The Brownian Web

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Abstract

Arratia, and later Tóth and Werner, constructed random processes that formally correspond to coalescing one-dimensional Brownian motions starting from every space-time point. We extend their work by constructing and characterizing what we call the *Brownian Web* as a random variable taking values in an appropriate (metric) space whose points are (compact) sets of paths. This leads to general convergence criteria and, in particular, to convergence in distribution of coalescing random walks in the scaling limit to the Brownian Web.

Introduction

Construct random paths in the plane, as follows. Take the square lattice consisting of all points $(\sqrt{2}m, \sqrt{2}n)$ with m, n integers and rotate it by 45 degrees resulting in all points (i, j) with i, j integers and $i + j$ even. Imagine a walker at spatial location i at time j deciding to move right or left at unit speed between times j and $j + 1$ if the outcome of a fair coin toss is heads ($\Delta_{i,j} = +1$) or tails ($\Delta_{i,j} = -1$), with the coin tosses independent for different space-time points (i, j). Figure 1 depicts a simulation of the resulting paths.

The path of a walker starting from y_0 at time s_0 is the graph of a simple symmetric one-dimensional random walk, $Y_{y_0,s_0}(t)$. At integer times, $Y_{y_0,s_0}(t)$ is the solution of the simple stochastic difference equation,

$$Y(j+1) - Y(j) = \Delta_{Y(j),j}, \quad Y(s_0) = y_0. \tag{0.1}$$

Note that the paths of distinct walkers starting from different (y_0, s_0)'s are automatically *coalescing* — i.e., they are independent of each other until they coalesce (i.e., become identical) upon meeting at some space-time point.

After rescaling to spatial steps of size δ and time steps of size δ^2, a single rescaled random walk (say, starting from 0 at time 0) $Y^{(\delta)}_{0,0}(t) = \delta Y_{0,0}(\delta^{-2}t)$ converges as $\delta \to 0$ to a standard

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Brownian motion $B(t)$. More precisely, by the Donsker invariance principle [1], the distribution of $Y_{0,0}^{(\delta)}$ on the space of continuous paths converges weakly as $\delta \to 0$ to standard Wiener measure.

The invariance principle is also valid for continuous time random walks, where the move from i to $i \pm 1$ takes an exponentially distributed time (see the discussion following Remark 2.2 below for more details). In continuous time, coalescing random walks are at the heart of Harris's graphical representation of the (one-dimensional) voter model [2] and their scaling limits arise naturally in the physical context of (one-dimensional) aging [3]. Like for a single random walk, finitely many rescaled coalescing walks in discrete or continuous time (with rescaled space-time starting points) converge in distribution to finitely many coalescing Brownian motions. In this paper, we present results concerning the convergence in distribution of the collection of the rescaled coalescing walks from *all* the starting points; detailed proofs will be published elsewhere [4].

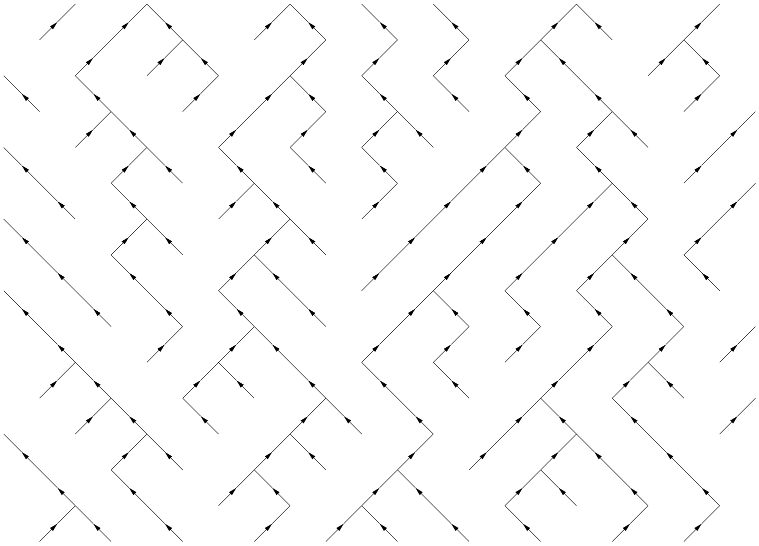

Figure 1: Coalescing random walks in discrete time; the horizontal coordinate is space and the vertical one is time.

Our results come in two parts:

(1) characterization (and construction) of the limiting object, which we call the standard *Brownian Web (BW)*, and

(2) general convergence criteria, which are then applied to coalescing random walks.

