The Brownian Web: Characterization and Convergence

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Abstract

The Brownian Web (BW) is the random network formally consisting of the paths of coalescing one-dimensional Brownian motions starting from every space-time point in $\mathbb{R} \times \mathbb{R}$. We extend the earlier work of Arratia and of Tóth and Werner by providing characterization and convergence results for the BW distribution, including convergence of the system of all coalescing random walks to the BW under diffusive space-time scaling. We also provide characterization and convergence results for the Double Brownian Web, which combines the BW with its dual process of coalescing Brownian motions moving backwards in time, with forward and backward paths "reflecting" off each other. For the BW, deterministic space-time points are almost surely of "type" $(0,1)$ — *zero* paths into the point from the past and exactly *one* path out of the point to the future; we determine the Hausdorff dimension for all types that actually occur: dimension 2 for type $(0,1)$, $3/2$ for $(1,1)$ and $(0,2)$, 1 for $(1,2)$, and 0 for $(2,1)$ and $(0,3)$.

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1 Introduction

In this paper, we present a number of results concerning the characterization of and convergence to a striking stochastic object called the *Brownian Web* (as well as similar results for the closely related *Double Brownian Web*). Several of the main results were previously announced, with sketches of the proofs, in [1].

Roughly speaking, the Brownian Web is the collection of graphs of coalescing one-dimensional Brownian motions (with unit diffusion constant and zero drift) starting from all possible starting points in one plus one dimensional (continuous) space-time. This object was originally studied more than twenty years ago by Arratia [2], motivated by asymptotics of one-dimensional voter models, and then about five years ago by Tóth and Werner [3], motivated by the problem of constructing continuum "self-repelling motions." Our own interest in this object arose because of its relevance to "aging" in statistical physics models of one-dimensional coarsening [4, 5]—which returns us to Arratia's original context of voter models, or equivalently coalescing random walks in one dimension. This motivation leads to our primary concern with weak convergence results, which in turn requires a careful choice of space for the Brownian Web so as to obtain useful characterization criteria for its distribution. We continue the introduction by discussing coalescing random walks and their scaling limits.

Let us begin by constructing random paths in the plane, as follows. Consider the two-dimensional lattice of all points (i, j) with i, j integers and $i+j$ even. Let a walker at spatial location i at time j move right or left at unit speed between times j and $j+1$ if the outcome of a fair coin toss is heads ($\Delta_{i,j} = +1$) or tails ($\Delta_{i,j} = -1$), with the coin tosses independent for different space-time points (i, j). Figure 1 depicts a simulation of the resulting paths.

The path of a walker starting from y_0 at time s_0 is the graph of a simple symmetric one-dimensional random walk, $Y_{y_0,s_0}(t)$. At integer times, $Y_{y_0,s_0}(t)$ is the solution of the simple stochastic difference equation,

$$Y(j+1) - Y(j) = \Delta_{Y(j),j}, \quad Y(s_0) = y_0. \tag{1.1}$$

Furthermore the paths of distinct walkers starting from different (y_0, s_0)'s are automatically *coalescing* — i.e., they are independent of each other until they coalesce (i.e., become identical) upon meeting at some space-time point.

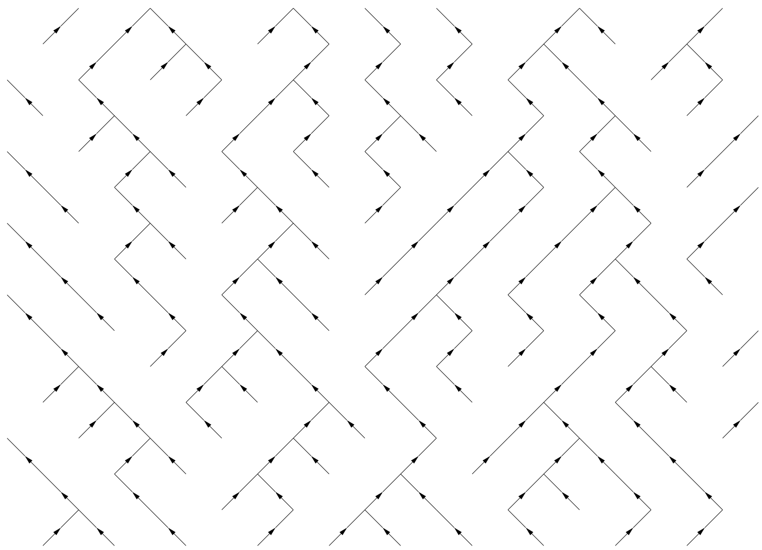

Figure 1: Coalescing random walks in discrete time; the horizontal coordinate is space and the vertical one is time.

If the increments $\Delta_{i,j}$ remain i.i.d., but take values besides ± 1 (e.g., ± 3), then one obtains non-simple random walks whose paths can cross each other in space-time, although they still coalesce when they land on the same space-time lattice site. Such systems with crossing paths will be discussed in Section 6.

After rescaling to spatial steps of size δ and time steps of size δ^2, a single rescaled random walk (say, starting from 0 at time 0) $Y_{0,0}^{(\delta)}(t) = \delta Y_{0,0}(\delta^{-2}t)$ converges as $\delta \to 0$ to a standard Brownian motion $B(t)$. That is, by the Donsker invariance principle [6], the distribution of $Y_{0,0}^{(\delta)}$ on the space of continuous paths converges as $\delta \to 0$ to standard Wiener measure.

The invariance principle is also valid for continuous time random walks, where the move from i to $i \pm 1$ takes an exponentially distributed time . In continuous time, coalescing random walks are at the heart of Harris's graphical representation of the (one-dimensional) voter model [7] and their scaling limits arise naturally in the physical context of (one-dimensional) aging (see, e.g., [4, 5]). Of course, finitely many rescaled coalescing walks in discrete or continuous time (with rescaled space-time starting points) converge in distribution to finitely many coalescing Brownian motions. In this paper, we present results concerning the convergence in distribution of the complete collection of the rescaled coalescing walks from *all* the starting points.

Our results are in two main parts:

(1) characterization (and construction) of the limiting object, the standard *Brownian Web (BW)*, and

(2) convergence criteria, which are applied, in this paper, to coalescing random walks.

As a cautionary remark, we point out that the scaling limit motivating our convergence results does not belong to the realm of hydrodynamic limits of particle systems but rather to the realm of invariance principles.

A key ingredient of the characterization and construction (see, e.g., Theorem 2.1) is the choice of a space for the Brownian web; this is the BW analogue of the space of continuous paths for Brownian motion. The convergence criteria and application (see, e.g., Theorems 2.3 and 7.1 below) are the BW analogues of Donsker's invariance principle. Like Brownian motion itself, we expect that the Brownian web and its variants (see, e.g., Sec. 5) will be quite ubiquitous as scaling limits, well beyond the context of coalescing random walks (and our sufficient conditions for convergence); one situation where this occurs is for two-dimensional "Poisson webs" [8]. Another example is in the area of river basin modelling. In [9], coalescing random walks were proposed as a model of a drainage network. Some of the questions about scaling in such models may find answers in the context of their scaling limits. For more on the random walk and other models for river basins, see [10].

Much of the construction of the Brownian web (but without characterization or convergence results) was already done in the groundbreaking work of Arratia [2, 11] (see also [12, 13]) and then in the work of Tóth and Werner [3] (see also [14] and [15]; in the latter reference, the Brownian web is introduced in relation to *Black Noise*). Arratia, Tóth and Werner all recognized that in the limit $\delta \to 0$ there would be (nondeterministic) space-time points (x, t) starting from which there are multiple limit paths and they provided various conventions (e.g., semicontinuity in x) to avoid such multiplicity. Our main contribution vis-a-vis construction is to accept the intrinsic nonuniqueness by choosing an appropriate metric space in which the BW takes its values. Roughly speaking, instead of using some convention to obtain a process that is a *single-valued* mapping from each space-time starting point to a single path from that starting point, we allow *multi-valued* mappings; more accurately, our BW value is the collection of *all* paths from all starting points. This choice of space is very much in the spirit of earlier work [16, 17, 18] on spatial scaling limits of critical percolation models and spanning trees, but modified for our particular space-time setting; the directed (in time) nature of our paths considerably simplifies the topological setting compared to [16, 17, 18].

