On Schrödinger operators with singular potentials and magnetic fields.

Vassili N. Kolokoltsov

Nottingham Trent University, School of Computing and Mathematics
Burton Street, Nottingham NG1 4BU, UK
and
Institute for Information Transmission Problems RAN, Moscow, Russia
email vassili.kolokoltsov@ntu.ac.uk

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Abstract. *A formal Schrödinger operator of the form* $H = (i^{-1}(\partial/\partial x) + A(x))^2 + V(x)$ *in* $\mathbf{R}^d$ *is considered, where* A *is a bounded measurable vector-valued function and* $V(x)$ *and* $div\, A$ *are both measures satisfying certain additional conditions. It is shown that one can give meaning to such an operator as a self-adjoint lower bounded operator in* $L^2(\mathbf{R}^d)$. *The corresponding heat kernel is constructed and its small time asymptotics is obtained. A rigorous Feynman path integral representation for the solutions of the heat and Schrödinger equations with the generator* H *is given.*

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1. Introduction.

A formal Schrödinger operator with a magnetic field is condsidered, namely the operator

$$H = \left(-i\frac{\partial}{\partial x} + A(x)\right)^2 + V(x), \tag{1.1}$$

where $\frac{\partial}{\partial x} = (\frac{\partial}{\partial x_1}, ... \frac{\partial}{\partial x_n})$ is the gradient operator in $\mathbf{R}^d$, under the following conditions:

C1) the magnetic vector-potential $A = (A^1, ..., A^d)$ is a bounded measurable $\mathbf{R}^d$-valued function on $\mathbf{R}^d$,

C2) the potential V and the divergence $divA = \sum_{j=1}^d \frac{\partial A^j}{\partial x^j}$ of A (defined in the sense of distributions) are both Borel measures,

C3) if $d > 1$ there exist $\alpha > d - 2$ and $C > 0$ such that for all $x \in \mathbf{R}^d$ and $r \in (0, 1]$

$$|divA|(B_r(x)) \leq Cr^\alpha, \quad |V|(B_r(x)) \leq Cr^\alpha, \tag{1.2}$$

where $|V|$, $|div\, A|$ denote the total variations of the (possibly non-positive) measures V and $div\, A$ respectively, and $B_r(x)$ denotes the ball in $\mathbf{R}^d$ of the radius r centered at x; if $d = 1$, then the same holds for $\alpha = 0$.

In Section 2 we show how to give meaning to the formal operator (1.1) as a self-adjoint operator in $L^2(\mathcal{R}^d)$ by means of a direct construction of the corresponding semigroup $\exp\{-tH\}$. Moreover, we give global estimates for the corresponding heat kernel and its local (in small time) multiplicative asymptotics. Our method is based on the standard perturbation theory around the free evolution $e^{t\Delta}$. However, the presence of a singular magnetic field complicates the estimates essentially.

Our main results are obtained in Section 3, where we represent the corresponding Schrödinger semigroup $\exp\{-iHt\}$ in terms of a rigorously defined path integral generalising the construction from [K3],[K4] to the case of nontrivial magnetic fields.

In Section 4 we obtain technical estimates which are used in Sections 2 and 3. Sections 5 and 6 are devoted to short discussions. In Section 5 we discuss the notion of the dimensionality of measures and its connection with our main assumption C3). In Section 6 possible generalisations of assumptions C1)-C3) are indicated, under which the main results of this paper can be still proved with minor modifications.

Some discussion of the relevant results is in order. To begin with let us notice that we contribute to several topics of the theory of Schrödinger equations, namely the heat kernel and fundamental solution estimates, singular potentials, magnetic fields and path integration technique. There appeared recently a lot of publications devoted to each of this topics. Therefore, though we present a rather extensive bibliography, it is far from being complete.

The notion of singular Schrödinger equations is very wide. Several authors call the potentials singular if they are given by strongly non-regular functions (say, locally unbounded and discontinuous). The estimates for the heat kernels of the corresponding Schrödinger operators

$$H = -\Delta/2 + V(x) \tag{1.3}$$

(i.e. integral kernels of the operators e^{-tH}), are discussed in many publications (see [Ars], [LM], [Zh] and references therein). More recently, the notion of a singular Schrödinger operator was started to indicate a (formal) expression (1.3), where the potential V is a certain generalised function (distribution). The most popular examples of such Schrödinger operators are given by systems with δ-interaction, i.e. with potentials supported by points or smooth submanifolds (see e.g. monographs [AGHH], [Kosh] and references therein), because such potentials appear in different physical models. Here we are interested in more general potentials. In [BM], the generalised Kato class of potentials (which consists of Radon measures with certain conditions) was introduced and local heat kernel estimates were obtained for the corresponding Schrödfinger semigroups using a generalisation of the probabilistic technique developed in [Si]. In [Br1], one can find several results on the spectral analysis of singular Schrödinger equations and an extensive review of different approaches to the analysis of singular Schrödinger equations, which include non-standard analysis, the theory of Dirichlet forms, the theory of quadratic form, the theory of self-adjoint extensions of Krein and von Neumann and others. A useful method of smooth approximation was developed in [AFHKL], which allows to give a rigorous meaning to operators (1.3) in rather general situations (in particular, when the corresponding form is not closable). Also an approximation technique was further developed in [BFT]. A deep study of Schrödinger operators on Riemannian manifolds (with a bounded below Ricci curvature) with potentials being almost general measures is given in [St]. The measures in [St] have neither to be regular nor to be σ-finite, the only requirement is that they do not charge polar sets.

Usually, quite essential simplifications occur when applying general tecniques to one-dimensional situations (see e.g. [Br2],[SS2]).

In [DS], the study of the Schrödinger equations with potentials supported by a smooth surface in the presence of magnetic fields was initiated. However, no general results seem to be obtained up to now on Schrödinger equation with a magnetic field and with a potential V being a distribution. The case of magnetic fields and potentials from the Kato class (not distributions) were considered in [BHL], where the self-adjointness was proved by probabilistic technique of [Si]. Concerning heat kernel estimates for these operators we refer to [LT], [LM] and references therein.

We turn now to the discussion of path integral representations. In the most of known approaches to the mathematically rigorous construction of the Feynman integral, one defines this integral not as a genuine integral over a bona fide σ-additive measure, but as some generalised functional on an appropriate space of functions, which can be defined, for example, as the limit of certain discrete approximations, by means of analytical continuation, by extensions of Parceval's identity and by related axiomatic definitions, and more recently by means of the white noise analysis (see e.g. reviews in [ABB], [K2], [SS1], for some recent results in [Ich], [Lo], [AKS2], [TZ], and for the discussion of Dirac equation in [Za]). These approaches still cover only a very restrictive class of potentials, for example, singular potentials were considered only by white noise analysis approach but only in one-dimensional case (see e.g. [AKK] and references therein).