A key ingredient of the characterization and construction (see Theorem 1.1) is the choice of a space for the Brownian web; this is the BW analogue of the space of continuous paths for Brownian motion. The convergence criteria and application (see Theorems 2.1 and 2.3 below) are the BW analogues of Donsker's invariance principle. Like Brownian motion itself, we expect that the Brownian web and its variants (see, e.g., Remark 1.4) will be quite ubiquitous as scaling limits, well beyond the context of coalescing random walks and our sufficient conditions for convergence.

Much of the construction of the Brownian web was already done in the groundbreaking work of Arratia [5, 6] and then in work of Tóth and Werner [7] (see also [8]). They all recognized that in the limit $\delta \to 0$ there would be (nondeterministic) space-time points (x, t) starting from which there are multiple limit paths and they provided various conventions (e.g., semicontinuity in x) to avoid such multiplicity. Our main contribution vis-a-vis construction is to accept the intrinsic nonuniqueness by choosing an appropriate metric space in which the BW takes its values. Roughly speaking, instead of using some convention to obtain a process that is a *single-valued* mapping from each space-time starting point to a single path from that starting point, we allow *multi-valued* mappings; more accurately, our BW value is the collection of *all* paths from all starting points. This choice of space is very much in the spirit of earlier work [9, 10, 11] on spatial scaling limits of critical percolation models and spanning trees, but modified for our particular space-time setting.

1 Brownian Web: Characterization

We begin by defining three metric spaces: $(\bar{\mathbb{R}}^2, \rho)$, (Π, d) and $(\mathcal{H}, d_{\mathcal{H}})$. The elements of the three spaces are respectively: points in space-time, paths with specified starting points in space-time and collections of paths with specified starting points. The BW will be an $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued random variable, where $\mathcal{F}_{\mathcal{H}}$ is the Borel σ-field associated to the metric $d_{\mathcal{H}}$.

$(\bar{\mathbb{R}}^2, \rho)$ is the completion (or compactification) of $\mathbb{R}^2$ under the metric ρ, where

$$\rho((x_1, t_1), (x_2, t_2)) = \left| \frac{\tanh(x_1)}{1 + |t_1|} - \frac{\tanh(x_2)}{1 + |t_2|} \right| \vee |\tanh(t_1) - \tanh(t_2)|. \tag{1.1}$$

$\bar{\mathbb{R}}^2$ may be thought as the set of (x, t) in $[-\infty, \infty] \times [-\infty, \infty]$ with all points of the form $(x, -\infty)$ identified (and similarly for (x, ∞)). More precisely, it is the image of $[-\infty, \infty] \times [-\infty, \infty]$ under the mapping

$$(x, t) \rightsquigarrow (\Phi(x, t), \Psi(t)) \equiv \left(\frac{\tanh(x)}{1 + |t|}, \tanh(t) \right). \tag{1.2}$$

For $t_0 \in [-\infty, \infty]$, let $C[t_0]$ denote the set of functions f from $[t_0, \infty]$ to $[-\infty, \infty]$ such that $\Phi(f(t), t)$ is continuous. Then define

$$\Pi = \bigcup_{t_0 \in [-\infty, \infty]} C[t_0] \times \{t_0\}, \tag{1.3}$$

where $(f, t_0) \in \Pi$ then represents a path in $\bar{\mathbb{R}}^2$ starting at $(f(t_0), t_0)$. For$(f, t_0) \in \Pi$, we denote by $\hat{f}$ the function that extends f to all $[-\infty, \infty]$ by setting it equal to $f(t_0)$ for $t < t_0$. Then we take

$$d((f_1, t_1), (f_2, t_2)) = (\sup_t |\Phi(\hat{f}_1(t), t) - \Phi(\hat{f}_2(t), t)|) \vee |\Psi(t_1) - \Psi(t_2)|. \tag{1.4}$$

(Π, d) is a complete separable metric space.

Let now $\mathcal{H}$ denote the set of compact subsets of (Π, d), with $d_{\mathcal{H}}$ the induced Hausdorff metric, i.e.,

$$d_{\mathcal{H}}(K_1, K_2) = \sup_{g_1 \in K_1} \inf_{g_2 \in K_2} d(g_1, g_2) \vee \sup_{g_2 \in K_2} \inf_{g_1 \in K_1} d(g_1, g_2). \tag{1.5}$$

$(\mathcal{H}, d_{\mathcal{H}})$ is also a complete separable metric space.

