2}\,ds$ for $\xi\neq 0$, we have $\widehat{u}(\xi) = \int_0^\infty e^{-s|\xi|^2}\nabla\cdot\widehat{(u\nabla\varphi)}\,ds$, and (2.9) holds true.

Now, assume that u satisfies (2.9). Proceeding in the reverse order, we arrive at $\widehat{u}(\xi) = |\xi|^{-2}\nabla\cdot(\widehat{u\nabla\varphi})$. Multiplying this by $\left(1-e^{-t|\xi|^2}\right)$ we come back to (2.8) expressed in the Fourier variables. A similar reasoning for the Navier–Stokes system is in [11, Sec. 6]. □

Our previous results imply that, in particular, there is a neighborhood of the origin in $\mathcal{PM}^{d-2}$ such that the trivial solution 0 is the only stationary solution of (1.1) in that neighborhood.

3. Stability of solutions. This section is devoted to a proof of a stability result for (1.1)–(1.2) which is in the spirit of those in the papers [23], [11] for the Navier–Stokes system, and [16] for nonlocal parabolic problems like (1.1).

Theorem 3.1 *Assume that u and v are solutions of the Cauchy problem (1.1)–(1.2) corresponding to the initial data $u(0)$, $v(0)\in\mathcal{PM}^{d-2}$ such that*

$$\max\{\|u(0)\|_{\mathcal{PM}^{d-2}}, \|v(0)\|_{\mathcal{PM}^{d-2}}\} < 1/(4K),$$

constructed in Theorem 2.1. Then the relation

$$\lim_{t\to\infty}\left\|e^{t\Delta}(u(0)-v(0))\right\|_{\mathcal{PM}^{d-2}} = 0 \tag{3.1}$$

implies

$$\lim_{t\to\infty}\|u(t)-v(t)\|_{\mathcal{PM}^{d-2}} = 0.$$

This result means that if the difference of the solutions of the heat equation issued from $u(0)$, $v(0)$ becomes negligible as $t\to\infty$ (e.g., if the difference of the initial data $u(0)-v(0)$ is not too singular), then the solutions of the nonlinear problem $u(t)$, $v(t)$ behave similarly for large times. It can be interpreted as a kind of asymptotic stability result if the choice of $v(0)$ is restricted to the initial data in a neighborhood of $u(0)$ satisfying additionally (3.1). It is easy to verify that the condition (3.1) is satisfied if, e.g., $|\xi|^{d-2}(\widehat{u}(\xi,0)-\widehat{v}(\xi,0))\to 0$ as $\xi\to 0$.

Proof. Recall first that, by Theorem 2.1,

$$\max\left\{\sup_{t\geq 0}\|u(t)\|_{\mathcal{PM}^{d-2}}, \sup_{t\geq 0}\|v(t)\|_{\mathcal{PM}^{d-2}}\right\} < 2\varepsilon$$

if $\max\{\|u(0)\|_{\mathcal{PM}^{d-2}}, \|v(0)\|_{\mathcal{PM}^{d-2}}\} < \varepsilon$ for all $0<\varepsilon\leq 1/(4K)$. Consider the difference of the formulas (2.3) written for $u(t)$ and $v(t)$, and compute its norm as an element of the space $\mathcal{PM}^{d-2}$:

$$\begin{aligned}\|u(t)-v(t)\|_{\mathcal{PM}^{d-2}} &\le \left\|e^{t\Delta}(u(0)-v(0))\right\|_{\mathcal{PM}^{d-2}}\\ &+ 4\varepsilon K\sup_{\xi\in I\!\!R^d}\int_0^{\delta t}|\xi|^2e^{-(t-s)|\xi|^2}\|u(s)-v(s)\|_{\mathcal{PM}^{d-2}}\,ds \qquad (3.2)\\ &+ 4\varepsilon K\sup_{\xi\in I\!\!R^d}\int_{\delta t}^{t}|\xi|^2e^{-(t-s)|\xi|^2}\|u(s)-v(s)\|_{\mathcal{PM}^{d-2}}\,ds,\end{aligned}$$

where $0<\delta<1$ is to be determined later. Changing the variables $s=t\tau$, $\tau\in[0,1]$, we use the identity