The Donsker invariance principle implies that the distribution of any continuous (in the sup-norm metric) functional of $Y_{0,0}^{(\delta)}$ converges to that for Brownian motion. The classic example of such a functional is the random walk maximum, $\sup_{0\leq t\leq 1} Y_{0,0}^{(\delta)}(t)$. An analogous example for coalescing random walks is the maximum over all rescaled walks starting at (or passing through) some vertical (time-like) interval, i.e., the maximum value (for times $t \in [s,1]$) over walks touching any space-time point of the form $(0,s)$ for some $s \in [0,1]$. In this case, the functional is not quite continuous for our choice of metric space, but it is continuous almost everywhere (with respect to the Brownian web measure), which is sufficient.

The rest of the paper is organized as follows. In Section 2, we state two of the main results of the paper: Theorem 2.1 is a characterization of the Brownian Web and Theorem 2.3 is a convergence theorem for the important special case where, even before taking a limit, all paths are noncrossing. Section 3 contains the proof of the initial characterization result, Theorem 2.1, as well as an alternative characterization, Theorem 3.6 in which a kind of separability condition is replaced by a minimality condition. In Section 4, we present other characterization results (Theorems 4.5 and 4.6) based on certain counting random variables, which will be needed for the derivation of our main convergence results. Section 5 concerns the construction and characterization of the *Double Brownian Web* which is the union of the Brownian Web with its dual system of coalescing Brownian motions going backwards in time. Our analysis of the Double Brownian Web relies heavily on the work of [14]. Section 5 also contains an analysis of the typology of generic and exceptional space-time points for the Brownian web, as in [3], and of the Hausdorff dimension of the various types. In Section 6, we extend our convergence results to cover the case of crossing paths; the proof of the noncrossing result, Theorem 2.3, is given here as a corollary of the more general result. In Section 7 we apply our (non-crossing) convergence results to the case of coalescing random walks. There are three appendices, the first covers issues of mesurability, the second issues of compactness and tightness, and the third gives a Hausdorff dimension result about Brownian motion graphs that is used in Section 5.

2 Some Main Results

We begin by defining three metric spaces: $(\bar{\mathbb{R}}^2, \rho)$, (Π, d) and $(\mathcal{H}, d_{\mathcal{H}})$. The elements of the three spaces are respectively: points in space-time, paths with specified starting points in space-time and collections of paths with specified starting points. The BW will be an $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued random variable, where $\mathcal{F}_{\mathcal{H}}$ is the Borel σ-field associated to the metric $d_{\mathcal{H}}$.

$(\bar{\mathbb{R}}^2, \rho)$ is the completion (or compactification) of $\mathbb{R}^2$ under a metric ρ given below (see (3.8) and (3.9)). $\bar{\mathbb{R}}^2$ may be thought as the set of (x, t) in $[-\infty, \infty] \times [-\infty, \infty]$ with all points of the form $(x, -\infty)$ identified (and similarly for (x, ∞)).

For $t_0 \in [-\infty, \infty]$, let $C[t_0]$ denote the set of functions f from $[t_0, \infty]$ to $[-\infty, \infty]$ such that $(1 + |t|)^{-1} \tanh(f(t))$ is continuous. Then define

$$\Pi = \bigcup_{t_0 \in [-\infty, \infty]} C[t_0] \times \{t_0\}, \tag{2.1}$$

where $(f, t_0) \in \Pi$ represents a path in $\bar{\mathbb{R}}^2$ starting at $(f(t_0), t_0)$. For(f, t_0) in Π, we denote by $\hat{f}$ the function that extends f to all $[-\infty, \infty]$ by setting it equal to $f(t_0)$ for $t < t_0$. Then we define a suitable distance d (see (3.11)) such that (Π, d) is a complete separable metric space.

Finally, $\mathcal{H}$ denotes the set of compact subsets of (Π, d), with $d_{\mathcal{H}}$ the induced Hausdorff metric (see (3.12)) and $\mathcal{F}_{\mathcal{H}}$ the Borel σ-algebra. $(\mathcal{H}, d_{\mathcal{H}})$ is also a complete separable metric space. For an $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued random variable $\bar{\mathcal{W}}$ (or its distribution μ), we define the *finite-dimensional distributions* of $\bar{\mathcal{W}}$ as the induced probability measures $\mu_{(x_1,t_1;\ldots;x_n,t_n)}$ on the subsets of paths starting from any finite deterministic set of points $(x_1, t_1), \ldots, (x_n, t_n)$ in $\mathbb{R}^2$. There are several ways in which the Brownian web can be characterized; they differ from each other primarily in the type of extra condition required beyond the finite-dimensional distributions (which are those of coalescing Brownian motions). In the next theorem, the extra condition is a type of Doob separability property (see, e.g., Chap. 3 of [19]). Variants are stated later either using a minimality property (Theorem 3.6) or a counting random variable (Theorems 4.5 and 4.6). Theorem 4.6 is the one most directly suited to the convergence results of Section 6.

The events and random variables appearing in the next two theorems are $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-measurable. This claim follows straightforwardly from Proposition A.2 in Appendix A. The proofs of the two theorems are given, respectively, in Sections 3 and 6.

Theorem 2.1 *There is an $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued random variable $\bar{\mathcal{W}}$ whose distribution is uniquely determined by the following three properties.*

- *(o) from any deterministic point (x, t) in $\mathbb{R}^2$, there is almost surely a unique path $W_{x,t}$ starting from (x, t).*
- *(i) for any deterministic $n, (x_1, t_1), \ldots, (x_n, t_n)$, the joint distribution of $W_{x_1,t_1}, \ldots, W_{x_n,t_n}$ is that of coalescing Brownian motions (with unit diffusion constant), and*
- *(ii) for any deterministic, dense countable subset $\mathcal{D}$ of $\mathbb{R}^2$, almost surely, $\bar{\mathcal{W}}$ is the closure in $(\mathcal{H}, d_{\mathcal{H}})$ of $\{W_{x,t} : (x, t) \in \mathcal{D}\}$.*

Remark 2.2 *One can choose a* single *dense countable $\mathcal{D}_0$ and in (o), (i) and (ii) restrict to space-time starting points from that $\mathcal{D}_0$. Different characterization theorems for the Brownian web with alternatives for (ii) are given in Sections 3 and 4. We note that there are natural $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued random variables satisfying (o) and (i) but not (ii). An instance of such a random variable (closely related to the double Brownian web) will be studied elsewhere, and shown to arise as the scaling limit of stochastic flows, extending earlier work of Piterbarg [20].*

The next theorem is restricted to noncrossing processes, but a somewhat more general result is given in Section 6.

Definition 2.1 *For $t > 0$, $t_0, a, b \in \mathbb{R}$, $a < b$, let $\eta(t_0, t; a, b)$ be the number of* distinct *points in $\mathbb{R} \times \{t_0 + t\}$ that are touched by paths in $\bar{\mathcal{W}}$ which also touch some point in $[a, b] \times \{t_0\}$. Let also $\hat{\eta}(t_0, t; a, b) = \eta(t_0, t; a, b) - 1$.*

We note that by duality arguments (see Remark 5.1), it can be shown that for deterministic t_0, t, a, b, this $\hat{\eta}$ is equidistributed with the number of distinct points in $[a, b] \times \{t_0 + t\}$ that are touched by paths in $\bar{\mathcal{W}}$ which also touch $\mathbb{R} \times \{t_0\}$.

Theorem 2.3 *Suppose $\mathcal{X}_1, \mathcal{X}_2, \ldots$ are $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued random variables with noncrossing paths. If, in addition, the following three conditions are valid, then the distribution μ_n of $\mathcal{X}_n$ converges to the distribution $\mu_{\bar{W}}$ of the standard Brownian web.*

- (I_1) *There exist such $\theta_n^y \in \mathcal{X}_n$ satisfying: for any deterministic $y_1, \ldots, y_m \in \mathcal{D}$, $\theta_n^{y_1}, \ldots, \theta_n^{y_m}$ converge in distribution as $n \to \infty$ to coalescing Brownian motions (with unit diffusion constant) starting at $y_1, \ldots, y_m$.*
- (B_1) $\forall t > 0,\ \limsup_{n\to\infty} \sup_{(a,t_0)\in\mathbb{R}^2} \mu_n(\hat{\eta}(t_0, t; a, a+\epsilon) \geq 1) \to 0$ *as* $\epsilon \to 0+$*;*
- (B_2) $\forall t > 0,\ \epsilon^{-1}\limsup_{n\to\infty} \sup_{(a,t_0)\in\mathbb{R}^2} \mu_n(\hat{\eta}(t_0, t; a, a+\epsilon) \geq 2) \to 0$ *as* $\epsilon \to 0+$*.*

Convergence of coalescing random walks (in discrete and continuous time) (see Theorem 7.1) is obtained as a corollary to Theorem 2.3.