First attempts to construct the Feynman integral as a bona fide intergral over a ginuine measure were erronious (see [GYa]) and lead to understanding that there is no direct generalisation of Wiener measure that can give an analog of Feynman-Kac formula for the case of Schrödinger operators (see e.g. [RS]). An approach to the path integral that allows to express it in terms of the Wiener measure was proposed in [Do] and is based on the idea of the rotation of the classical trajectories in complex domains, which requires very restrictive analytic assumptions on the potentials (see e.g. [H1],[H2] for recent results in this direction).

Another approach to the construction of a genuine Feynman measure, initiated in [MCh], [M], defines the solutions to the Schrödinger equation in momentum repesentation as expectations with respect to a certain compound Poisson process, or as integrals over a measure concentrated on piecewise constant paths. Though this method was succesfully applied to different models (see e.g. [GK] for many particle problems, [Se] for simple quantum field models, [CheQ] for computational aspects and tunneling problems, [Gav] and [KY] for Dirac equations), the restriction on interaction forces were always very strong, for example, for a usual Schrödinger equation, this approach was used only in the case of potentials which are Fourier transforms of finite measures. However in [K3],[K4] following this trend, a construction was given that covered already essentially more general potentials. To achieve this, one uses a coordinate representation for the Schrödinger equation (and not the momentum representation as in [MCh]) and also regularise the Schrödinger equation by introducing either complex times $t(1 - i\epsilon)$ or a continuous non-demolition observation of the quantum system under consideration with a small coupling constant λ (it was seemingly first noted in [Me] that such a regularisation can help to overcome the difficulties in mathematical construction of the Feynmann integral). Our path integral is defined as an absolutely convergent integral in the sense of Riemann or Lebesgue for positive ϵ or λ respectively. Moreover, the corresponding measure on path space does not depend on ϵ or λ (and is actually a measure on the Cameron-Martin space of paths having square integrable derivatives). Hence, the limit as $\epsilon \to 0$ or $\lambda \to 0$ respectively (which exists in many situations) giving the solution to the initial non-regularised Schrödinger equation can be considered as an improper (infinite-dimensional) Riemann integral. However, in one-dimensional case, no regularisation is needed, and the fundamental solution to the Schrödinger equation is obtained as an absolutely convergent infinite-dimensional integral (defined rigorously in the sense of Lebesgue). Moreover, it turns out that our path integral has a natural representation in Fock space, and therefore can be expressed in terms of the expectation of a certain functional with respect to the Wiener process, see [K3]. As we noted above, in this paper (which represents an improved version of the author's preprint [K1]) we develop further the approach from [K3],[K4] using for simplicity only regularisation by complex times.

2. Self-adjointness and asymptotics for the heat kernel.

Throughout the paper we assume the conditions $C1) - C3)$ to be valid.

In order to give meaning to operator (1.1) we are going to construct the corresponding semigroup e^{-tH}. Moreover, for our purposes, it will be useful to consider the following more general formal Cauchy problem

$$\frac{\partial u}{\partial t} = -DHu, \quad u|_{t=0} = u_0, \tag{2.1}$$

where the (generalised) diffusion coefficient D is an arbitrary complex number such that $\epsilon = ReD \geq 0$, $|D| > 0$. In the interaction representation, equation (2.1) takes the form

$$u(t) = e^{-Dt\Delta/2} u_0 - D \int_0^t e^{-D(t-s)\Delta/2} (W - 2i(A, \nabla)) u(s)\, ds, \tag{2.2}$$

where

$$W(x) = V(x) + |A(x)|^2 - i\, divA(x). \tag{2.3}$$

More precisely, W is the measure, which is the sum of the measure $V - i\, divA$ and the measure having the density $|A|^2$ with respect to Lebesgue measure. The letter W in (2.2) stands for the operator of multiplication by W.

It is well known (and can be seen by simple calculations) that problems (2.1) and (2.2) are formally equivalent. In some situations, equation (2.2) is more convenient than (2.1) to deal with, because it can often be solved by iterations. In the case of the Green function $G^D(t, x, y)$ (or fundamental solution) of equation (2.2), i.e. its solution with the

Dirac initial condition $u_0(x) = \delta(x-y)$, the iteration procedure leads to the following representation:

$$G^D(t,x,y) = \sum_{k=0}^{\infty} I_k^D(t,x,y) \tag{2.4}$$

with $I_0^D(t,x,y) = G_{free}^D(t,x,y)$ and with other I_k^D, $k>1$, being defined inductively by the formula

$$I_k^D(t,x,y) = -D\int_0^t \int_{\mathbf{R}^d} I_{k-1}^D(t-s,x,\xi)(W(d\xi)+2i\frac{(A,\xi-y)}{Ds}\,d\xi)G_{free}^D(s,\xi-y)\,ds, \tag{2.5}$$

where G_{free}^D is the Green function of the "free" equation (2.1) (i.e. with $V=0$):

$$G_{free}^D(t,x-y) = (2\pi tD)^{-d/2}\exp\{-\frac{(x-y)^2}{2Dt}\}. \tag{2.6}$$

More precisely, it is clear that if the series on the r.h.s. of (2.4) is convergent and for its sum G^D the r.h.s. of equation (2.2) makes sense, then G^D gives a solution (which may be not unique, a priori) of (2.2) with $u_0(x) = \delta(x-y)$.

The main technical result of the paper is the following.

Proposition 1. *If $\epsilon = ReD > 0$, then all terms of series (2.4) are well defined as absolutely convergent integrals, the series itself is absolutely convergent and its sum $G^D(t,x,y)$ is continuous in $x,y\in\mathbf{R}^d$, $t>0$ (and D) and satisfies the following estimate*

$$|G^D(t,x,y)| \le KG_{free}^{|D|^2/\epsilon}(t,x,y)\}\exp\{B|x-y|\} \tag{2.7}$$

uniformly for $t\le t_0$ with any fixed t_0, where B,K are constants. Moreover, the integral operators

$$(U^D(t)u_0)(t,x) = \int u_G^D(t,x,y)u_0(y)\,dy \tag{2.8}$$

defining the solutions to equation (2.2) for $t\in[0,t_0]$ form a uniformly bounded family of operators $L_2(\mathbf{R}^d)\mapsto L_2(\mathbf{R}^d)$. Furthermore, if D is real, i.e. $D=\epsilon>0$, then there exists a constant $\omega>0$ such that the Green function G^ϵ has the asymptotic representation

$$G^\epsilon(t,x,y) = G_{free}^\epsilon(t,x,y)(1+O(t^\omega)+O(|x-y|)) \tag{2.9}$$

for small t and $x-y$. In case of vanishing A, the multiplier $\exp\{B|x-y|\}$ *in (2.7) can be omitted, and the term $O(|x-y|$ in (2.9) can be dropped. In this case, formula (2.9) gives global (uniform for all x,y) small time asymptotics for G^ϵ.*

Remark. The statement and the proof of this proposition still holds if A is supposed to be complex valued, which gives in particular the estimates for the heat kernel of a diffusion equation with singular drifts and sources generalising some results of [ADB] obtained for the case of V being a δ-function.