Before stating our characterization theorem for the Brownian web, we need some definitions. For an $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued random variable $\bar{W}$ (or its distribution μ), we define the *finite-dimensional distributions* of $\bar{W}$ as the induced probability measures $\mu_{(x_1,t_1;\dots;x_n,t_n)}$ on the subsets of paths starting from any finite deterministic set of points $(x_1, t_1), \dots, (x_n, t_n)$ in $\mathbb{R}^2$. There are several ways in which the Brownian web can be characterized; they differ from each other primarily in the type of extra condition required beyond the finite-dimensional distributions. The characterization of the next theorem, or more precisely a variant discussed later in Remark 1.3, is the one most directly suited to the convergence results of Section 2; an alternative characterization in which the extra condition is a type of Doob separability property (see, e.g., Chap. 3 of [12]) is discussed in Remark 1.2. For the next theorem, we also define, for $t \geq 0$ and $a \leq b$, the ($\{0, 1, \dots, \infty\}$-valued) random variable $\eta(t_0, t; a, b)$ as the number of *distinct* points in $\mathbb{R} \times \{t_0 + t\}$ that are touched by paths in $\bar{W}$ which also touch some point in $[a, b] \times \{t_0\}$.

Theorem 1.1 *There is an $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued random variable $\bar{W}$ whose distribution μ is uniquely determined by the following two properties:*

(i) its finite-dimensional distributions are those of coalescing Brownian motions (with unit diffusion constant), and

(ii) for $-\infty < t_0 < \infty$, $0 < t < \infty$, $-\infty < a \leq b < \infty$,

$$E_{\mu}(\eta(t_0, t; a, b)) = 1 + \frac{b - a}{\sqrt{\pi t}}. \tag{1.6}$$

Remark 1.2 *Implicit in condition (i) of the theorem is that starting from any deterministic point, there is almost surely only a single path in $\bar{W}$. Condition (ii) can be replaced by the separability property that there is a deterministic dense countable set $\mathcal{D}$ of space-time starting points, such that almost surely, $\bar{W}$ is the closure in (Π, d) of the set of paths starting from the points of $\mathcal{D}$. It should be noted that $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued random variables, satisfying condition (i) but not condition (ii) or its separability alternative, can occur naturally. Such a process (closely related to the "Double Brownian Web" of Remark 1.4 below), where the counting variable η is infinite with strictly positive probability, will be studied elsewhere and shown to arise as the scaling limit of stochastic flows, extending earlier work of Piterbarg [13].*

Sketch of Proof of Theorem 1.1. The construction of the Brownian web (i.e., the existence of such a $\bar{W}$) begins as in [6, 7] with the construction of a set W of coalescing Brownian paths starting from a deterministic dense countable set $\mathcal{D}$ of space-time starting points. This skeleton $W = \{\tilde{W}_1, \tilde{W}_2, \dots,\}$ is a random subset of Π that is constructed by deterministically ordering the points of $\mathcal{D}$ as $(x_1, t_1), (x_2, t_2), \dots$, then defining $W_j = (x_j + B_j(t - t_j), t_j) \in \Pi$ where the B_j's are independent standard Brownian motions, and finally using the ordering to inductively define $\tilde{W}_j \in \Pi$ by following W_j until it meets some $\tilde{W}_k$ with $k < j$ after which point it follows $\tilde{W}_k$.

The next several steps of the construction are to show that the closure $\bar{W}$ in (Π, d) of this BW-skeleton is compact, that the distribution of $\bar{W}$ does not depend on the choice of $\mathcal{D}$ or its ordering, and that $\bar{W}$ satisfies (i) and (ii) of Theorem 1.1 above. The compactness can be proved in a number of ways; one of these is to verify a condition, as in (2.1) below, but with μ_δ replaced by the distribution of $\{\tilde{W}_1, \ldots, \tilde{W}_m\}$ and the sup over δ replaced by a sup over m (and then argue as at the beginning of the proof of Theorem 2.1 below, eventually invoking the Arzelà-Ascoli theorem). To verify the said condition, one argues as in the last two paragraphs of the proof of Theorem 2.3 below. The argument actually involves only a single bound like (2.6), which is obtained in the same way as in the proof of Theorem 2.3. We remark that by considering the quantity $\tilde{g}(t, u)$, as in (2.1), but with $u = t^\xi$, one can show not just compactness of $\bar{W}$, but also Hölder continuity with any exponent $\xi < 1/2$ for *all* the paths of $\bar{W}$.