$$\sup_{\xi\in I\!\!R^d}|\xi|^2e^{-(1-\tau)t|\xi|^2}=((1-\tau)t)^{-1}\sup_{\zeta\in I\!\!R^d}|\zeta|^2e^{-|\zeta|^2}$$

to estimate the first integral term in (3.2) by

$$\begin{aligned}&4\varepsilon K\sup_{\xi\in I\!\!R^d}\int_0^{\delta}t|\xi|^2e^{-(1-\tau)t|\xi|^2}\|u(s)-v(s)\|_{\mathcal{PM}^{d-2}}\,d\tau\\ \le\ &4\varepsilon K\int_0^{\delta}(1-\tau)^{-1}\|u(s)-v(s)\|_{\mathcal{PM}^{d-2}}\,d\tau. \qquad (3.3)\end{aligned}$$

The second integral term in (3.2) is simply bounded from above by

$$\begin{aligned}&4\varepsilon K\left(\sup_{\xi\in I\!\!R^d}\int_{\delta t}^{t}|\xi|^2e^{-(t-s)|\xi|^2}\,ds\right)\sup_{\delta t\le s\le t}\|u(s)-v(s)\|_{\mathcal{PM}^{d-2}}\\ \le\ &4\varepsilon K\sup_{\delta t\le s\le t}\|u(s)-v(s)\|_{\mathcal{PM}^{d-2}} \qquad (3.4)\end{aligned}$$

because $\sup_{\xi\in I\!\!R^d}\int_{\delta t}^{t}|\xi|^2e^{-(t-s)|\xi|^2}\,ds=\sup_{\xi\in I\!\!R^d}\left(1-e^{-(1-\delta)t|\xi|^2}\right)=1$.

Now, observe that by the assumptions of Theorem 3.1 the function $g(t)\equiv\left\|e^{t\Delta}(u(0)-v(0))\right\|_{\mathcal{PM}^{d-2}}$ is bounded on $[0,\infty)$ and $\lim_{t\to\infty}g(t)=0$. Hence, applying (3.3) and (3.4) to (3.2), we arrive at

$$\begin{aligned}\|u(t)-v(t)\|_{\mathcal{PM}^{d-2}} &\le g(t)+4\varepsilon K\int_0^{\delta}(1-\tau)^{-1}\|u(t\tau)-v(t\tau)\|_{\mathcal{PM}^{d-2}}\,d\tau\\ &\quad+4\varepsilon K\sup_{\delta t\le s\le t}\|u(s)-v(s)\|_{\mathcal{PM}^{d-2}} \qquad (3.5)\end{aligned}$$

valid for all $t\ge 0$. Let

$$A=\limsup_{t\to\infty}\|u(t)-v(t)\|_{\mathcal{PM}^{d-2}}=\lim_{I\!\!N\ni k\to\infty}\sup_{t\ge k}\|u(t)-v(t)\|_{\mathcal{PM}^{d-2}}\in[0,\infty).$$

We should prove that $A=0$. Let us begin with the obvious inequality

$$\sup_{t\ge k}\int_0^{\delta}(1-\tau)^{-1}\|u(t\tau)-v(t\tau)\|_{\mathcal{PM}^{d-2}}\,d\tau\le\int_0^{\delta}(1-\tau)^{-1}\sup_{t\ge k}\|u(t\tau)-v(t\tau)\|_{\mathcal{PM}^{d-2}}\,d\tau,$$

and apply the Lebesgue Dominated Convergence Theorem to get

$$\limsup_{t\to\infty}\int_0^\delta (1-\tau)^{-1}\|u(t\tau)-v(t\tau)\|_{\mathcal{PM}^{d-2}}\,d\tau \le A\int_0^\delta (1-\tau)^{-1}\,d\tau = -A\log(1-\delta).$$

Concerning the third term in (3.5) we note that

$$\sup_{t\ge k}\ \sup_{\delta t\le s\le t}\|u(s)-v(s)\|_{\mathcal{PM}^{d-2}} \le \sup_{\delta k\le s<\infty}\|u(s)-v(s)\|_{\mathcal{PM}^{d-2}},$$

and thus

$$\limsup_{t\to\infty}\ \sup_{\delta t\le s\le t}\|u(s)-v(s)\|_{\mathcal{PM}^{d-2}} \le A.$$