3 Construction and initial characterizations

In this section we give a complete construction of the Brownian web which yields a proof of Theorem 2.1. Then we give in Theorem 3.6 a somewhat different characterization of the BW distribution.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space where an i.i.d. family of standard Brownian motions $(B_j)_{j\geq 1}$ is defined. Let $\mathcal{D} = \{(x_j, t_j),\ j \geq 1\}$ be a countable dense set in $\mathbb{R}^2$. Let W_j be a Brownian path starting at position x_j at time t_j. More precisely,

$$W_j(t) = x_j + B_j(t - t_j),\ t \geq t_j. \tag{3.1}$$

We now construct coalescing Brownian paths out of the family of paths $(W_j)_{j\geq 1}$ by specifying coalescing rules. When two paths meet for the first time, they coalesce into a single path, which is that of the Brownian motion with the lower label. We denote the coalescing Brownian paths by $\tilde{W}_j$, $j \geq 1$. Notice the strong Markov property of Brownian motion allows for a lot of freedom in giving a coalescing rule. Any rule, even non local, that does not depend on the realization of the (W_j)'s *after* the time of coalescence will yeld the same object in distribution

We now give a more careful inductive definition of the coalescing paths. First we set

$$\tilde{W}_1 = W_1. \tag{3.2}$$

For $j \geq 2$, let $\tilde{W}_j$ be the mapping from $[t_j, \infty)$ to $\mathbb{R}$ defined as follows. Let

$$\begin{aligned} \tau_j &= \inf\{t \geq t_j : W_j(t) = \tilde{W}_i(t) \text{ for some } 1 \leq i \leq j-1\}, && (3.3)\\ I_j &= \min\{1 \leq i \leq j-1 : W_j(\tau_j) = \tilde{W}_i(\tau_j)\}, && (3.4) \end{aligned}$$

and define

$$\begin{aligned} \tilde{W}_j(t) &= W_j(t), \text{ if } t_j \leq t \leq \tau_j \\ &= \tilde{W}_{I_j}(t), \text{ if } t > \tau_j. \end{aligned} \tag{3.5}$$

We then define the Brownian web skeleton $\mathcal{W}(\mathcal{D})$ with *starting set* $\mathcal{D}$ by

$$\begin{aligned} \mathcal{W}_k &= \mathcal{W}_k(\mathcal{D}) = \{\tilde{W}_j : 1 \leq j \leq k\} && (3.6)\\ \mathcal{W} &= \mathcal{W}(\mathcal{D}) = \bigcup_k \mathcal{W}_k && (3.7) \end{aligned}$$

$(\bar{\mathbb{R}}^2, \rho)$ is the completion (or compactification) of $\mathbb{R}^2$ under the metric ρ, where

$$\rho((x_1, t_1), (x_2, t_2)) = \left| \frac{\tanh(x_1)}{1 + |t_1|} - \frac{\tanh(x_2)}{1 + |t_2|} \right| \vee |\tanh(t_1) - \tanh(t_2)|. \tag{3.8}$$

$\bar{\mathbb{R}}^2$ may be thought as the image of $[-\infty, \infty] \times [-\infty, \infty]$ under the mapping

$$(x, t) \rightsquigarrow (\Phi(x, t), \Psi(t)) \equiv \left(\frac{\tanh(x)}{1 + |t|}, \tanh(t) \right). \tag{3.9}$$

For $t_0 \in [-\infty, \infty]$, let $C[t_0]$ denote the set of functions f from $[t_0, \infty]$ to $[-\infty, \infty]$ such that $\Phi(f(t), t)$ is continuous. Then define

$$\Pi = \bigcup_{t_0 \in [-\infty, \infty]} C[t_0] \times \{t_0\}, \tag{3.10}$$

where $(f, t_0) \in \Pi$ represents a path in $\bar{\mathbb{R}}^2$ starting at $(f(t_0), t_0)$. For(f, t_0) in Π, we denote by $\hat{f}$ the function that extends f to all $[-\infty, \infty]$ by setting it equal to $f(t_0)$ for $t < t_0$. Then we take

$$d((f_1, t_1), (f_2, t_2)) = (\sup_t |\Phi(\hat{f}_1(t), t) - \Phi(\hat{f}_2(t), t)|) \vee |\Psi(t_1) - \Psi(t_2)|. \tag{3.11}$$

(Π, d) is a complete separable metric space.

Let now $\mathcal{H}$ denote the set of compact subsets of (Π, d), with $d_{\mathcal{H}}$ the induced Hausdorff metric, i.e.,

$$d_{\mathcal{H}}(K_1, K_2) = \sup_{g_1 \in K_1} \inf_{g_2 \in K_2} d(g_1, g_2) \vee \sup_{g_2 \in K_2} \inf_{g_1 \in K_1} d(g_1, g_2). \tag{3.12}$$

$(\mathcal{H}, d_{\mathcal{H}})$ is also a complete separable metric space.

Definition 3.1 *$\bar{\mathcal{W}}(\mathcal{D})$ is the closure in (Π, d) of $\mathcal{W}(\mathcal{D})$.*

Proposition 3.1 *$\bar{\mathcal{W}}(\mathcal{D})$ satisfies properties (o) and (i) of Theorem 2.1; i.e., its finite dimensional distributions (whether from points in $\mathcal{D}$ or not) are those of coalescing Brownian motions.*

Proof We will prove the proposition in the special case of the distribution of path(s) from the single point at the origin $(0,0)$. The proof easily extends to the case of finitely many space-time points. Let $\{(x_n^r, t_n^r)\}_{n=1}^{\infty}$ and $\{(x_n^l, t_n^l)\}_{n=1}^{\infty}$ be two sequences of points from $\mathcal{D}$ satisfying the following conditions. There exist positive constants $c_1, c_2 \leq 1/2$ such that

$$\begin{gathered} -c_1/n^2 < x_n^l < 0 < x_n^r < c_2/n^2, \\ -|x_n^l|^3 < t_n^l < 0, -|x_n^r|^3 < t_n^r < 0. \end{gathered}$$

Let $\tilde{W}_{n,l}$ and $\tilde{W}_{n,r}$ be the coalescing Brownian motions starting from (x_n^l, t_n^l) and (x_n^r, t_n^r) respectively and let $B(t)$ be a standard Brownian motion (which for convenience we take as defined on $(\Omega, \mathcal{F}, \mathbb{P})$, e.g. by letting $B = B_1$).

Using the reflection principle we obtain

$$\begin{aligned} & \mathbb{P}\left(\max_{t_n^l \leq s \leq 0} \tilde{W}_{n,l}(s) \geq 0\right) = 2\mathbb{P}\left(B(|t_n^l|) \geq |x_n^l|\right) \\ = \;& 2\mathbb{P}\left(B(1) \geq |x_n^l|/|t_n^l|^{1/2}\right) \leq 2\mathbb{P}\left(B(1) \geq 1/|x_n^l|^{1/2}\right) \\ \leq \;& 2\mathbb{P}\left(B(1) \geq \sqrt{2}\, n\right) \leq e^{-n^2}. \end{aligned} \tag{3.13}$$

Proceeding along the same lines we can prove that

$$\mathbb{P}\left(\min_{t_n^r \leq s \leq 0} \tilde{W}_{n,r}(s) \leq 0\right) \leq e^{-n^2}. \tag{3.14}$$

Using scaling properties of Brownian motion, we obtain

$$
\begin{aligned}
& \mathbb{P}\left(\tilde{W}_{n,l}(0) \le -2c_1/n^2\right) \le \mathbb{P}\left(B(|t_n^l|) \ge c_1/n^2\right) \\
= \;& \mathbb{P}\left(B(1) \ge \frac{c_1}{n^2}\frac{1}{|t_n^l|^{1/2}}\right) \le \mathbb{P}\left(B(1) \ge \frac{c_1}{n^2}\frac{n^3}{c_1^{3/2}}\right) \\
= \;& \mathbb{P}\left(B(1) \ge n/c_1^{1/2}\right) \le \frac{1}{2}\, e^{-n^2/(2c_1)} \le e^{-n^2}. \qquad (3.15)
\end{aligned}
$$

Similarly,

$$\mathbb{P}\left(\tilde{W}_{n,r}(0) \ge 2c_2/n^2\right) \le e^{-n^2}.$$

Now, let $\tau_n = \max\{t_n^l, t_n^r\}$. Then, for all $n \ge 1$ we have the following inequalities and equalities (where we use the reflection principle in one place).