The lack of dependence of the distribution on the choice of $\mathcal{D}$ or its ordering follows fairly directly after verifying property (i) for $\bar{W}$. Property (i) itself follows by a trapping argument about a deterministic point $(\bar{x}, \bar{t})$ (and similarly for finitely many points) and any sequence $(\bar{x}_i, \bar{t}_i) = (x_{j(i)}, t_{j(i)})$ from $\mathcal{D}$ converging to $(\bar{x}, \bar{t})$ as $i \to \infty$: that (even if $j(i)$ is nondeterministic and regardless of whether $t_{j(i)} > \bar{t}$ or $t_{j(i)} \leq \bar{t}$) for large i, $\tilde{W}_{j(i)}$ is (with probability very close to one) trapped between $\tilde{W}_k$ and $\tilde{W}_{k'}$ for deterministic k, k' with $x_k < \bar{x} < x'_k$, $t_k < \bar{t}$, $t_{k'} < \bar{t}$, x_k and x'_k close to $\bar{x}$ and $t_k, t_{k'}$ even closer to $\bar{t}$ so that $\tilde{W}_{j(i)}$ (with probability very close to one) quickly coalesces with both $\tilde{W}_k$ and $\tilde{W}_{k'}$; thus $\tilde{W}_{j(i)}$ converges almost surely as $i \to \infty$ to a path (independent of the specific sequence $(\bar{x}_i, \bar{t}_i)$) that is distributed as a Brownian motion starting from $(\bar{x}, \bar{t})$.

Verifying property (ii) is somewhat indirect. First, one shows that the random variable η for our constructed $\bar{W}$ is almost surely finite with a finite mean which, by the translation invariance in space and time that results from the lack of dependence on $\mathcal{D}$, must be of the form $\Lambda(b - a, t)$. Second, the specific evaluation of Λ as given on the righthand side of (1.6) is carried out. As the explicit expression for Λ will not actually be used in our convergence results of the next section (see Remark 1.3), we can and will use those convergence results in the evaluation of Λ.

The first part of the verification of (ii) is a consequence of an inequality,

$$P(\eta(t_0, t; a, b) \geq k) \leq [P(\eta(t_0, t; a, b) \geq 2)]^{k-1} = [\Theta(b - a, t)]^{k-1}, \tag{1.7}$$

where $\Theta(b - a, t)$ is the probability that two independent Brownian motions starting at a distance $b - a$ apart at time zero will have met by time t (which itself can be expressed in terms of a single Brownian motion). The inequality in (1.7) is first derived for finite subsets $\{\tilde{W}_1, \tilde{W}_2, \ldots, \tilde{W}_m\}$ of the skeleton, and thus for the whole skeleton. For the whole skeleton and its closure $\bar{W}$, the equality in (1.7) is seen to be valid by choosing the countable set $\mathcal{D}$ so that its first two points are (a, t_0) and (b, t_0). Then the inequality is extended to $\bar{W}$ from the skeleton by a limit/approximation argument which uses that $\{K \in \mathcal{H} : \tilde{\eta}(K) \geq k\}$ is open in $(\mathcal{H}, d_{\mathcal{H}})$, where $\tilde{\eta}$ is the modification of η that counts points in $\mathbb{R} \times \{t_0 + t\}$ touched by paths in K which touch $(a - \tilde{\varepsilon}_1, b + \tilde{\varepsilon}_1) \times \{t_0 + \tilde{\varepsilon}_2\}$ and start earlier than $t_0 + \tilde{\varepsilon}_2$.

The second part of the verification of (ii), in which Λ is explicitly evaluated, is a consequence of all the following: a result of Bramson and Griffeath [14] on the large-time asymptotics of mean interparticle distance in coalescing random walks, the conversion of that result by standard arguments to asymptotics for the mean of the rescaled random walk version of the counting variable η, convergence of the distribution of η in the scaling limit (see Remark 2.2), and finally the analogue of (1.7) for coalescing walks (see (2.5)) which implies uniform integrability of η as $\delta \to 0$ and hence convergence of the mean of η.

It remains to show that conditions (i) and (ii) for a measure μ' on $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$ together imply that μ' equals the distribution μ of the constructed Brownian web $\bar{W}$. Let us denote by X' the $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued random variable distributed by μ' and by η' the counting random variable appearing in condition (ii) for μ'. Choose some deterministic dense countable subset $\mathcal{D}$ and consider the countable collection W^* of paths of X' starting from $\mathcal{D}$. By condition (i), W^* is equidistributed with our constructed Brownian skeleton W (based on the same $\mathcal{D}$) and hence the closure $\bar{W}^*$ of W^* in (Π, d) is a subset of X' that is equidistributed with our constructed Brownian web $\bar{W}$. To complete the proof, we will use condition (ii) to show that $X' \setminus \bar{W}^*$ is almost surely empty by using the fact that the counting variable η^* for $\bar{W}^*$ already satisifies condition (ii) since $\bar{W}^*$ is distributed as a Brownian web. If $X' \setminus \bar{W}^*$ were nonempty (with strictly positive probability), then there would have to be some rational t_0, t, a, b for which $\eta' > \eta^*$. But then

$$P(\eta'(t_0, t; a, b) > \eta^*(t_0, t; a, b)) \; > \; 0 \tag{1.8}$$

for some rational t_0, t, a, b and so condition (ii) for η' would *not* be valid for that t_0, t, a, b.