Finally, (3.5) leads to $A \le 4\varepsilon K(1-\log(1-\delta))A$. Consequently, if we take $\varepsilon < (4K(1-\log(1-\delta))^{-1}$ then necessarily $A = \lim_{t\to\infty}\|u(t)-v(t)\|_{\mathcal{PM}^{d-2}} = 0$. The penultimate condition can be satisfied for each $\varepsilon < 1/(4K)$ choosing suitably small $\delta > 0$. □

Remark. Similar results can be proved for solutions u, v of a nonhomogeneous version of the equation (1.1), $u_t = \Delta u + \nabla\cdot(u\nabla\varphi) + F(t)$, $\nabla\varphi = \nabla E_d * u$, and $v_t = \Delta v + \nabla\cdot(v\nabla\psi) + G(t)$, $\nabla\psi = \nabla E_d * v$, with $\lim_{t\to\infty}\|F(t)-G(t)\|_{\mathcal{PM}^{d-4}} = 0$, as was for the Navier–Stokes system with external forces in [11]. Of course, the regularity of those solutions depends on the initial data and the forces. We skip the details because the nonhomogeneous equations for gravitational attraction of particles, unlike the Navier–Stokes equations, do not have relevant physical interpretations except for very specific F and G. Here, the crucial estimate is

$$\left\|\int_0^t e^{(t-s)\Delta}F(s)\,ds\right\|_{\mathcal{PM}^{d-2}} \le \sup_{0\le s\le t}\|F(s)\|_{\mathcal{PM}^{d-4}}. \tag{3.6}$$

Note that more generally than (2.6), one can prove that for all $c_2 \ge c_1$

$$\sup_{t>0}\left\|t^{(c_2-c_1)/2}e^{t\Delta}u(0)\right\|_{\mathcal{PM}^{c_2}} \le C\|u(0)\|_{\mathcal{PM}^{c_1}}, \tag{3.7}$$

and thus for $0 \le b < 1$

$$t^b\left\|\int_0^t e^{(t-s)\Delta}F(s)\,ds\right\|_{\mathcal{PM}^{c_1+2}} \le C\sup_{0\le s\le t}s^b\|F(s)\|_{\mathcal{PM}^{c_1}}. \tag{3.8}$$

The argument follows from the representation

$$|\xi|^{c_2}e^{-t|\xi|^2}\widehat{u}(\xi,0) = \left|t^{1/2}\xi\right|^{c_2-c_1} t^{-(c_2-c_1)/2}|\xi|^{c_1}e^{-t|\xi|^2}\widehat{u}(\xi,0),$$

$\sup_{s\ge 0} s^{c_2-c_1}e^{-s^2} < \infty$, and an easy integration over $[0,t/2]\cup[t/2,t]$, resp.

□

4. Smoothing effect. This section is devoted to an analysis of the parabolic regularization effect in the scale of pseudomeasure spaces.

Let for $1 < a < d-1$, $\mathcal{Y}_a$ denote the space $\mathcal{X} \cap \{v : (0,\infty) \to \mathcal{PM}^a : |\!|\!|v|\!|\!|_a \equiv \sup_{t>0} t^{1+(a-d)/2} \|v(t)\|_{\mathcal{PM}^a} < \infty\}$. $\mathcal{Y}_a$ is normed by the quantity $\|v\|_{\mathcal{Y}_a} = |\!|\!|v|\!|\!|_{d-2} + |\!|\!|v|\!|\!|_a$. Of course, $\mathcal{Y}_{d-2} \equiv \mathcal{X}$ with this definition. It is easy to check the following property

Proposition 4.1 *If $d \ge 3$, $1 < a < d-1$, then there exists a constant $L = L(a,d)$ such that for all $u \in \mathcal{X}$ and $v \in \mathcal{Y}_a$*

$$\|B(u,v)\|_{\mathcal{Y}_a} \le L \|u\|_{\mathcal{X}} \|v\|_{\mathcal{Y}_a}$$

holds for the quadratic form B defined in (2.4).