Proposition 1 is proved in Section 4.

Next, we are going to show that the operators $U^\epsilon(t)$, $\epsilon > 0$, form a strongly continuous operator semigroup in $L_2(\mathbf{R}^d)$. To this end, we first regularise A and V using the following standard procedure. Let us choose an infinitely smooth non-negative even function ϕ on $\mathbf{R}^d$ with a support in the ball $B_1(0)$ and such that $\int \phi(x)\,dx = 1$. Let $\phi_a(x) = a^{-d}\phi(x/a)$, $a > 0$, and

$$A_a(x) = \int \phi_a(x-y)A(y)\,dy, \quad A_a^2(x) = \int \phi_a(x-y)A^2(y)\,dy, \quad V_a(x) = \int \phi_a(x-y)V(dy). \tag{2.10}$$

Define

$$\begin{aligned} H_a &= \left(-i\frac{\partial}{\partial x} + A_a(x)\right)^2 + A_a^2(x) - (A_a(x))^2 + V(x) \\ &= -\Delta + 2i(A_a(x), \nabla) + W_a(x), \end{aligned} \tag{2.11}$$

where

$$W_a(x) = \int \phi_a(x-y)W(y)\,dy = A_a^2(x) - i\,div\,A_a(x) + V_a(x). \tag{2.12}$$

It is easy to see that A_a and V_a satisfies the assumptions C1)-C3) with the same constants α and C. Therefore Proposition 1 is still valid for the solution of equation (2.2) with W, A replaced by W_a, A_a for all a (with all estimates being uniform in a). On the other hand, since A_a and V_a are smooth bounded functions, the operator H_a is known to be self-adjoint and bounded below, and moreover, the family of operators $U_a^D(t)$ with the kernels G_a^D giving solutions to the corresponding equation (2.2) forms a semigroup, which coincides with the semigroup e^{-tDH_a} generated by DH_a.

At the end of Section 4, we shall prove the following statement.

Proposition 2. *For any D with $\epsilon = ReD > 0$ and all $x, y \in \mathbf{R}^d$, $t > 0$*

$$\lim_{a\to 0} G_a^D(t,x,y) = G^D(t,x,y)$$

uniformly on compact sets of parameters and $U_n^D(t) \to U^D(t)$, as $n \to \infty$, strongly in $L_2(\mathbf{R}^d)$ sense.

This Proposition implies, in particular, that $U^\epsilon(t)$ for real ϵ forms a strongly continuous semigroup of bounded self-adjoint operators. In particular, there exist positive constants M and b such that

$$\|U^\epsilon(t)\| \le Me^{b\epsilon t}. \tag{2.13}$$

The generator of any strongly continuous semigroup of operators in a Hilbert space is known to be a densely defined and closed operator. Let us denote by $\tilde{H}$ the generator of the semigroup $U^1(t)$ so that $U^\epsilon(t) = \exp\{-t\epsilon\tilde{H}\}$. Since all operators $U^\epsilon(t)$ are self-adjoint, it follows that $\tilde{H}$ is symmetric. Moreover, the condition (2.13) implies that any $\lambda < -b$ belongs to the resolvent set of $\tilde{H}$, and therefore, by the main criterion of self-adjointness (see e.g. [RS]), the operator $\tilde{H}$ is self-adjoint and bounded below. Thus we obtained the following result.

Theorem 1. *The family (2.8) of operators $U^D(t)$ (giving solutions to equation (2.2), which is formally equivalent to the evolutionary equation (2.1) with the formal generator*

DH) coincides with the semigroup $\exp\{-tD\tilde{H}\}$ *defined by means of the functional calculus for a self-adjoint bounded below operator* $\tilde{H}$*. For the integral kernel of the operators* $\exp\{-tD\tilde{H}\}$ *the estimates (2.7) and (2.9) hold.*

Since $e^{-tH_n} \to e^{-t\tilde{H}}$ and formally $H_n \to H$, it is consistent with the intuitive meaning of the formal operator $\tilde{H}$ to consider $\tilde{H}$ as the rigorous self-adjoint version of the formal expression H. Further on we shall denote $\tilde{H}$ again by H.

3. Path integral representation.

To construct a path integral representation for the solution $U^D(t)u_0$ of equation (1.1), we shall construct a measure on a path space that is supported on the set of continuous piecewise linear paths. Denote this set by CPL. Let $CPL^{x,y}(0,t)$ denote the class of paths $q:[0,t]\mapsto \mathcal{R}^d$ from CPL joining x and y in time t, i.e. such that $q(0)=x$, $q(t)=y$. By $CPL^{x,y}_n(0,t)$ we denote its subclass consisting of all paths from $CPL^{x,y}(0,t)$ that have exactly n jumps of their derivative. Clearly, each $CPL^{x,y}_n(0,t)$ is parametrised by the simplex

$$Sim^n_t=\{s_1,...,s_n: 0<s_1<s_2<...<s_n\leq t\}$$

of the times of jumps $s_1,...,s_n$ of the derivatives of a path and by n positions $q(s_j)$, $j=1,...,n$, of this path at these points. In other words, an arbitrary path in $CPL^{x,y}_n(0,t)$ has the form

$$q(s)=q^{s_1...s_n}_{\eta_1...\eta_n}(s)=\eta_j+(s-s_j)\frac{\eta_{j+1}-\eta_j}{s_{j+1}-s_j},\quad s\in[s_j,s_{j+1}] \tag{3.1}$$

(where it is assumed that $s_0=0, s_{n+1}=t, \eta_0=x, \eta_{n+1}=y$). Obviously,

$$CPL^{x,y}(0,t)=\cup_{n=0}^\infty CPL^{x,y}_n(0,t).$$

To any $\mathbf{R}^{d+1}$-valued Borel measure $M=(\mu,\nu)=(\mu,\nu^1,...,\nu^d)$ on $\mathbf{R}^d$ there corresponds a (σ-finite) complex measure M^{CPL} on $CPL^{x,y}(0,t)$, which is defined as the sum of the measures M^{CPL}_n on the finite-dimensional spaces $CPL^{x,y}_n(0,t)$ such that M^{CPL}_0 is just the unit measure on the one-point set $CPL^{x,y}_0(0,t)$ and each M^{CPL}_n, $n>0$, is defined in the following way: if Φ is a functional on $CPL^{x,y}(0,t)$, then

$$\int_{CPL^{x,y}_n(0,t)}\Phi(q(.))M^{CPL}_n(dq(.))=\int_{Sim^n_t}ds_1...ds_n$$

$$\times\int_{\mathcal{R}^d}...\int_{\mathcal{R}^d}\left(\mu+2i\frac{(\eta_2-\eta_1,\nu)}{s_2-s_1}\right)(d\eta_1)...\left(\mu+2i\frac{(y-\eta_n,\nu)}{t-s_n}\right)(d\eta_n)\Phi(q(.)). \tag{3.2}$$