Remark 1.3 *The proof of Theorem 1.1 makes clear that the idea behind (i) and (ii) together implying uniqueness of the distribution is that (i) implies sufficiently many paths and (ii) implies no extraneous ones. Thus condition (i) can be weakened to the existence of a subset of paths distributed as the coalescing Brownian motions of the skeleton* W *(for any deterministic dense countable* $\mathcal{D}$*) and condition (ii) can also be modified, e.g., by replacing the equality in (1.6) by an inequality (*$\leq$*) and by replacing an (in)equality for the mean by one for the distribution. Similarly, in applying our characterization results to obtain convergence criteria as we do in Theorem 2.1, an explicit expression for the mean as given in the righthand side of (1.6) or an explicit expression for the distribution is not needed; i.e., to verify that an* X' *is equidistributed with our explicitly constructed Brownian web* $\bar{W}$*, condition (ii) for the* η' *of* X' *can be replaced by the condition that the distribution of* η' *equal (or only is stochastically dominated by) the distribution of the* η *of* $\bar{W}$*.*

Remark 1.4 *In the graphical representation of Harris for the one-dimensional voter model [2], coalescing random walks forward in time and coalescing dual random walks backward in time (with forward and backward walks not crossing each other) are constructed simultaneously (see, e.g., the discussion in [3]). The simultaneous construction of forward and (dual) backward Brownian motions was emphasized in [7, 8] and their approach and results can be applied to extend both our characterization and convergence results to the* Double Brownian Web *(DBW) which includes simultaneously the forward BW and its dual backward BW. We note that in the DBW, the* η *of (1.6) equals* $1 + \eta^{\mathrm{dual}}$*, where* η^{dual} *is the number of distinct points in* $[a, b] \times \{t_0\}$ *touched by backward paths which also touch* $\mathbb{R} \times \{t_0 + t\}$*.*

Remark 1.5 *As in [7], space-time points* (x, t) *can be characterized by the number of locally disjoint paths* m_{in} *(resp.,* m_{out}*) of the BW entering (resp., leaving) that point from earlier* *(resp., to later) times. The corresponding dual BW characterization has* $m_{\text{in}}^{\text{dual}} = m_{\text{out}} - 1$ *and* $m_{\text{out}}^{\text{dual}} = m_{\text{in}} + 1$. *Generic (e.g., deterministic) points have* $(m_{\text{in}}, m_{\text{out}}) = (0, 1)$. *Almost surely, there are nongeneric points of type* $(0,2), (0,3), (1,1), (1,2)$ *and* $(2,1)$ *but no others. We note that as in [7], ruling out points of higher type uses improvements of (1.7) for* $k > 2$. *Type* $(2,1)$ *(resp.,* $(0,3)$*) points are those where coalescing (resp., dual coalescing) occurs. Type* $(1,2)$ *points are particularly interesting in that the single incident path continues along exactly one of the two outward paths — with the choice determined intrinsically rather than by some convention.*

2 Convergence to the Brownian Web

Let X_δ be an $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued random variable indexed by $\delta \in (0,1]$, with distribution μ_δ. We present criteria sufficient to insure convergence in distribution as $\delta \to 0$ of X_δ to the Brownian web $\bar{W}$, in the setting where the X_δ's have coalescing paths; for simplicity, we will not present here more general criteria that do not require the coalescing property. We next introduce the various conditions on μ_δ which together will imply convergence.

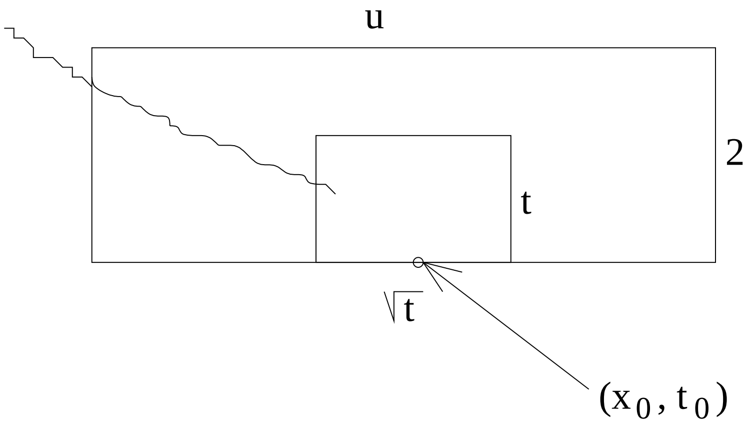

Figure 2: Schematic diagram of a path causing the unlikely event $A_{t,u}(x_0, t_0)$ to occur.