$$
\begin{aligned}
& \mathbb{P}\left(\inf_{\tau_n \le s \le 1/n} |\tilde{W}_{n,l}(s) - \tilde{W}_{n,r}(s)| > 0\right) \\
\le \;& \mathbb{P}\left(\tilde{W}_{n,l}(0) \le -2c_1/n^2\right) + \mathbb{P}\left(\tilde{W}_{n,r}(0) \ge 2c_2/n^2\right) \\
+ \;& \mathbb{P}\left(\inf_{0 \le s \le 1/n} |\tilde{W}_{n,l}(s) - \tilde{W}_{n,r}(s)| > 0, \tilde{W}_{n,l}(0) > -\frac{2c_1}{n^2}, \tilde{W}_{n,r}(0) < \frac{2c_2}{n^2}\right) \\
\le \;& 2e^{-n^2} + 1 - \mathbb{P}\left(\inf_{0 \le s \le 1/n}\left[\sqrt{2}B(s) + 2(c_1+c_2)/n^2\right] > 0\right) \\
= \;& 2e^{-n^2} + 1 - \mathbb{P}\left(\sup_{0 \le s' \le 2/n} B(s') < 2(c_1+c_2)/n^2\right) \\
= \;& 2e^{-n^2} + 1 - 2\mathbb{P}\left(B(2/n) > 2(c_1+c_2)/n^2\right) \\
= \;& 2e^{-n^2} + 1 - 2\mathbb{P}\left(B(1) > \frac{2(c_1+c_2)/n^2}{\sqrt{2/n}}\right) \\
\le \;& C/n^{3/2}, \qquad (3.16)
\end{aligned}
$$

for some constant C. We also observe that

$$
\begin{aligned}
\mathbb{P}\left(\max_{t_n^l \le s \le 1/n} \tilde{W}_{n,l}(s) \ge n^{-1/4}\right) &= 2\mathbb{P}\left(B(-t_n^l + 1/n) \ge n^{-1/4} + |x_n^l|\right) \\
&\le 2\mathbb{P}\left(B(1) \ge n^{-1/4}/\sqrt{-t_n^l + 1/n}\right) \\
&\le e^{-C'n^{1/2}} \qquad (3.17)
\end{aligned}
$$

for some C'. Similarly,

$$\mathbb{P}\left(\inf_{t_n^l<s<1/n}\tilde{W}_{n,r}(s)\le -n^{-1/4}\right)\le e^{-C'n^{1/2}}. \tag{3.18}$$

Let $\tilde{W}_{n,l}(0)=l_n$ and $\tilde{W}_{n,r}(0)=r_n$. Put $S_n=\inf\{s:\tilde{W}_{n,l}(s)=\tilde{W}_{n,r}(s)\}$ and let $M_n=\tilde{W}_{n,l}(S_n)$. Using the Borel-Cantelli lemma, from (3.13), (3.14), (3.16), (3.17) and (3.18) we have that almost surely for all but finitely many n's, $l_n<0$ and $r_n>0$, $S_n<1/n$, and $|M_n|<n^{-1/4}$. This implies the following: Let Δ_n be the interior of the "triangular" space-time region with vertices $(x_n^l,t_n^l),(x_n^r,t_n^r),(M_n,S_n)$, base $[x_n^l,x_n^r]$ and sides $\tilde{W}_{n,l},\tilde{W}_{n,r}$. There exists an almost surely finite random variable N such that for all $k>N$ any coalescing Brownian motion $\tilde{W}_i$ starting from point $(x_i,t_i)\in\mathcal{D}$ and $(x_i,t_i)\in\Delta_k$, $\tilde{W}_i(s)=\tilde{W}_{k,l}(s)=\tilde{W}_{k,r}(s)$ for all $s>S_k$. Since (M_n,S_n) converges to $(0,0)$ almost surely we have, for every sequence $((x_i,t_i))_i$ in $\mathcal{D}$ converging to $(0,0)$, that the coalescing Brownian motions starting from (x_i,t_i) converge (in the path space metric) to a unique coalescing Brownian motion starting from $(0,0)$, independent of the sequence. This proves Proposition 3.1.

The next result is contained in Proposition B.7.

Proposition 3.2 *$\bar{\mathcal{W}}(\mathcal{D})$ is almost surely a compact subset of* (Π,d).

Remark 3.3 *Almost surely, $\bar{\mathcal{W}}(\mathcal{D})=\lim_{k\to\infty}\mathcal{W}_k(\mathcal{D})$, where the limit is taken in $\mathcal{H}$.*

Remark 3.4 *It can be shown by the methods discussed in Remark B.2 of the appendix that, almost surely, all paths in $\bar{\mathcal{W}}(\mathcal{D})$ are Hölder continuous with exponent α, for any $\alpha<1/2$.*

Proposition 3.5 *The distribution of $\bar{\mathcal{W}}(\mathcal{D})$ does not depend on $\mathcal{D}$ (including its order). Furthermore, $\bar{\mathcal{W}}(\mathcal{D})$ satisfies property* (*ii*) *of Theorem 2.1.*

Proof Given two countable dense sets in $\mathbb{R}^2$, $\mathcal{D}_1$ and $\mathcal{D}_2$, we will couple versions of $\bar{\mathcal{W}}(\mathcal{D}_1)$ and $\bar{\mathcal{W}}(\mathcal{D}_2)$ and show that they are equal. Let $\mathcal{W}_k(\mathcal{D}_2)$ be as in the construction of the Brownian web with starting set $\mathcal{D}_2$ and let $\mathcal{W}'_k(\mathcal{D}_2)$ be the paths of $\bar{\mathcal{W}}(\mathcal{D}_1)$ starting from the first k elements of $\mathcal{D}_2$. By Proposition 3.1, $\mathcal{W}_k(\mathcal{D}_2)$ and $\mathcal{W}'_k(\mathcal{D}_2)$ have the same distribution (namely, that of coalescing Brownian motions starting from the first k elements of $\mathcal{D}_2$). From Proposition B.8,

$$\bar{\mathcal{W}}(\mathcal{D}_2)\sim\bar{\mathcal{W}}'(\mathcal{D}_2)\equiv\overline{\cup_k\mathcal{W}'_k(\mathcal{D}_2)}\subseteq\bar{\mathcal{W}}(\mathcal{D}_1).$$

By using property (o) for $\bar{\mathcal{W}}(\mathcal{D}_1)$ (more generally, by Proposition 3.1), we claim that the paths in $\bar{\mathcal{W}}(\mathcal{D}_1)$ starting from $\mathcal{D}_1$ belong to $\bar{\mathcal{W}}'(\mathcal{D}_2)$ (i.e., can be approximated arbitrarily well by paths in $\mathcal{W}'_k(\mathcal{D}_2)$ for sufficiently large k). To see this, let (x^1, t^1) be some spacetime point in $\mathcal{D}_1$ and let (x^2_j, t^2_j) be a sequence in $\mathcal{D}_2$ converging to (x^1, t^1); then the unique (by property (o)) paths $W[x^2_j, t^2_j]$ in $\bar{\mathcal{W}}(\mathcal{D}_1)$ starting from (x^2_j, t^2_j) converge (by property (o)) to $W[x^1, t^1]$. Our claim follows since $W[x^2_j, t^2_j]$ is in $\mathcal{W}'_k(\mathcal{D}_2)$ for a sufficiently large k. It then follows that *all* paths in $\bar{\mathcal{W}}(\mathcal{D}_1)$ can be so approximated and hence $\bar{\mathcal{W}}(\mathcal{D}_1) \subseteq \bar{\mathcal{W}}'(\mathcal{D}_2)$. Thus $\bar{\mathcal{W}}(\mathcal{D}_1) = \bar{\mathcal{W}}'(\mathcal{D}_2) \sim \bar{\mathcal{W}}(\mathcal{D}_2)$. This proves both the distributional independence on $\mathcal{D}$ of $\bar{\mathcal{W}}(\mathcal{D})$ and the validity of property (ii).

Proof of Theorem 2.1 The existence part of the theorem is immediate from Propositions 3.1 and 3.5. The uniqueness may be argued as follows.