Remark. Nothing is changed if $CPL^{x,y}_n(0,t)$ is defined as the set of paths with no more than n jumps of their derivative. In fact, the measure M^{CPL}_n of the subset $CPL^{x,y}_{n-1}(0,t)\subset CPL^{x,y}_n(0,t)$ vanishes anyway, because if there is no jump, say, at the moment s_j it means that $(\eta_j-\eta_{j-1})(s_{j+1}-s_{j-1})=(\eta_{j+1}-\eta_{j-1})(s_j-s_{j-1})$, therefore s_j can be only one point, and the Lebesgue measure has no atoms.

Now, let $M=(W, A\,dx/D)$.

Theorem 2. *For any D with $\epsilon=ReD>0$, the Green function G^D of equation (2.1) has the following path integral representation:*

$$G^D(t,x,y)=\int_{CPL^{x,y}(0,t)}\Phi_D(q(.))\exp\{-\int_0^t\dot{q}^2(s)\,ds/2D\}M^{CPL}(dq(.)), \tag{3.3}$$

with $q(s)$ given by (3.1) and

$$\Phi_D(q(.)) = D^n \prod_{j=1}^{n+1} (2\pi(s_j - s_{j-1})D)^{-d/2}. \tag{3.4}$$

For any $u_0 \in L^2(\mathcal{R}^d)$ the solution $u(t,s)$ of the Cauchy problem (2.1) with $D = i$ has the form

$$u(t,x) = \lim_{\epsilon \to 0_+} \int_{CPL^{x,y}(0,t)} \int_{\mathcal{R}^d} u_0(y)\Phi_{i+\epsilon}(q(.)) \exp\{-\int_0^t \dot{q}^2(s)\, ds/2(i+\epsilon)\} M^{CPL}(dq(.))dy, \tag{3.5}$$

where the limit is understood in L^2-sense.

Due to the definition of M^{CPL}, the integral (3.3), if it exists, is the convergent sum of finite-dimensional integrals, which are all absolutely convergent. But one sees directly that these integrals are exactly the same as the integrals $I_k(t,x,y)$ of series (2.4), (2.5). This implies the validity of (3.3), since it is shown in Proposition 1 that the series (2.4), (2.5) is absolutely convergent and all its terms are absolutely convergent integrals. Formula (3.5) follows immediately from (3.3) and the observation that, due to the functional calculus, for any bounded below self-adjoint operator

$$\exp\{-itH\} = \lim_{\epsilon \to 0} \exp\{-(i+\epsilon)tH\}$$

strongly for all positive t.

As shown in [K3], in case $A = 0$ and $d = 1$, formula (3.3) still holds for $D = i$ (i.e. for vanishing ϵ), so that in this case the Green function of the Schrödinger equation itself exists and is represented by a convergent path integral (without any regularisation). This is one of the performances of the fact mentioned in the introduction that in one-dimensional situation essential simplifications usually occur. In general, the question of existence of a pointwise limit for G^D as $D \to i$ is very subtle and is not considered here. It is reasonable to suggest (as a conjecture) that it exists if the fundamental solution (or the Green function) G^i of the Schrödinger semigroup exists as a continuous function. This latter question is quite non-trivial and only recently some results were obtained that include rather general potentials, see [Ya2], (respectively [Ya3]) for the case of the absence of magnetic fields and general smooth (respectively singular) potentials, and [Ya1] for the case of smooth magnetic fields.

In case $A = 0$, integral (3.3) has a simple probabilistic interpretation in terms of an expectation with respect to a compound Poisson process. The following statement is a direct consequence of Theorem 2 and the standard properties of Poisson processes.

Theorem 3. *Suppose the vector potential A vanishes and V satisfies assumption C3). Suppose additionally that V has no atom at the origin and is a finite positive measure so that $\lambda_V = V(\mathcal{R}^d) > 0$. Let the paths of CPL are parametrised by (3.1). Let E_t denote the expectation with respect to the process of jumps η_j which are identically independently distributed according to the probability measure V/λ_V, and which occur at times s_j from*

$[0, t]$ *distributed according to Poisson process of intensity* λ_V. *Then the integral (3.3) can be written in the form*

$$G^D(t,x,y) = e^{t\lambda_V} E_t \left(\Phi_D(q(.)) \exp\{- \int_0^t \dot{q}^2(s)\, ds/2D\} \right). \tag{3.6}$$

Similar representation surely holds for formula (3.5).

Let us note also that the restriction on V being finite (used in Theorem 3) is not essential, because clearly an arbitrary V has a density with respect to a certain finite (positive) measure $\tilde{V}$. Hence, one can include this density in the integrand and work with the Poisson process defined by the Lévy measure $\tilde{V}$.

4. Proof of Propositions 1 and 2.

Throughout this section the letter C will be reserved for constants, which may be different in different places.

For the complex measure W given by (2.3), let us define its total variation as the positive measure $|W| = |V| + |A|^2 + |div\, A|$. We start with a simple general estimate for $|W|$, given in Lemma 1 below. Notice first that (1.2) holds for the measure $|W|$ as well. Moreover, this estimate holds also for all $r \leq r_0$ for an arbitrary r_0. In what follows, α will always denotes the parameter from (1.2) (in particular, $\alpha = 0$ in the case $d = 1$).

Next, it is easy to see that (1.2) implies that for all R

$$|W|(B_R(x)) \leq C(\max(1, R))^d. \tag{4.1}$$

Let us choose arbitrary positive numbers t_0 and ϵ and a number $\delta \in (0, 1/2)$ such that the constant

$$\omega = \min(1/2 - \delta, \alpha(\frac{1}{2} - \delta) - \frac{d}{2} + 1)$$

is positive (this is possible due to the assumption on α).

Lemma 1. *The following estimate holds*

$$\int_{\mathbf{R}^d} |\xi|^p G^{\epsilon}_{free}(t, \xi)|W|(d\xi) \leq C^{p/2}((tp)^{p/2} + t^{(p+1)\omega - 1}), \tag{4.2}$$

where the free Green function G^D_{free} *is given in (2.6) and* $C = C(d, \epsilon, t_0)$ *does not depend on* p *and* $t \leq t_0$.