The first condition will guarantee tightness of the μ_δ's. Let $R(x_0, t_0; u, t)$ denote the rectangle $[x_0 - u/2, x_0 + u/2] \times [t_0, t_0 + t]$ in $\mathbb{R}^2$. We call $\{x_0 \pm u/2\} \times [t_0, t_0 + t]$ its right and left boundaries. For $t > 0, u > \sqrt{t}$, define $A_{t,u}(x_0, t_0)$ to be the event (in $\mathcal{F}_{\mathcal{H}}$) that K (in $\mathcal{H}$) contains a path touching both $R(x_0, t_0; \sqrt{t}, t)$ and (at a later time) the left or right boundary of the bigger rectangle $R(x_0, t_0; u, 2t)$; see Figure 2. Our tightness condition is

$$(T_1) \quad \tilde{g}(t,u) \equiv t^{-3/2} \sup_{\delta > 0} \sup_{x_0, t_0} \mu_\delta(A_{t,u}(x_0, t_0)) \to 0 \text{ as } t \to 0+ \text{ for fixed } u > 0 \,. \qquad (2.1)$$

Our second condition will guarantee a weakened version of (i) in Theorem 1.1 (see Remark 1.3) for any limit μ of μ_δ. Let $\mathcal{D}$ be any deterministic countable dense set of points in $\mathbb{R}^2$. The condition concerns the existence for each $\delta > 0$ and $y \in \mathcal{D}$ of measurable (on the probability space of X_δ) single-path valued random variables $\theta_\delta^y(\omega) \in X_\delta(\omega)$:

(I_1) There exist such $\theta_\delta^y \in X_\delta$ satisfying: for any deterministic $y_1, \ldots, y_m \in \mathcal{D}$, $\theta_\delta^{y_1}, \ldots, \theta_\delta^{y_m}$ converge in distribution as $\delta \to 0$ to coalescing Brownian motions (with unit diffusion constant) starting at $y_1, \ldots, y_m$.

Our next two conditions will together guarantee (when X_δ is coalescing) a version of (ii) in Theorem 1.1 (see Remark 1.3). For $-\infty < t_0 < \infty$ and $0 < t < \infty$,

$$(B_1) \qquad \limsup_{\delta \to 0} \sup_{a \in \mathbb{R}} \mu_\delta(\eta(t_0, t; a, a+\epsilon) \geq 2) \to 0 \text{ as } \epsilon \to 0+; \tag{2.2}$$

$$(B_2) \quad \epsilon^{-1} \limsup_{\delta \to 0} \sup_{a \in \mathbb{R}} \mu_\delta(\eta(t_0, t; a, a+\epsilon) \geq 3) \to 0 \text{ as } \epsilon \to 0+. \tag{2.3}$$

Theorem 2.1 *Suppose* X_δ *for* $0 < \delta \leq 1$ *are* $(\mathcal{H}, \mathcal{F}_\mathcal{H})$*-valued random variables with coalescing paths. If* T_1, I_1, B_1 *and* B_2 *all hold, then the distributions* μ_δ *of* X_δ *converge weakly as* $\delta \to 0$ *to the distribution* $\mu_{\bar{\mathcal{W}}}$ *of the standard Brownian web.*

Sketch of Proof of Theorem 2.1. We first explain why T_1 implies tightness. Let $g_\delta(t, u)$ denote the sup over x_0, t_0 of $\mu_\delta(A_{t,u})$ as in (2.1). This represents an upper bound on the μ_δ-probability that there is some path (f, t^*) passing through some point $(x', t') = (f(t'), t')$ in the deterministic $\sqrt{t} \times t$ rectangle $R(x_0, t_0; \sqrt{t}, t)$ located at any (x_0, t_0), such that for some $t'' \in [t', t'+t]$ the spatial increment $|f(t'') - f(t')| \geq u$ even though the time increment is $\leq t$. Now taking a large $L \times T$ space-time rectangle centered at the origin and covering it with $O(LT/t^{3/2})$ $\sqrt{t} \times t$ small rectangles, we see that $LT\,\tilde{g}(t, u)$ represents an upper bound on the μ_δ-probability (for any δ) that some path has $|f(t'') - f(t')| \geq u$ while $t'' - t' \leq t$ with $(f(t'), t')$ *anywhere* in the large rectangle. We next choose sequences $u_n \to 0$, $L_n \to \infty$, $T_n \to \infty$ and then $t_n \to 0$ sufficiently rapidly that $L_n T_n \tilde{g}(t_n, u_n)$ is summable. Now moving to the compactified space-time $\bar{\mathbb{R}}^2$ (and using the notation of (1.2)), it follows that there are sequences $\phi_n, \psi_n \to 0$ so that for large enough n, with μ_δ-probability close to one (for any δ), $|\Psi(t'') - \Psi(t')| \leq \psi_n$ implies $\Phi(f(t''), t'') - \Phi(f(t'), t')| \leq \phi_n$. This equicontinuity with probability close to one (for any δ) combined with a version of the Arzelà-Ascoli theorem leads to the paths, as elements of (Π, d), belonging to a compact subset $K_{\tilde{\varepsilon}}$ of (Π, d) with μ_δ-probability $\geq 1 - \tilde{\varepsilon}$ (for any δ), which implies tightness because the collection of compact subsets of $K_{\tilde{\varepsilon}}$ is itself a compact set in $(\mathcal{H}, d_\mathcal{H})$.