Proof. Note that

$$\begin{aligned} \left|\widehat{(u\nabla\psi)}(\xi,t)\right| &\le \left(\int |\xi-\zeta|^{2-d}|\zeta|^{-(a+1)}\,d\zeta\right) \|u(t)\|_{\mathcal{PM}^{d-2}} \|v(t)\|_{\mathcal{PM}^a} \\ &= C|\xi|^{1-a} \|u(t)\|_{\mathcal{PM}^{d-2}} \|v(t)\|_{\mathcal{PM}^a}, \end{aligned}$$

where $\nabla\psi = \nabla E_d * v$. Then we estimate

$$\begin{aligned} &|\xi|^a \left| \int_0^t i\xi e^{-(t-s)|\xi|^2} \widehat{(u\nabla\psi)}(\xi,s)\,ds \right| \\ &\le C \int_0^t |\xi|^2 e^{-(t-s)|\xi|^2} \|u(s)\|_{\mathcal{PM}^{d-2}} \|v(s)\|_{\mathcal{PM}^a}\,ds \\ &\le C \left(\int_0^t |\xi|^2 e^{-(t-s)|\xi|^2} s^{(d-a)/2-1}\,ds \right) \|u\|_{\mathcal{X}} \|v\|_{\mathcal{Y}_a}. \end{aligned}$$

Since $1 < a < d-1$ we have

$$\sup_{t>0,\,\xi\in\mathbb{R}^d} \int_0^t |\xi|^2 e^{-(t-s)|\xi|^2} s^{(d-a)/2-1}\,ds < \infty$$

which is an easy consequence of the estimates of the integrals $\int_0^{t/2} \ldots\,ds$ and $\int_{t/2}^t \ldots\,ds$. □

Corollary 4.1 *Given $d \ge 4$, $1 < a < d-1$ and $u(0) \in \mathcal{PM}^{d-2}$ of small norm there exists a solution of (1.1)–(1.2) such that $u \in \mathcal{Y}_a$, i.e.*

$$|\!|\!|u|\!|\!|_a = \sup_{t>0} t^{1+(a-d)/2} \|u(t)\|_{\mathcal{PM}^a} < \infty.$$

The proof follows from the well-known Kato's scheme involving two norms: $\mathcal{X}$ and $\mathcal{Y}_a$, cf. [9] or [2]. Indeed, a combination of the estimates of the nonlinear term in Proposition 2.1 (valid for $d \ge 4$) and in Proposition 4.1 (valid even for $d \ge 3$) shows the continuity of B in $(\mathcal{X} \cap \mathcal{Y}_a)^2$.

This implies, in turn, that the fixed point argument argument in Lemma 2.2 applies supplying us with a solution in $\mathcal{X} \cap \mathcal{Y}_a$. $\square$

Remark. This corollary shows that small solutions constructed in Theorem 2.1 enjoy *a posteriori* some regularity properties in the scale of pseudomeasures spaces, e.g., they belong to $\mathcal{PM}^a$ with $d-2 < a < d-1$ (which gives a decay of $\widehat{u}(\xi, t)$ in ξ and t), and to $\mathcal{PM}^a$ with $1 < a < d-2$ (which shows that $\widehat{u}(\xi, t)$ has a weaker singularity at $\xi = 0$ than *a priori* assumed for $u \in \mathcal{X}$).

Let us remark that the two norms approach by Kato imposes *a priori* a regularization effect on solutions we look for. In other words, they are considered as fluctuations around the solution of the heat equation $e^{t\Delta}u(0)$. By that construction, the solutions appear to be unique locally in the space of more regular functions.

The approach with the only one norm in Theorem 2.1 gives the local uniqueness in the larger space which, in our case, contains genuinely singular solutions (like $\tilde{u}$) which are not smoothed out by the action of the nonlinear semigroup associated with (1.1). Since $e^{t\Delta}\tilde{u}$ is smooth, this means that $\tilde{u}$ as a (stationary) solution of (1.1) cannot be considered as a perturbation of $e^{t\Delta}\tilde{u}$ (at least for $d > 4$).