Remark. In case where W is Lebesgue measure the same estimate holds, but the second term on the r.h.s. of (4.2) can be omitted.

Proof. This is obtained by simple calculations. One decomposes the integral on the l.h.s. of (4.2) into the sum $I_1 + I_2$ by decomposing the domain of integration into the union of non- intersecting domains D_1 and D_2, where $D_1 = B_R$ is the ball of the radius $R = \sqrt{\epsilon}\, t^{1/2 - \delta}$ centered at the origin. Estimating G^{ϵ}_{free} by $O(t\epsilon)^{-d/2}$ and using (1.2) one derives that the second term on the r.h.s. of (4.2) gives an estimate for I_1. Next,

$$I_2 = \sum_{k=1}^{\infty} \int_{B_{(k+1)R} \setminus B_{kR}} |\xi|^p G^{\epsilon}_{free}(t, \xi)|W|(d\xi),$$

which, due to (4.1), can be estimated (up to a constant multiplier) by

$$\sum_{k=1}^{\infty}(k+1)^p R^p(\epsilon t)^{-d/2}\exp\{-\frac{k^2R^2}{2t\epsilon}\}[\max(1,(k+1)R)]^d$$

$$\leq 2^p t^{-d/2}(\sum_{k:kR\leq 1}\exp\{-\frac{k^2R^2}{2t\epsilon}\}+\sum_{k:kR>1}(kR)^{p+d}\exp\{-\frac{k^2R^2}{2t\epsilon}\}).$$

The first sum in brackets can be estimated (up to a constant) by its first term

$$\exp\{-R^2/2t\epsilon\}=\exp\{-t^{-2\delta}/2\},$$

which gives an exponentially small contribution, and the second sum does not exceed (due to the well known connection between the sums and integrals of monotonic functions, and again up to a constant) the expression

$$\max_{x\geq 0}(x^{p+d}\exp\{-\frac{x^2}{2t\epsilon}\})+\frac{1}{R}\int_0^{\infty}x^{p+d}\exp\{-\frac{x^2}{2t\epsilon}\}\,dx$$

$$=((p+d)t\epsilon/e)^{(p+d)/2}+(t\epsilon)^{(p+d)/2}t^{\delta}\Gamma((p+d+1)/2).$$

This clearly implies (4.2), which completes the proof of the Lemma.

The main ingredient in the proof of Proposition 1 is the following.

Lemma 2. *Let $\tilde{I}_k^D$ denote the integrals obtained from the integrals I_k^D of (2.5) by replacing all integrands by its magnitudes. The integrals $\tilde{I}_k^D$ (and therefore $|I_k^D|$ as well) satisfy the following estimates:*

$$\tilde{I}_k^D(t,x,y)\leq\sum_{l=0}^{k}\frac{B^k}{l![(k-l)!]^{\omega}}|x-y|^l(t^{\omega})^{k-l}G_{free}^{|D|^2/\epsilon}(t,x-y) \tag{4.3}$$

with a certain constant $B>0$.

Proof. For $k=0$ the estimate is evident. We shall prove it now for $k+1$ assuming that it holds for k. From (2.5) one has

$$\tilde{I}_{k+1}^D(t,x,y)\leq C\int_0^t ds\int_{\mathbf{R}^d}\tilde{I}_k^D(t-s,x,\xi)(|W|(d\xi)+\frac{|\xi-y|}{s}\,d\xi)G_{free}^{|D|^2/\epsilon}(s,\xi-y) \tag{4.4}$$

with C not depending on k,t,s,x,y. Since

$$\frac{(x-\xi)^2}{t-s}+\frac{(\xi-y)^2}{s}=\frac{t}{s(t-s)}(\xi-\xi_0)^2+\frac{(x-y)^2}{t}$$

with

$$\xi_0=\xi_0(t,s;x,y)=\frac{sx+(t-s)y}{t},$$

one obtains from (4.3), (4.4) that

$$\tilde{I}^D_{k+1}(t,x,y) \le JB^k G^{|D|^2/\epsilon}_{free}(t,x-y), \tag{4.5}$$

where

$$J \le C \int_0^t ds \int_{\mathbf{R}^d} \sum_{l=0}^k \frac{1}{l![(k-l)!]^\omega} |x-\xi|^l ((t-s)^\omega)^{k-l}$$
$$\times G^{|D|^2/\epsilon}_{free}\left(\frac{s(t-s)}{t}, \xi-\xi_0\right) (|W|(d\xi) + \frac{|\xi-y|}{s} d\xi). \tag{4.6}$$

Clearly

$$x-\xi = (\xi_0-\xi) + \frac{t-s}{t}(x-y), \quad \xi - y = (\xi-\xi_0) + \frac{s}{t}(x-y). \tag{4.7}$$

The first of these equalities yields the estimate

$$|x-\xi|^l \le \sum_{m=0}^l \binom{l}{m} |\xi-\xi_0|^{l-m} |x-y|^m \frac{(t-s)^m}{t^m}.$$

Using also the second equality in (4.7), then shifting the vartiable of integration ξ in integral (4.6) by ξ_0 and denoting the shifted measure $|W|$ again by $|W|$ (notice that estimates (1.2) are translation invariant and therefore Lemma 1 will still hold for this new measure $|W|$) one obtains

$$J \le C \sum_{m=0}^k |x-y|^m \int_0^t ds \int_{\mathbf{R}^d} \left(\frac{t-s}{t}\right)^m \sum_{l=m}^k \frac{1}{l!} \binom{l}{m} |\xi|^{l-m}$$
$$\times \frac{(t-s)^{\omega(k-l)}}{[(k-l)!]^\omega} G^{|D|^2/\epsilon}_{free}\left(\frac{s(t-s)}{t}, \xi\right) \left[|W|(d\xi) + \left(\frac{|\xi|}{s} + \frac{|x-y|}{t}\right) d\xi)\right]$$
$$= C \sum_{m=0}^k |x-y|^m \int_0^t ds \int_{\mathbf{R}^d} \left(\frac{t-s)}{t}\right)^m \sum_{p=0}^{k-m} \frac{1}{p!} |\xi|^p$$
$$\times \frac{(t-s)^{\omega(k-p-m)}}{[(k-p-m)!]^\omega} G^{|D|^2/\epsilon}_{free}\left(\frac{s(t-s)}{t}, \xi\right) \left[|W|(d\xi) + \left(\frac{|\xi|}{s} + \frac{|x-y|}{t}\right) d\xi)\right].$$