Tightness implies that every subsequence of μ_δ has a sub-subsequence converging weakly to some μ. To complete the proof, we need to show that any such μ equals $\mu_{\bar{\mathcal{W}}}$. To do this, we will show that μ satisifies the two characterization properties of Theorem 1.1, as modified in Remark 1.3. Combining condition I_1 with convergence in distribution (along a subsequence) of X_δ to some X distributed by μ, we see that we can realize X on some probability space so that it contains paths starting from the points of $\mathcal{D}$ distributed as coalescing Brownian motions. This is just the desired (weakened version of) property(i) in Theorem 1.1. Indeed, this shows that X contains an X' that has the Brownian web distribution.

To complete the proof we use conditions B_1 and B_2. Note first that by limit/approximation arguments these two conditions (without the $\limsup$ over δ) are valid with μ replacing μ_δ. For fixed t_0, t, a, b, we now consider $M+1$ equally spaced points, $z_j = (a + j(b-a)/M, t_0)$ for $j = 0, \ldots, M$. For the random X we will denote the counting variable $\eta(t_0, t; a, b)$ by η, and the corresponding variable for X' by η'. We also want to count the number of points on $\mathbb{R} \times \{t_0 + t\}$ that are touched by paths of X that also touch $\{z_0, \ldots, z_M\}$ and we will denote these variables for X and X' by η_M and η'_M. Of course, $\eta \geq \eta_M$ and $\eta \geq \eta'_M$. By condition B_1 (for μ) applied to small intervals about each of the z_j's, it follows that $\eta_M = \eta'_M$ almost surely. Applying condition B_2 (for μ) to the M spatial intervals $[z_{j-1}, z_j]$ of length $\epsilon = (b-a)/M$, and using the coalescing (or at least non-crossing) property of X that it inherits from the X_δ's, it follows that

$$P(\eta > \eta'_M) \ = \ P(\eta > \eta_M) \to 0 \text{ as } M \to \infty. \tag{2.4}$$

Thus $P(\eta > \eta') = 0$ so that the distribution of η' is stochastically dominated by (and hence equal to) the distribution of η. This gives the desired (modified version of) property(ii) in Theorem 1.1 and completes the proof.

Remark 2.2 *The arguments used in the proof of Theorem 2.1 also show that under the same four conditions T_1, I_1, B_1 and B_2, the counting random variables $\eta_\delta(t_0, t; a, b)$ for X_δ converge in distribution to the Brownian web counting variable $\eta_{\bar{W}}(t_0, t; a, b)$. If one also has uniform integrability as $\delta \to 0$, then the means converge to the mean of $\eta_{\bar{W}}$.*

To apply Theorem 2.1 to random walks, we begin by precisely defining Y (resp., $\tilde{Y}$), the set of all discrete (resp., continuous) time coalescing random walks on $\mathbb{Z}$. The sets of rescaled walks, $Y^{(\delta)}$ and $\tilde{Y}^{(\delta)}$, are then obtained by the usual rescaling of space by δ and time by δ^2. The (main) paths of Y are the discrete-time random walks Y_{y_0, s_0}, as described in the Introduction and shown in Figure 1, with $(y_0, s_0) = (i_0, j_0) \in \mathbb{Z} \times \mathbb{Z}$ arbitrary except that $i_0 + j_0$ must be even. Each random walk path goes from (i, j) to $(i \pm 1, j + 1)$ linearly. In addition to these, we add some boundary paths so that Y will be a compact subset of Π. These are all the paths of the form (f, s_0) with $s_0 \in \mathbb{Z} \cup \{-\infty, \infty\}$ and $f \equiv \infty$ or $f \equiv -\infty$. Note that for $s_0 = -\infty$ there are two different paths starting from the single point at $s_0 = -\infty$ in $\bar{\mathbb{R}}^2$.