In that way, if one considers the functions $u_\varepsilon(x, 0) = \varepsilon\tilde{u}(x) = 2(d-2)\varepsilon|x|^{-2}$, $0 \le \varepsilon \le 1$, as the initial data (1.2) for (1.1), then for small ε the system (1.1) has a (self-similar) solution which is even more regular than *a priori* expected (see Corollary 2.1), and for $\varepsilon = 1$ the system (1.1) has a discontinuous solution. Thus, a kind of loss of smoothness phenomenon for solutions with large initial data for (1.1)–(1.2) appears.

The solutions considered above are, in fact, smooth in x for all $t > 0$. Indeed, $u(t) \in \mathcal{PM}^a$ with each $a < d-1$ implies that $\widehat{u}(t) \in L^p_\xi$ for each $2 > p > d/(d-1)$ (check the decay rate of $\widehat{u}$ in ξ at infinity), and the Hausdorff–Young inequality assures us that $u(t) \in L^q_x$ with q satisfying $1/p + 1/q = 1$, i.e., in particular, for some $q > d/2$. In view of [2, Prop. 1], this suffices to obtain smooth solutions for $t > 0$ because initial data (1.2) in $L^q(\mathbb{R}^d)$ with $q > d/2$ lead to C^∞ solutions of (1.1)–(1.2). That argument can be made precise using the result on the interpolation of $\mathcal{PM}$ norms in

Lemma 4.1 *Let $d \ge 4$, $a \in (d-2, d-1)$ and $q \in \left(\frac{d}{2}, \frac{d}{d-a}\right)$. Then the following inequality*

$$\|v\|_q \le C\|v\|_{\mathcal{PM}^a}^{\frac{2-d/q}{a-d+2}} \|v\|_{\mathcal{PM}^{d-2}}^{\frac{a-d+d/q}{a-d+2}}$$

holds for some $C = C(a, q, d)$ and all $v \in \mathcal{PM}^{d-2} \cap \mathcal{PM}^a$.

Proof. Let p satisfy the relation $1/p + 1/q = 1$ so that by the Hausdorff–Young inequality

$$\begin{aligned}\|v\|_q^p &\le C\int_{|\xi|\le R}|\widehat{v}(\xi)|^p\,d\xi + C\int_{|\xi|>R}|\widehat{v}(\xi)|^p\,d\xi\\ &\le C\|v\|^p_{\mathcal{PM}^{d-2}}\int_{|\xi|\le R}|\xi|^{-p(d-2)}\,d\xi + C\|v\|^p_{\mathcal{PM}^a}\int_{|\xi|>R}|\xi|^{-ap}\,d\xi\\ &= CR^{d-p(d-2)}\|v\|^p_{\mathcal{PM}^{d-2}} + CR^{d-ap}\|v\|^p_{\mathcal{PM}^a}.\end{aligned}$$

After the optimization with $R = (\|v\|_{\mathcal{PM}^a}/\|v\|_{\mathcal{PM}^{d-2}})^{1/(a-d+2)}$ this gives the conclusion. □

Note again that the two norms approach by Kato in [17] for the problem (1.1)–(1.2) as was in [2] automatically excludes the solutions with singularities like $c|x|^{-2}$ because the second norm of such a function is infinite, see [2, Th. 1].

Since we have been mainly interested in positive solutions of the problem (1.1)–(1.2) which are physically significant, we did not recall in this paper yet that there are plenty of singular stationary solutions of (1.1) that change the sign. They could be obtained from those in [1] after a suitable rescaling.

Remark. A related phenomenon of a loss of smoothness of solutions with the initial data of growing size has been studied for the Navier–Stokes equations with external forces in [11] where the role of the singular solution $\tilde{u}$ is played by a family of homogeneous of degree -1 stationary solutions of the nonhomogeneous Navier–Stokes system (with $(C\delta, 0, 0)$ forces) constructed in [26], see also [18, Ch. II, Sec. 23]. □

Finally, examples of blowing up solutions of (1.1)–(1.2) which cease to exist after a finite time $T > 0$ considered in [4], [5] show that not only the global existence of solutions for small initial data in Theorem 2.1 is in certain sense an optimal result, but also the structure of the whole set of local in time solutions of (1.1) is much more complicated. Namely, there are solutions enjoying the parabolic smoothing, other that loose the regularity, and other that cannot be continued for all $t \ge 0$. Moreover, we expect that (1.1) with $u(x,0) = \varepsilon\tilde{u}(x) = \varepsilon 2(d-2)|x|^{-2}$, $\varepsilon > 1$, has no solutions at all, even in an ultraweak sense, because there is a phenomenon of an instantaneous blow-up (the blow-up time $T = T_\varepsilon \to 0$ as $\varepsilon \nearrow 1$).