Take any $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued random variable $\mathcal{X}$ with properties $(o), (i)$ and (ii), and fix a deterministic countable dense subset $\mathcal{D} = \{(x_1, t_1), (x_2, t_2), \ldots\}$ of $\mathbb{R}^2$. Let $\bar{\mathcal{W}} = \bar{\mathcal{W}}(\mathcal{D})$, the version of the Brownian web constructed at the beginning of this section of the paper, using $\mathcal{D}$ (but whose distribution does not depend on $\mathcal{D}$, by Proposition 3.5). From (i) and Proposition 3.1, $\mathcal{X}_n := \{X_{x_i,t_i} : i = 1, \ldots, n\}$, where $X_{x,t}$ is the path of $\mathcal{X}$ starting at (x, t) (almost surely unique by (o)), is equidistributed with $\mathcal{W}_n := \{W_{x_i,t_i} : i = 1, \ldots, n\}$ for all $n \geq 1$. Now (ii) and Remark 3.3 imply or say that $\mathcal{X} = \lim_{n\to\infty} \mathcal{X}_n$ and $\bar{\mathcal{W}} = \lim_{n\to\infty} \mathcal{W}_n$. So, they have the same distribution and the proof is complete.

The next theorem provides an alternative characterization to Theorem 2.1. Other characterizations that will be used for our convergence results, are presented in Sec. 3.

Definition 3.2 *(Stochastic ordering)* $\mu_1 << \mu_2$ *if for* g *any bounded measurable function on* $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$ *that is* increasing *(i.e.,* $g(K) \leq g(K')$ *when* $K \subseteq K'$*),* $\int g d\mu_1 \leq \int g d\mu_2$.

Theorem 3.6 *There is an* $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$*-valued random variable* $\bar{\mathcal{W}}$ *whose distribution is uniquely determined by properties* $(o), (i)$ *of Theorem 2.1 and*

- (ii') *if* $\mathcal{W}^*$ *is any other* $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$ *- valued random variable satisfying (o) and (i), then* $\mu_{\bar{\mathcal{W}}} << \mu_{\mathcal{W}^*}$.

Proof of Theorem 3.6 The existence part of the theorem is immediate from Theorem 2.1. The uniqueness may be argued as follows. Fix a deterministic countable dense subset $\mathcal{D} = \{(x_1, t_1), (x_2, t_2), \ldots\}$ of $\mathbb{R}^2$, let $\mathcal{W}^*$ be the Brownian web constructed from $\mathcal{D}$ and let $W^*_{x,t}$ for $(x,t) \in \mathcal{D}$ be the path from (x,t) in $\mathcal{W}^*$. Since $\bar{\mathcal{W}} \supset \overline{\{W_{x,t},\, (x,t) \in \mathcal{D}\}}$, where $W_{x,t}$ is the path from (x,t) in $\bar{\mathcal{W}}$, and, from (o) and (i), $\mathcal{W}_n := \{W_{x_i,t_i},\, i = 1, \ldots, n\}$ has the same distribution as $\mathcal{W}^*_n := \{W^*_{x_i,t_i},\, i = 1, \ldots, n\}$ for all $n \geq 1$, we have that $\mu_{\mathcal{W}^*} << \mu_{\bar{\mathcal{W}}}$. This and (ii') imply that $\mu_{\mathcal{W}^*} = \mu_{\bar{\mathcal{W}}}$, and the proof is complete.

4 Characterization via counting

In this section, we give other characterizations of the Brownian web that will be used for our convergence theorem. They will be given in terms of the counting random variables η and $\hat{\eta}$ defined in Definition 2.1. We begin with some properties of the Brownian web as constructed in Section 1.

Proposition 4.1 *For a Brownian skeleton $\mathcal{W}(\mathcal{D})$, the corresponding counting random variable $\hat{\eta}_{\mathcal{D}} = \hat{\eta}_{\mathcal{D}}(t_0, t; [a,b])$ satisfies*

$$\begin{aligned} \mathbb{P}(\hat{\eta}_{\mathcal{D}} \geq k) &\leq \mathbb{P}(\hat{\eta}_{\mathcal{D}} \geq k-1)\mathbb{P}(\hat{\eta}_{\mathcal{D}} \geq 1) \qquad (4.1) \\ &\leq (\mathbb{P}(\hat{\eta}_{\mathcal{D}} \geq 1))^k = (\Theta(b-a,t))^k, \qquad (4.2) \end{aligned}$$

where $\Theta(b-a,t)$ is the probability that two independent Brownian motions starting at a distance $b-a$ apart at time zero will not have met by time t (which itself can be expressed in terms of a single Brownian motion). Thus, $\hat{\eta}_{\mathcal{D}}$ is almost surely finite and $\mathbb{E}(\hat{\eta}_{\mathcal{D}}) < \infty$.

Proof We note that it is sufficient to prove the inequalities for $\mathcal{W}_n(\mathcal{D}) = \{\tilde{W}_j : 1 \leq j \leq n\}$ for all n. Morevover for a given n if we prove the result for n coalescing random walks, then the result for n Brownian motions will follow from the scaling limit of the random walks. Therefore we now consider n discrete time simple symmetric coalescing random walks on $\mathbb{Z}$: $X_1, X_2, \ldots, X_n$, starting from $l_1 < l_2 < \ldots < l_n$, respectively.

We observe these paths between time zero and time $T \in \mathbb{N}$. We define $\eta(X_1, \ldots, X_j)$ to be the number of walkers remaining at time T from the initial j at time zero (i.e., the number of distinct values taken by $X_1(T)$,...,

$X_j(T)$) and $\eta_T = \eta(X_1, \ldots, X_n)$. We write $X_j - X_i > 0$ to denote that $X_j(s) - X_i(s) > 0$ for every $s = 0, \ldots, T$.

We are interested in $\mathbb{P}(\eta_T \geq k+1 | \eta_T \geq k)$. For $k \leq M \leq n$, consider

$$\mathbb{P}(X_1 = \xi_1, \ldots, X_{M-1} = \xi_{M-1}, Y_M - \xi_{M-1} > 0, Y_n - Y_M > 0),$$

where $\xi_1, \xi_2, \ldots, \xi_{M-1}$ are non-crossing random walk paths starting at l_1, l_2,..., l_{M-1} and Y_M and Y_n are independent simple random walks starting at l_M and l_n (with no coalescing properties). With $\xi_1, \xi_2, \ldots, \xi_{M-1}$ fixed and such that $\eta(\xi_1, \xi_2, \ldots, \xi_{M-1}) = k-1$, we regard only Y_M and Y_n as random and use the facts that Y_M and $-Y_n$ are independent and that the events $A = \{Y_M - \xi_{M-1} > 0\}$ and $B = \{Y_n - Y_M > 0\}$ are respectively increasing and decreasing in their dependence on $\{Y_M, -Y_n\}$. It then follows by the FKG inequalities (for independent variables) that $\mathbb{P}(A \cap B) \leq \mathbb{P}(A)\mathbb{P}(B)$. Letting B' denote the event that $Y_n - Y_1 > 0$, we obviously have that $\mathbb{P}(B) \leq \mathbb{P}(B')$ so that $\mathbb{P}(A \cup B) \leq \mathbb{P}(A)\mathbb{P}(B')$. Now averaging the latter inequality over $M, \xi_1, \xi_2, \ldots, \xi_{k-2}$, and noting that $\mathbb{P}(B') = \mathbb{P}(\eta_T \geq 2)$, we get

$$\mathbb{P}(\eta_T \geq k+1) \leq \mathbb{P}(\eta_T \geq k)\mathbb{P}(\eta_T \geq 2)).$$

This completes the proof of the first inequality of Proposition 4.1. The second inequality follows immediately and we leave the equality as an exercise for the reader.

Remark 4.2 *The argument in the above proof relies on a negative dependence (anti-FKG) property of the point process for which $\hat{\eta}_{\mathcal{D}}$ is the cardinality. This property suggests that the tail decay of $\hat{\eta}_{\mathcal{D}}$ is at at least as fast as Poissonian (which would correspond to the case of independence). By analogy with the number of crossings in the scaling limit of percolation and other statistical mechanics models [17], one could expect the actual decay to be Gaussian.*

The next proposition is a consequence of the one just before; it will be used in this section and also later in Section 5.