Therefore,

$$\tilde{I}^D_{k+1}(t,x,y) \le B^k G^{|D|^2/\epsilon}_{free}(t,x-y) \sum_{l=0}^{k+1} \beta_l |x-y|^l, \tag{4.8}$$

where

$$\beta_{k+1} \le \frac{C}{k!t^{k+1}} \int_0^t ds \int_{\mathbf{R}^d} (t-s)^k G^{|D|^2/\epsilon}_{free}\left(\frac{s(t-s)}{t}, \xi\right) d\xi$$
$$\le \frac{C}{k!t^{k+1}} \int_0^t (t-s)^k \, ds = \frac{C}{(k+1)!}, \tag{4.9}$$

and for $l < k+1$

$$\beta_l \leq \frac{C}{l!t^l} \int_0^t ds \int_{\mathbf{R}^d} (t-s)^l G_{free}^{|D|^2/\epsilon} \left(\frac{s(t-s)}{t}, \xi \right)$$

$$\times \left(\sum_{p=0}^{k-l} \frac{|\xi|^p}{p!} \frac{(t-s)^{\omega(k-p-l)}}{[(k-p-l)!]^\omega} [|W|(d\xi) + \frac{|\xi|}{s} d\xi] + \frac{l}{t-s} \sum_{p=0}^{k+1-l} \frac{|\xi|^p}{p!} \frac{(t-s)^{\omega(k+1-p-l)}}{[(k+1-p-l)!]^\omega} d\xi \right).$$

The second sum here can be rewritten as

$$\frac{(t-s)^{\omega(k+1-l)}}{[(k+1-l)!]^\omega} + \sum_{p=0}^{k-l} \frac{|\xi|^{p+1}}{(p+1)!} \frac{(t-s)^{\omega(k-p-l)}}{[(k-p-l)!]^\omega}.$$

Consequently

$$\beta_l \leq \frac{K}{l!t^l[(k+1-l)!]^\omega} \int_0^t ds \int_{\mathbf{R}^d} (t-s)^{l+\omega(k-l)} G_{free}^{|D|^2/\epsilon} \left(\frac{s(t-s)}{t}, \xi \right) \{ l(t-s)^{\omega-1}$$

$$+ \sum_{p=0}^{k-l} \left[\frac{(k+1-l)!}{(k-p-l)!} \right]^\omega (t-s)^{-\omega p} \left[\frac{|\xi|^p}{p!} |W|(d\xi) + \frac{|\xi|^{p+1}}{(p+1)!} \left(\frac{p+1}{s} + \frac{l}{t-s} \right) d\xi \right] \}. \quad (4.10)$$

By Stirling's formula

$$\frac{(k+1-l)!}{(k-p-l)!} \leq C^p \sqrt{\frac{k+1-l}{k-p-l}} (k-l)^{1+p}$$

for $p < k-l$. Hence the sum in (4.10) can be estimated by the sum

$$\sum_{p=0}^{k-l} \frac{C^p}{p^p} k^{\omega(1+p)} \left[\frac{k+1-l}{k+1-p-l} \right]^{\omega/2} (t-s)^{-\omega p} \left[|\xi|^p |W|(d\xi) + |\xi|^{p+1} \left(\frac{1}{s} + \frac{l}{t-s} \right) d\xi \right].$$

Therefore, one derives from Lemma 1 that β_l does not exceed

$$\frac{C}{l!t^l} \frac{1}{[(k+1-l)!]^\omega} \int_0^t ds (t-s)^{l+\omega(k-l+1)-1} \{ l + \sum_{p=0}^{k-l} \frac{C^p}{p^p} \frac{k^{\omega(1+p)}}{(t-s)^{\omega(p+1)-1}} \left[\frac{k+1-l}{k+1-p-l} \right]^{\omega/2}$$

$$\times \left[\left(\frac{(t-s)s}{t} \right)^{p/2} p^{p/2} + \left(\frac{(t-s)s}{t} \right)^{(p+1)\omega-1} + \left(\frac{(t-s)s}{t} \right)^{(1+p)/2} p^{p/2} \left(\frac{1}{s} + \frac{l}{t-s} \right) \right] \}.$$

Consequently

$$\beta_l \leq \frac{C}{l!t^l} \frac{(t^\omega)^{k+1-l}}{[(k+1-l)!]^\omega} (1+\gamma_l) \quad (4.11)$$

with

$$\gamma_l = \sum_{p=0}^{k-l} C^p k^{\omega(1+p)} \left[\frac{k+1-l}{k+1-p-l}\right]^{\omega/2}$$

$$\times[B(l+\omega(k-l)+1, p/2+1)p^{-p/2} + B(l+\omega(k-l+1),(p+1)\omega)p^{-p}$$

$$+B(l+\omega(k-l)+3/2,(p+1)/2)p^{-p/2} + lB(l+\omega(k-l)+1/2,(p+3)/2)p^{-p/2}], \quad (4.12)$$

where $B(u,v) = \Gamma(u)\Gamma(v)/\Gamma(u+v)$ denotes the Euler beta-function.

We are going to show that γ_l is uniformly bounded. By Stirling's formula we can write

$$B(l+\omega(k-l)+1, p/2+1)p^{-p/2}$$

$$\leq C^p \exp\{l+\omega(k-l))\ln(l+\omega(k-l)) - (l+\omega(k-l)+1+p/2)\ln(l+\omega(k-l)+1+p/2)\}$$

$$\leq C^p \exp\{l+\omega(k-l))\ln(l+\omega(k-l)) - (l+\omega(k-l)+1+p/2)\ln(l+\omega(k-l))\} \leq C^p k^{-1-p/2}$$

(let us recall that the constant C can be different in different places), and similarly

$$B(l+\omega(k-l+1),(p+1)\omega)p^{-p} \leq C^p p^{-(1-\omega)} k^{-\omega(1+p)},$$

$$B(l+\omega(k-l)+3/2,(p+1)/2)p^{-p/2} \leq C^p k^{-(1+p)/2},$$

$$lB(l+\omega(k-l)+1/2,(p+3)/2)p^{-p/2} \leq lC^p k^{-(3+p)/2}.$$

Thus, the first and the fourth term in square brackets in (4.12) can be estimated by the third one and we obtain

$$\gamma_l \leq \sum_{p=0}^{k-l} C^p \left[\frac{k+1-l}{k-p-l}\right]^{\omega/2} (p^{-p(1-\omega)} + k^{-(1+p)(1/2-\omega)}).$$

The sum here can be divided into the two sums in such a way that in the first sum the parameter p ranges from 0 to $(k-l)/2$ and in the second sum it ranges from $(k-l)/2$ to $k-l$. In the first sum $(k+1-l)/(k+1-l-p) \leq 2$, and therefore this first sum is clearly bounded. To simplify the second sum we denote $m = (k-l)/2$ and change the index of summation p to $q = p-m$. This yields the estimate

$$\gamma_l \leq C + \sum_{q=0}^{m} C^{m+q} m^{\omega/2}((m+q)^{-(m+q)(1-\omega)} + k^{-(1+m+q)(1/2-\omega)}).$$