The continuous time $\tilde{Y}$ can be defined similarly, except that here y_0 is any $i_0 \in \mathbb{Z}$ and s_0 is arbitrary in $\mathbb{R}$. Continuous time walks are normally seen as jumping from i to $i \pm 1$ at the times $T_k^{(i)} \in (-\infty, \infty)$ of a rate one Poisson process. If the jump is, say, to $i + 1$, then our polygonal path will have a linear segment between $(i, T_k^{(i)})$ and $(i + 1, T_{k'}^{(i+1)})$, where $T_{k'}^{(i+1)}$ is the first Poisson event at $i + 1$ after $T_k^{(i)}$. Furthermore, if $T_k^{(i_0)} < s_0 < T_{k+1}^{(i_0)}$, then there will be a constant segment in the path before the first nonconstant linear segment. If $s_0 = T_k^{(i_0)}$, then we take two paths: one with an initial constant segment and one without.

Theorem 2.3 *Each of the collections of rescaled coalescing random walk paths, $Y^{(\delta)}$ (in discrete time) and $\tilde{Y}^{(\delta)}$ (in continuous time) converges in distribution to the standard Brownian web as $\delta \to 0$.*

Sketch of Proof of Theorem 2.3. By Theorem 2.1, it suffices to verify conditions T_1, I_1, B_1 and B_2. We will save the tightness condition T_1 for last as it is the messiest to verify, at least in the continuous time case of $\tilde{Y}^{(\delta)}$.

Condition I_1 is basically a consequence of the Donsker invariance principle, as already noted in the Introduction. Conditions B_1 and B_2 follow from the coalescing walks version of the inequality of (1.7), which is

$$\mu_\delta(\eta(t_0, t; a, a + \epsilon) \geq k) \ \leq \ [\mu_\delta(\eta(t_0, t; a, a + \epsilon) \geq 2)]^{k-1}. \tag{2.5}$$

Taking the sup over a and the lim sup over δ and using standard random walk arguments produces an upper bound of the from $C_k(\epsilon/\sqrt{t})^{k-1}$ which yields B_1 and B_2 as desired.

It remains to verify T_1. We will sketch the arguments for the continuous time $\tilde{Y}^{(\delta)}$; the discrete time $Y^{(\delta)}$ is easier and corresponds to a portion of the continuous time arguments. As in the proof of Theorem 2.1, we denote by $g_\delta(t,u)$ the sup over x_0, t_0 of $\mu_\delta(A_{t,u})$, where μ_δ now denotes the distribution of $\tilde{Y}^{(\delta)}$. For the continuous time case (and for $u \le 1$ and $\sqrt{t}$ much smaller than u), we will obtain a δ-independent bound on $g_\delta(t,u)$ that will yield T_1 by first obtaining, as we explain below, separate upper bounds in three regions of δ-values that depend on t, u:

$$C_1 \exp(-C_2 u/\sqrt{t}) \quad \text{for } D_0/\sqrt{t} \le \delta^{-1}, \tag{2.6}$$

$$(2t/\delta^2)^{u/(3\delta)} \quad \text{for } 6/u \le \delta^{-1} \le D_0/\sqrt{t}, \tag{2.7}$$

$$C_3\,(t/u^2)^2 \quad \text{for } \delta^{-1} \le 6/u. \tag{2.8}$$

Together, these bounds (with D_0 chosen appropriately) yield

$$g_\delta(t,u) \le C_4 t^2/u^4, \tag{2.9}$$

which gives T_1 as desired.

The first region of δ-values corresponds to a spatial interval of width $\sqrt{t}$ being multiple lattice spacings δ wide and a spatial interval of width u being multiple $\sqrt{t}$-intervals wide. The bound (2.6) comes about because the event $A_{t,u}$ is prevented if between the small rectangle and both the left and right boundaries of the larger rectangle (see Figure 2), there is a random walk path that stays within some spatial $\sqrt{t}$-interval between times t_0 and $t_0 + 2t$. The second region corresponds to two (or more) spatial lattice sites between the small rectangle and the left (or right) boundary of the larger rectangle. The bound here comes from preventing $A_{t,u}$ by having a random walk path stay between two adjacent spatial lattice sites between times t_0 and $t_0 + 2t$. The third bound comes from preventing $A_{t,u}$ by not having the Poisson process occurences at adjacent spatial lattice sites, $T_k^{(i)}$ and $T_{k'}^{(i\pm1)}$, too close together in time. This completes our sketch of the proof.

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