Proposition 4.3 *Almost surely, for every $\epsilon > 0$ and every $\theta = (f(s), t_0)$ in $\bar{W}(\mathcal{D})$, there exists a path $\theta_\epsilon = (g(s), t'_0)$, in the skeleton $\mathcal{W}(\mathcal{D})$ such that $g(s) = f(s)$ for all $s \geq t_0 + \epsilon$.*

Proof Let $\epsilon > 0$ be given. Since $\bar{\mathcal{W}}$ is the closure (in (Π, d)) of $\mathcal{W}(\mathcal{D})$, we have that there exists a sequence $\{\theta_n = (g_n(s), t'_n)\}$ of paths in $\mathcal{W}(\mathcal{D})$

with $t'_n < t_0 + \frac{\epsilon}{2}$, such that $d(\theta, \theta_n) \to 0$ as $n \to \infty$. There also exists an integer-valued random variable N such that $f(t_0 + \frac{\epsilon}{2}) \in [N, N+1]$. Now by Proposition 4.1 it follows that $\mathbb{P}(\cup_{k\in\mathbb{Z}}\{\hat{\eta}_D(t_0 + \frac{\epsilon}{2}, \frac{\epsilon}{2}; k, k+3) < \infty\}) = 1$. Therefore we have $\hat{\eta}_D(t_0 + \frac{\epsilon}{2}, \frac{\epsilon}{2}; N-1, N+2)$ is almost surely finite. $\theta_n \to \theta$ implies that $g_n(t_0 + \frac{\epsilon}{2})$ is eventually in $[N-1, N+2]$. Now since the paths are coalescing and $\hat{\eta}_D(t_0 + \frac{\epsilon}{2}, \frac{\epsilon}{2}; N-1, N+2)$ is almost surely finite we easily see that $\theta_n(t_0 + \epsilon)$ is eventually constant. This implies that $\theta(t_0 + \epsilon) = \theta_n(t_0 + \epsilon)$ for large enough n proving the proposition.

Proposition 4.4 *Let $\hat{\eta} = \hat{\eta}(t_0, t; a, b)$ be the counting random variable for $\bar{\mathcal{W}}(\mathcal{D})$, then $\mathbb{P}(\hat{\eta} \geq k) \leq (\Theta(b-a,t))^k$, and thus $\hat{\eta}$ is almost surely finite with finite expectation. Furthermore, $\hat{\eta} = \hat{\eta}_\mathcal{D}$ almost surely and thus*

$$\begin{aligned} \mathbb{P}(\hat{\eta} \geq k) &\leq \mathbb{P}(\hat{\eta} \geq k-1)\mathbb{P}(\hat{\eta} \geq 1) && (4.3)\\ &\leq (\mathbb{P}(\hat{\eta} \geq 1))^k = (\Theta(b-a,t))^k. && (4.4) \end{aligned}$$

Proof Choose the set $\mathcal{D}$ so that its first two points are (a, t_0) and (b, t_0). We first prove that $\mathbb{P}(\hat{\eta} \geq k) \leq (\Theta(b-a,t))^k$. From Proposition 4.1 we know that this claim is true on the skeleton. Define $\tilde{\eta}(t_0, t; a, b, \epsilon_1, \epsilon_2)(K) =$ cardinality $\{y \in \mathbb{R}$: there exists a path $(f(s), s_0)$ in K with $s_0 < t_0 + \epsilon_2$, satisfying $f(t_0 + \epsilon_2) \in (a - \epsilon_1, b + \epsilon_1)$ and $f(t_0 + t) = y\}$. $\tilde{\eta}_D$ is defined similarly by restricting the path to be a skeletal path. We observe that $\{K \in \mathcal{H} : \tilde{\eta}_D \geq k\}$ is an open subset of $\mathcal{H}$. Therefore,

$$\begin{aligned} \mathbb{P}(\eta(t_0, t; a, b) \geq k) &\leq \mathbb{P}(\tilde{\eta}(t_0, t; a, b, \epsilon_1, \epsilon_2) \geq k) && (4.5)\\ &= \mathbb{P}(\tilde{\eta}_D(t_0, t; a, b, \epsilon_1, \epsilon_2) \geq k) && (4.6)\\ &\leq \mathbb{P}(\eta_D(t_0 + \epsilon_2, t - \epsilon_2; a - \epsilon_1, b + \epsilon_1) \geq k) && (4.7)\\ &\leq (\Theta(b - a + 2\epsilon_1, t + 2\epsilon_2))^{k-1}, && (4.8) \end{aligned}$$

where the equality follows from Proposition 4.3 and the last inequality follows from inequality (4.2). Letting ϵ_1 and $\epsilon_2 \to 0$ we see that

$$\mathbb{P}(\eta(t_0, t; a, b) \geq k) \leq (\Theta(b-a,t))^{k-1}.$$

Therefore,

$$\mathbb{P}(\hat{\eta}(t_0, t; a, b) \geq k) = \mathbb{P}(\eta(t_0, t; a, b) \geq k+1) \leq (\Theta((b-a), t))^k. \quad (4.9)$$

Using the above inequality for $k = 2$ we get

$$\mathbb{P}(\hat{\eta}(t_0, t; a, b) \geq 2) \leq (\Theta((b-a), t))^2 \leq C(b-a)^2, \tag{4.10}$$

where C depends only on t. From inequality (4.10) it readily follows that at any deterministic time almost surely there are no triple (or higher multiplicity) points. That is,

$$\mathbb{P}(K \in \mathcal{H} : \hat{\eta}(t_0, t; c, c) \geq 2 \text{ for some c } \in \mathbb{R}) = 0.$$

Now we want to show that $\hat{\eta} = \hat{\eta}_D$ almost surely. If $\hat{\eta} \neq \hat{\eta}_D$, then there exists an $x \in (a, b)$ such that there is a path $(f(s), t_0) \in \bar{\mathcal{W}}(\mathcal{D})$ with $f(t_0) = x$ satisfying the property that if $(g(s), s_0) \in \mathcal{W}(\mathcal{D})$ with $s_0 \leq t_0$ and $g(t_0) \in (a, b)$ then $g(s) \neq f(s)$ for all $s \in [t_0, t_0 + t]$.

Since $\mathcal{D}$ is dense in $\mathbb{R}^2$, for all $n \in \mathbb{N}$ we find paths $(g_n(s), s_n), (h_n(u), u_n)$ in $\mathcal{W}(\mathcal{D})$ such that $s_n \in (t_0 - 1/n, t_0), g_n(s_n) \in (x - 2/n, x - 1/n), u_n \in (t_0 - 1/n, t_0), h_n(u_n) \in (x + 1/n, x + 2/n)$. Moreover for large enough n (depending on x) almost surely we can choose these paths such that $g_n(t_0) \in (a, x)$ and $h_n(t_0) \in (x, b)$. Letting $n \to \infty$ it is not hard to show that x is a triple (or higher multiplicity) point. To see this, note that by Prop. 4.3, not only do $g_n(s)$ and $h_n(s)$ differ from $f(s)$ for $s \in [t_0, t_0 + t]$ but also for each $s \in (t_0, t_0 + t]$ there are only finitely many possible values for g_n and h_n (as n varies) and hence any (subsequence) limits of g_n or h_n differ from f for all $s \in (t_0 + t]$. Therefore we conclude that $\mathbb{P}(\hat{\eta} = \hat{\eta}_D) = 1$, thus proving the proposition.

Theorem 4.5 *Let $\mathcal{W}'$ be an $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued random variable; its distribution equals that of the (standard) Brownian web $\bar{\mathcal{W}}$ (as characterized by Theorems 2.1 and 3.6) if its finite dimensional distributions are coalescing (standard) Brownian motions (i.e. conditions (o) and (i) of Theorem 2.1 are valid) and*

(ii'') for all t_0, t, a, b, $\hat{\eta}_{\mathcal{W}'}$ is equidistributed with $\hat{\eta}_{\bar{\mathcal{W}}}$.

For purposes of proving our convergence results, we will use a modified version of the above characterization theorem in which conditions (o), (i), (ii'') are all weakened.

Theorem 4.6 *Let $\mathcal{W}'$ be an $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued random variable and let $\mathcal{D}$ be a countable dense deterministic subset of $\mathbb{R}^2$ and for each $y \in \mathcal{D}$, let $\theta^y \in \mathcal{W}'$ be some single (random) path starting at y. $\mathcal{W}'$ is equidistributed with the (standard) Brownian web $\bar{\mathcal{W}}$ if* *(i′) the θ^y's are distributed as coalescing (standard) Brownian motions and*

(ii‴) for all t_0, t, a, b, $\eta_{\mathcal{W}'} << \eta_{\bar{\mathcal{W}}}$, i.e. $\mathbb{P}(\eta_{\mathcal{W}'} \geq k) \leq \mathbb{P}(\eta_{\bar{\mathcal{W}}} \geq k)$ for all k.