The term $m^{\omega/2}$ here can be included in the multiplier C^{m+q} (by changing C appropriately). Next, the contribution of the second term in the bracket is bounded, because C/k is small for large k and the corresponding sum can be estimated by a convergent geometric progression. Hence,

$$\gamma_l \leq C + \sum_{q=0}^{\infty} C^{m+q}(m+q)^{-(m+q)(1-\omega)}.$$

Taking q_0 such that $q_0^{1-\omega} > C$ we get the estimate

$$\gamma_l \le C + \sum_{q=0}^{q_0} C^{m+q}(m+q)^{-(m+q)(1-\omega)} + \sum_{q=q_0}^{\infty} (C(m+q_0)^{-(1-\omega)})^{m+q}.$$

The second sum here is again bounded as a geometric progression and we have

$$\gamma_l \le C + q_0 C^{q_0} C^m m^{-m(1-\omega)},$$

which is bounded (uniformly in m). Hence γ_l is bounded, and from (4.8), (4.11) one obtains the required estimate for $\tilde{I}^D_{k+1}$, which completes the proof of the lemma.

Proof of Proposition 1. From Lemma 2 it follows that uniformly for $t \le t_0$

$$\sum_{k=0}^{\infty} \tilde{I}^D_k(t,x,y) \le G^{|D|^2/\epsilon}_{free}(t,x-y) \sum_{l=0}^{\infty} \frac{B^l |x-y|^l}{l!} \sum_{k=l}^{\infty} \frac{B^{k-l}(t^\omega)^{k-l}}{[(k-l)!]^\omega}$$

$$= G^{|D|^2/\epsilon}_{free}(t,x-y) \sum_{l=0}^{\infty} \frac{B^l |x-y|^l}{l!} f(t^\omega),$$

where

$$f(t^\omega) = \sum_{p=0}^{\infty} \frac{B^p (t^\omega)^p}{[p!]^\omega}$$

is an entire function of t^ω. This clearly implies the estimate (2.7) for the sum (2.4). In case $A = 0$ all calculations are essentially simplified and instead of estimate (4.3) one obtains the estimate

$$\tilde{I}_k(t,x,y) \le (Bt^\omega)^k/(k!)^\omega,$$

and therefore the multiplier $\exp\{B|x-y|\}$ does not appear in (2.7). Next, in case of real $D = \epsilon$, one has $G^{|D|^2/\epsilon}_{free} = G^{\epsilon}_{free}$, and the estimates of Lemma 2 implies (2.9). At last, straightforward calculations show that the integral operators with the integral kernels $G^{|D|^2/\epsilon}_{free}(t,x-y)\exp\{B|x-y|\}$ are uniformly bounded, and therefore estimate (2.7) implies the required statement about the boundedness of the operators (2.8).

Let us turn to the proof of Proposition 2.

Notice first that for an arbitrary differentiable function g one has

$$\int g(y)(W_a - W)(dy) = \int\int (g(y-z) - g(y)) W(dy) \phi_a(z)\, dz,$$

and therefore

$$|\int g(y)(W_a - W)(dy)| \le a \int \sup_{z \in B_a(y)} |g'(z)||W|(dy). \tag{4.13}$$

The same estimate holds for the vector-function A, i.e.

$$|\int g(y)(A_a - A)\, dy| \le a \int \sup_{z \in B_a(y)} |g'(z)||A| dy. \tag{4.14}$$

Let us denote by $I_k^{D,a}$ the integrals (2.5) where all W and A are replaced by their approximations W_a and A_a. Then clearly

$$|I_k^{D,a}(t,x,y) - I_k^D(t,x,y)| \leq \sum_{p=1}^k I_{k,p}^{D,a}(t,x,y), \tag{4.15}$$

where

$$I_{k,k}^{D,a}(t,x,y) = -D\int_0^t \int_{\mathbf{R}^d} I_{k-1}^D(t-s,x,\xi)$$
$$\times((W_a - W)(d\xi) + 2i\frac{(A_a - A), \xi - y)}{Ds}\, d\xi) G_{free}^D(s, \xi - y)\, ds \tag{4.16}$$

and for $k > p$ the integrals $I_{k,p}^{D,a}$ are defined inductively by the formulas

$$I_{k,p}^{D,a}(t,x,y) = -D\int_0^t \int_{\mathbf{R}^d} I_{k-1}^{D,a}(t-s,x,\xi)(W_a(d\xi) + 2i\frac{(A_a, \xi - y)}{Ds}\, d\xi) G_{free}^D(s, \xi - y)\, ds. \tag{4.17}$$

In order to prove Proposition 2, we need to estimate the l.h.s. of (4.15), and for this we need the estimates for the integrals $I_{k,p}^{D,a}$.

Lemma 3. *One has*

$$|I_{k,k}^{D,a}(t,x,y)| \leq \left(\frac{ak}{t}\right)^\nu \sum_{l=0}^k \frac{\tilde{B}^k}{l![(k-l)!]^\omega} |x-y|^l (t^\omega)^{k-l} G_{free}^{2|D|^2/\epsilon}(t, x-y), \tag{4.18}$$

where ν is any number from the interval $(0,\omega)$ and $\tilde{B}$ is a certain constant (depending on the choice of ν).

Remark. This is a very rough estimate, because we have $G_{free}^{2|D|^2/\epsilon}$ here as compared with $G_{free}^{|D|^2/\epsilon}$ in (4.3).

Proof. Decompose the integral in (4.16) into the sum $J_1+J_2+J_3$ of the three untegrals with the domains $D_1 = \{s : s < a\}$, $D_2 = \{s : (t-s) < a\}$, $D_3 = \{s : a \leq \min(s, t-s)\}$. In D_1 we use the estimate $1 \leq s^{-\nu}a^\nu$. Looking through the proof of Lemma 2 we see that the additional multiplier $s^{-\nu}$ in the integrand will give the additional multiplier $(k/t)^\nu$ in the estimate. The same happens with the integral over D_2, where we use the estimate $1 \leq (t-s)^{-\nu}a^\nu$ (the estimate is even better in this case, because it does not contain k^ν, which is not important now, but will be important in the proof of Lemma 4). Thus the estimate for $J_1 + J_2$ is the same as for I_k^D but with the additional multiplier $(ak/t)^\nu$ as required. It remains to estimate the integral J_3. To this end we shall use formulas (4.13), (4.14). The differentiation in ξ of the integrand in (4.16) gives two terms $J_3^1 + J_3^2$ that come from the differentiation of I_{k-1}^D and from the differentiation of the terms on the right of $(W_a - W)$ respectively. Let us consider only the contribution J_3^2 of the second term (the first is considered similarly). We have

$$J_3^2 \leq aC\int_{D_3}\int_{\mathbf{R}^d} |I_{k-1}^D(t-s,x,\xi)|$$

$$\times \left[(|W|(d\xi) + \frac{|\xi - y|}{s} \, d\xi)(\frac{|\xi - y|}{s} + a) + \frac{1}{s} \, d\xi \right] |G^D_{free}(s, \xi - y)| \, ds.$$