The Cauchy problem (1.1)–(1.2) has not been solved yet for $d = 3$. Below, we sketch an approach that permits us to construct solutions of (1.1)–(1.2) in another way – using the two norms technique of Kato (which is, as we have already remarked, intrinsically connected with proving an additional parabolic regularization effect).

The approach is based on the following estimates of the bilinear form B in $\mathcal{Y}_a$ spaces which extend those in Proposition 4.1 (we recall the first one for the reader's convenience).

Proposition 4.2 *Let $d \geq 3$, $1 < a < d-1$. Then there exists a constant $L = L(a,d)$ such that*

$$\|B(u,v)\|_{\mathcal{Y}_a} \leq L\|u\|_{\mathcal{X}}\|v\|_{\mathcal{Y}_a}$$

for every $u \in \mathcal{X} \equiv \mathcal{Y}_{d-2}$, $v \in \mathcal{Y}_a$. Moreover, if $(d-1)/2 < a < d-1$, then there exists another constant $\tilde{L} = \tilde{L}(a,d)$ such that

$$\|B(u,v)\|_{\mathcal{X}} \leq \tilde{L}\|u\|_{\mathcal{Y}_a}\|v\|_{\mathcal{Y}_a}$$

for every u, $v \in \mathcal{Y}_a$.

Proof. The reasoning is similar to that in Proposition 4.1 so we only sketch the calculations. Starting from

$$\left|\widehat{(u\nabla\psi)}(\xi,t)\right| \leq \left(\int |\xi-\zeta|^{-a}|\zeta|^{-(a+1)}\,d\zeta\right)\|u(t)\|_{\mathcal{PM}^a}\|v(t)\|_{\mathcal{PM}^a},$$

we see that the conditions $a+a+1-d>0$ and $a+1<d$ suffice to get the result in the second part of Proposition. □

An immediate consequence of the Kato construction, the estimates for the nonlinear term in Proposition 4.2 and the linear estimates (3.7) is the following existence result for solutions enjoying an additional regularization effect of the heat semigroup.

Theorem 4.1 *Let $d \geq 3$ and $(d-1)/2 < a < d-1$. For each initial data $u(0) \in \mathcal{PM}^{d-2}$ of small norm there exists a solution $u \in \mathcal{Y}_a$ normed by*

$$\|u\|_{\mathcal{Y}_a} \equiv \sup_{t\geq 0}\|u(t)\|_{\mathcal{PM}^{d-2}} + \sup_{t>0} t^{1+(a-d)/2}\|u(t)\|_{\mathcal{PM}^a}.$$

Proof. It is easy to see that, due to Proposition 4.2,

$$\|B(u,v)\|_{\mathcal{Y}_a} \leq L\|u\|_{\mathcal{Y}_a}\|v\|_{\mathcal{Y}_a},$$

and the scheme of the existence proof in Theorem 2.1 applies. □

Remark. Similar existence results can be proved in the framework of weak L^p spaces (i.e. the Marcinkiewicz spaces $L^{p,\infty}(I\!\!R^d)$ with $p = d/2$), improving, e.g., [2, Th. 1] in the case $d = 3$. The estimates of the nonlinear term B are, however, technical and more complicated in this case compared to the framework of pseudomeasures, cf. also [16].

As it concerns the local in time existence of solutions of the problem (1.1)–(1.2), let us notice that (besides [2], [4]) one can obtain results when more general spaces of pseudomeasures $\mathcal{Y}_{a,b}$ normed by the quantity $|\!|\!| v |\!|\!|_{a,b} \equiv \sup_{t>0} t^b \|v(t)\|_{\mathcal{PM}^a}$ are considered. The space $\mathcal{Y}_a$ defined in the beginning of Section 4 corresponds to $\mathcal{Y}_{a,b}$ with $b = 1 + (a-d)/2$. The estimates of the nonlinear term B resemble those in Proposition 4.2 but they are more elaborate, cf. also (3.7)–(3.8).

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