Proof We need to show that the above conditions together imply that μ', the distribution of $\mathcal{W}'$, equals the distribution μ of the constructed Brownian web $\bar{\mathcal{W}}$. Let η' the counting random variable appearing in condition (ii‴) for μ'. Choose some deterministic dense countable subset $\mathcal{D}$ and consider the countable collection $\mathcal{W}^*$ of paths of $\mathcal{W}'$ starting from $\mathcal{D}$. By condition (i′), $\mathcal{W}^*$ is equidistributed with our constructed Brownian skeleton $\mathcal{W}$ (based on the same $\mathcal{D}$) and hence the closure $\bar{\mathcal{W}}^*$ of $\mathcal{W}^*$ in (Π, d) is a subset of $\mathcal{W}'$ that is equidistributed with our constructed Brownian web $\bar{\mathcal{W}}$. To complete the proof, we will use condition (ii‴) to show that $\mathcal{W}' \setminus \bar{\mathcal{W}}^*$ is almost surely empty by using the fact that the counting random variable η^* for $\bar{\mathcal{W}}^*$ already satisfies condition (ii‴) since $\bar{\mathcal{W}}^*$ is distributed as a Brownian web. If $\mathcal{W}' \setminus \bar{\mathcal{W}}^*$ were nonempty (with strictly positive probability), then there would have to be some rational t_0, t, a, b for which $\eta' > \eta^*$. But then

$$\mathbb{P}(\eta'(t_0, t; a, b) > \eta^*(t_0, t; a, b)) \; > \; 0 \tag{4.11}$$

for some rational t_0, t, a, b, and this together with the fact that $\mathbb{P}(\eta' \geq \eta^*) = 1$ (which follows from $\bar{\mathcal{W}}^* \subset \mathcal{W}'$) would violate condition (ii‴) with those t_0, t, a, b. The proof is complete.

Remark 4.7 *The condition $\eta_{\mathcal{W}'} << \eta_{\bar{\mathcal{W}}}$ can be replaced by $\mathbb{E}(\eta_{\mathcal{W}'}) \leq \mathbb{E}(\eta_{\bar{\mathcal{W}}})$. We note that $\mathbb{E}(\eta_{\bar{\mathcal{W}}}) = 1 + (b-a)/\sqrt{\pi t}$, as given in [1] by a calculation stretching back to [21]. So, in particular, in the context of Theorem 4.6, if besides (i′), $\mathbb{E}(\eta_{\mathcal{W}'}) \leq 1 + (b-a)/\sqrt{\pi t}$ for all t_0, t, a, b, then $\mathcal{W}'$ is equidistributed with the Brownian web.*

5 Dual and Double Brownian Webs

In this section, we construct and characterize the *Double Brownian Web*, which combines the Brownian web with a *Dual Brownian Web* of coalescing Brownian motions moving backwards in time.

Remark 5.1 *In the graphical representation of Harris for the one-dimensional voter model [7], coalescing random walks forward in time and coalescing dual random walks backward in time (with forward and backward walks not* *crossing each other) are constructed simultaneously (see, e.g., the discussion in [4, 5]). Figure 2 provides an example in discrete time. Note that there is no crossing between forward and backward walks—a property that is preserved in the* Double Brownian Web *(DBW) scaling limit. The simultaneous construction of forward and (dual) backward Brownian motions was emphasized in [3, 14] and their approach and results can be applied to extend both our characterization and convergence results to the DBW which includes simultaneously the forward BW and its dual backward BW. We note that in the DBW, the η of Definition 2.1 equals (almost surely for deterministic t_0, t, a, b) $1 + \eta^{\text{dual}}$, where η^{dual} is the number of distinct points in $[a, b] \times \{t_0\}$ touched by backward paths which also touch $\mathbb{R} \times \{t_0 + t\}$.*

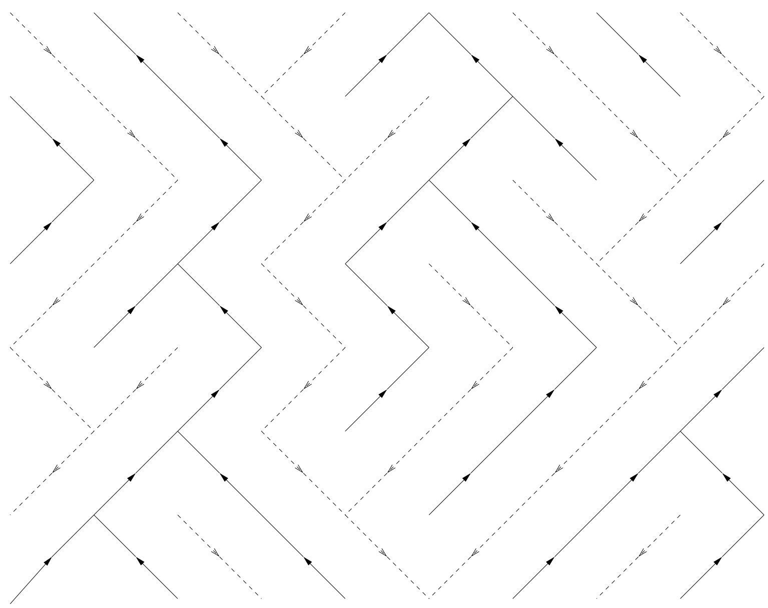

Figure 2: Forward coalescing random walks (full lines) in discrete time and their dual backward walks (dashed lines).

The forward and backward paths basically reflect off each other. In Section 7 (see Theorem 7.3), we show that the Double Brownian Web arises as the scaling limit of coalescing random walks forward in time together with their (graph theoretic) dual coalescing paths backward in time (see Figure 2). Our analysis will rely on a paper [14] of Soucaliuc, Tóth and Werner together with our results from earlier sections on the (forward) Brownian web.

We again begin with an (ordered) dense countable set $\mathcal{D} \subset \mathbb{R}^2$ but this time we need twice as many i.i.d. standard B.M.'s as before $B_1, B_1^b, B_2, B_2^b, \ldots$. We use this paths to construct forward and backward independent B.M.'s $W_1, W_1^b, W_2, W_2^b, \ldots$ starting from $(x_j, t_j) \in \mathcal{D}$ as follows.

$$\begin{aligned} W_j(t) &= x_j + B_j(t - t_j),\ t \geq t_j & (5.1)\\ W_j^b(t) &= x_j + B_j^b(t_j - t),\ t \leq t_j. & (5.2) \end{aligned}$$

Then we create coalescing and reflecting Brownian paths $\tilde{W}_1, \tilde{W}_1^b, \ldots$ inductively, as follows.

$$\begin{aligned} \tilde{W}_1 &= W_1; \quad \tilde{W}_1^b = W_1^b; & (5.3)\\ \tilde{W}_n &= CR(W_n; \tilde{W}_1, \tilde{W}_1^b, \ldots, \tilde{W}_{n-1}, \tilde{W}_{n-1}^b); & (5.4)\\ \tilde{W}_n^b &= CR(W_n^b; \tilde{W}_1, \tilde{W}_1^b, \ldots, \tilde{W}_{n-1}, \tilde{W}_{n-1}^b), & (5.5) \end{aligned}$$

where the operation CR is defined in [14], Subsubsection 3.1.4. We proceed to explain CR for the simplest case, in the definition of $\tilde{W}_2$.

As pointed out in [14], the nature of the reflection of a forward Brownian path $\tilde{W}$ off a backward Brownian path $\tilde{W}^b$ (or vice-versa) is special. It is actually better described as a push of $\tilde{W}$ off $\tilde{W}^b$ (see Subsection 2.1 in [14]). It does not have an explicit form in general, but in the case of one forward path and one backward path, the form is as follows. Following our notation and construction, we ignore $\tilde{W}_1$ and consider $\tilde{W}_1^b$ and $\tilde{W}_2$ in the time interval $[t_2, t_1]$ (we suppose $t_2 < t_1$; otherwise, $\tilde{W}_1^b$ and $\tilde{W}_2$ are independent). Given W_2 and $\tilde{W}_1^b$, for $t_2 \leq t \leq t_1$,

$$\tilde{W}_2(t) = \begin{cases} W_2(t) + \sup_{t_2 \leq s \leq t}(W_2(s) - \tilde{W}_1^b(s))^-, & \text{if } W_2(t_2) > \tilde{W}_1^b(t_2); \\ W_2(t) - \sup_{t_2 \leq s \leq t}(W_2(s) - \tilde{W}_1^b(s))^+, & \text{if } W_2(t_2) < \tilde{W}_1^b(t_2). \end{cases} \qquad (5.6)$$

After t_1, $\tilde{W}_2$ interacts only with $\tilde{W}_1$, by the usual coalescence.