Using the definition of the domain D_3 we can estimate here $a/s \leq (a/s)^\nu$. Then again going through all steps of the proof of Lemma 2 we obtain here the same estimate as for $|I^D_k|$ but with the multiplier $(ak/t)^\nu(1+|x-y|)$. Clearly one can get rid of $|x-y|$ replacing $G^{|D|^2/\epsilon}_{free}$ by $G^{2|D|^2/\epsilon}_{free}$ in the estimate of I_k, which leads to required estimate (4.18).

Lemma 4. *For $k \geq p$,*

$$|I^{D,a}_{k,p}(t, x, y)| \leq \left(\frac{ap}{t}\right)^\nu \sum_{l=0}^{k} \frac{\tilde{B}^k}{l![(k-l)!]^\omega} |x - y|^l (t^\omega)^{k-l} G^{2|D|^2/\epsilon}_{free}(t, x - y), \tag{4.19}$$

Proof. By formula (4.17) and again by the inspection of the proof of Lemma 2 one sees that the additional multiplier $t^{-\nu}$ (by which essentially $I^D_{k,k}$ differs from I^D_k) will just reproduce itself in each application of formula (4.17).

Proof of Proposition 2.

From (4.15) and Lemma 4 it follows that

$$|G^D_a(t, x, y) - G^D(t, x, y)| \leq K \left(\frac{a}{t}\right)^\nu G^{2|D|^2/\epsilon}_{free}(t, x, y) \exp\{\tilde{B}|x - y|\}, \tag{4.20}$$

which clearly implies Proposition 2.

5. A remark on the notion of dimensionality of a measure.

We discuss here the connection of our assumption C3) on the measure V with the similar assumption from [AFHKL]. Let μ be a finite (positive) Radon measure in $\mathbf{R}^d$. In [AFHKL], a number N was called admissible for μ if there is a constant $C(N)$ such that

$$\mu(B_r(x)) \leq C(N) r^N \tag{5.1}$$

for all r and almost all x with respect to μ. The least upper bound $dim(\mu)$ of all admissible numbers was called the dimensionality of μ.

It is easy to see that the restriction that (5.1) holds only for almost all x can be omitted.

Proposition 3. *Let (5.1) holds for almost all x (with respect to μ). Then (5.1) holds for all $x \in \mathbf{R}^d$.*

Proof. First suppose that x is an arbitrary point from the support of μ. Then there is a sequence of points x_n such that $\lim_{n\to\infty} x_n = x$ and for all x_n the inequality (5.1) holds. For any δ, $B_r(x) \subset B_{r+\delta}(x_n)$ for sufficiently large n, and thus (5.1) holds for x with $r + \delta$ instead of r on the r.h.s. But since δ is arbitrary small, it holds for $\delta = 0$ as well. Now, to prove (5.1) for all x (not necessarily from the support of μ) one needs to notice that for any d there is an integer $K(d)$ which is the upper bound for the cardinality of any finite subset $S = \{x_1, ..., x_n\}$ of B_r with the property that the distance between any two points x_i, x_j is not less than r. Therefore, the intersection of the ball $B_r(x)$ and the support of

μ can be covered by $K(d)$ balls centered at the points from the support of μ. Therefore, (5.1) holds for all x (with $C(N)$ replaced by $C(N)K(d)$).

Now, generalising the notion of dimensionality of a finite measure, let us say that a measure μ (not necessary finite now) has a local dimensionality $dim_{loc}(\mu)$, if this is the least upper bound of the numbers N such that (5.1) holds for $r \leq 1$ (or, equivalently, for $r \leq r_0$ with any fixed r_0) and for all x. Thus our assumption C3) in case $d > 1$ means that the measures V and $div\, A$ both have local dimensionalities greater than $d-2$.

If V is a positive finite measure of dimensionality greater that $d-2$ supported in a Lebesgue measure zero subset of $\mathbf{R}^d$, the self-adjointness of (1.3) in the sense of the closability of the corresponding form was proved in [AFHKL] (see Proposition 12 there) for small coupling constant, i.e. V replaced by λV with λ small enough.

6. Generalisations.

Our main assumptions C1)-C3) on A and V can be generalised in various direction without changing the main results of the paper essentially. Let us indicate here some possibilities. The details will be published elsewhere.

1) Instead of C1) one can assume that the function A belongs to the uniform local class L^p with $p > d$, i.e.

$$\int_{B_1(x)} |A|^p(y)\, dy \leq C$$

with a certain constant C uniformly for all x. (In particular, this generalises the assumptions of the well known criterion of self-adjointness from [LS] in case of dimensions $d \leq 3$.)

In this case, instead of the main estimate (2.7) for the corresponding series (2.4) one can prove that

$$|G^D(t,x,y)| \leq K G_{free}^{|D|^2/\epsilon}(t,x,y)\} \exp\{\frac{B|x-y|}{t^\nu}\}$$

with a certain $\nu < 1/2$, and all other results can be then generalised accordingly to this modification.

2) One can easily include in the consideration the case of harmonic oscillator and the presence of a constant magnetic field. More precisely, one can consider instead of the operator (1.1) the operator

$$H = \left(-i\frac{\partial}{\partial x} + A(x) + Mx\right)^2 + V(x) + (Gx,x),$$

where M and G are certain matrices such that G is non-negative, and A and V satisfy the conditions C1)-C3). Under these assumptions all considerations follow the same lines, but instead of the explicit formula (2.6) for G^D_{free} one uses the explicit expression for the heat kernel of the harmonic oscillator in a constant magnetic field. This generalisation includes, in particular, an interesting exactly solvable model describing the Aharonov-Bohm effect with δ-type interaction, which is considered in detail in [DS]. For vanishing A and M such a generalisation of our construction is given in [K4].

3) Finally, instead of adding to operator (1.2) the quadratic terms, as in 2) above, one can add to it any locally analytic (or, more generally, smooth) potential W. A path integral representation for the corresponding Schrödinger equation can be obtained by generalising the approach from [K3] developed there for analytic potentials without singular perturbations. The corresponding measure on path space will be concentrated on the set of locally classical paths defined by the Hamiltonian $p^2 + W(x)$.

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