We call $\mathcal{W}_n^D := \{\tilde{W}_1, \tilde{W}_1^b, \ldots, \tilde{W}_n, \tilde{W}_n^b\}$ *coalescing/reflecting forward and backward Brownian motions (starting at* $\{(x_1, t_1), \ldots, (x_n, t_n)\}$*).* We will also use the alternative notation $\mathcal{W}^D(\mathcal{D}_n)$ in place of $\mathcal{W}_n^D$, where $\mathcal{D}_n := \{(x_1, t_1), \ldots, (x_n, t_n)\}$.

Remark 5.2 *In Theorem 8 of [14], it is proved that the above construction is a.s. well-defined, gives a perfectly coalescing/reflecting system (see Subsubsection 3.1.1 in [14]), and for every* $n \geq 1$*, the distribution of* $\mathcal{W}_n^D$ *does not depend on the ordering of* $\mathcal{D}_n$*. It also follows from that result that* $\{\tilde{W}_1, \ldots, \tilde{W}_n\}$

and $\{\tilde{W}_1^b, \ldots, \tilde{W}_n^b\}$ *are separately forward and backward coalescing Brownian motions, respectively. Thus* $\{\tilde{W}_1, \tilde{W}_2, \ldots\}$ *and* $\{\tilde{W}_1^b, \tilde{W}_2^b, \ldots\}$ *are forward and backward Brownian web skeletons, respectively.*

Remark 5.3 *One can alternatively use a set* $\mathcal{D}^b$ *of starting points for the backward paths different than* $\mathcal{D}$ *rather than our choice above of* $\mathcal{D}^b = \mathcal{D}$.

We now define dual spaces of paths going backward in time (Π^b, d^b) and a corresponding $(\mathcal{H}^b, d_{\mathcal{H}^b})$ in an obvious way, so that they are the dual versions of (Π, d) and $(\mathcal{H}, d_{\mathcal{H}})$, and then define $\mathcal{H}^D = \mathcal{H} \times \mathcal{H}^b$ and

$$d_{\mathcal{H}^D}((K_1, K_1^b), (K_2, K_2^b)) = \max(d_{\mathcal{H}}(K_1, K_2), d_{\mathcal{H}^b}(K_1^b, K_2^b)).$$

As in the construction of the (forward) BW, we now define

$$\mathcal{W}_n^D(\mathcal{D}) = \{\tilde{W}_1, \ldots, \tilde{W}_n\} \times \{\tilde{W}_1^b, \ldots, \tilde{W}_n^b\}, \tag{5.7}$$

$$\mathcal{W}^D(\mathcal{D}) = \{\tilde{W}_1, \tilde{W}_2, \ldots\} \times \{\tilde{W}_1^b, \tilde{W}_2^b, \ldots\}, \tag{5.8}$$

$$\bar{\mathcal{W}}^D(\mathcal{D}) = \overline{\{\tilde{W}_1, \tilde{W}_2, \ldots\}} \times \overline{\{\tilde{W}_1^b, \tilde{W}_2^b, \ldots\}}. \tag{5.9}$$

The latter closures are in Π for the first factor and in Π^b for the second one.

From Remark 5.2, we have that

$$\bar{\mathcal{W}} := \overline{\{\tilde{W}_1, \tilde{W}_2, \ldots\}} \text{ and } \bar{\mathcal{W}}^b := \overline{\{\tilde{W}_1^b, \tilde{W}_2^b, \ldots\}}$$

are forward and backward Brownian webs, respectively. The next result follows from Proposition 3.2.

Proposition 5.4 *Almost surely,* $\bar{\mathcal{W}}^D(\mathcal{D}) \in \mathcal{H}^D$ *(i.e.* $\overline{\{\tilde{W}_1, \tilde{W}_2, \ldots\}}$ *and* $\overline{\{\tilde{W}_1^b, \tilde{W}_2^b, \ldots\}}$ *are compact).*

Remark 5.5 *It is immediate from this proposition that*

$$\bar{\mathcal{W}}^D(\mathcal{D}) = \lim_{n\to\infty} \mathcal{W}_n^D(\mathcal{D}),$$

where the limit is in the $d_{\mathcal{H}^D}$ metric.

Proposition 5.6 $\bar{\mathcal{W}}^D(\mathcal{D})$ *satisfies*

(o^D) *From any deterministic* (x, t) *there is almost surely a unique forward path and unique backward path.*

(i^D) *For any deterministic* $\mathcal{D}'_n := \{(y_1, s_1), \dots, (y_n, s_n)\}$ *the forward and backward paths from* $\mathcal{D}'_n$*, denoted* $\bar{\mathcal{W}}^D(\mathcal{D}, \mathcal{D}'_n)$*, are distributed as coalescing/reflecting forward and backward Brownian motions starting at* $\mathcal{D}'_n$*. In other words,* $\bar{\mathcal{W}}^D(\mathcal{D}, \mathcal{D}'_n)$ *has the same distribution as* $\mathcal{W}^D(\mathcal{D}'_n)$*.*

Proof (o^D) follows from the separate properties of $\bar{\mathcal{W}}(\mathcal{D})$ and of $\bar{\mathcal{W}}^b(\mathcal{D})$, since they are distributed respectively as forward and backward Brownian webs, as previously noted.

As to (i^D): for $k = 1, 2, \dots$ let $\mathcal{D}_n^{(k)} = \{(x_1^{(k)}, t_1^{(k)}), \dots, (x_n^{(k)}, t_n^{(k)})\} \subset \mathcal{D}$ be such that $(x_i^{(k)}, t_i^{(k)})$ converges to (y_i, s_i) for each $1 \leq i \leq n$ as $k \to \infty$. Since $\mathcal{W}^D(\mathcal{D}'_n)$ and $\mathcal{W}^D(\mathcal{D}_n^{(k)})$ (for every $k \geq 1$) are defined using the same (B_i, B_i^b)'s, we get from standard Brownian motion sample path properties that $\mathcal{W}^D(\mathcal{D}_n^{(k)}) \to \mathcal{W}^D(\mathcal{D}'_n)$ a.s. as $k \to \infty$, and thus

$$\mathcal{W}^D(\mathcal{D}_n^{(k)}) \to \mathcal{W}^D(\mathcal{D}'_n) \tag{5.10}$$

in distribution as $k \to \infty$. Now from Theorem 8 of [14], $\bar{\mathcal{W}}^D(\mathcal{D}, \mathcal{D}_n^{(k)})$ has the same distribution as $\bar{\mathcal{W}}^D(\mathcal{D}_n^{(k)})$. We proceed to show that

$$\bar{\mathcal{W}}^D(\mathcal{D}, \mathcal{D}_n^{(k)}) \to \bar{\mathcal{W}}^D(\mathcal{D}, \mathcal{D}'_n) \tag{5.11}$$

a.s. as $k \to \infty$, and (i^D) follows immediately from that and (5.10).

Denoting by $\bar{\mathcal{W}}(\mathcal{D}, \mathcal{D}_n^{(k)})$ and $\bar{\mathcal{W}}^b(\mathcal{D}, \mathcal{D}_n^{(k)})$ respectively the set of forward and backward paths of $\bar{\mathcal{W}}^D(\mathcal{D}, \mathcal{D}_n^{(k)})$, (5.11) is equivalent to the pair of a.s. limits (as $k \to \infty$):

$$\begin{aligned} \bar{\mathcal{W}}(\mathcal{D}, \mathcal{D}_n^{(k)}) &\to \bar{\mathcal{W}}(\mathcal{D}, \mathcal{D}'_n); && (5.12) \\ \bar{\mathcal{W}}^b(\mathcal{D}, \mathcal{D}_n^{(k)}) &\to \bar{\mathcal{W}}^b(\mathcal{D}, \mathcal{D}'_n). && (5.13) \end{aligned}$$

We argue (5.12) only; (5.13) is similar. Let $\gamma_i^{(k)}$ be the path in $\bar{\mathcal{W}}(\mathcal{D}, \mathcal{D}_n^{(k)})$ starting at $(x_i^{(k)}, t_i^{(k)})$ and γ_i be the path in $\bar{\mathcal{W}}(\mathcal{D}, \mathcal{D}'_n)$ starting at (y_i, s_i). It is enough to show that for every $i = 1, \dots, n$, $\gamma_i^{(k)} \to \gamma_i$ in Π as $k \to \infty$. Suppose not; then since $\gamma_i^{(k)} \in \bar{\mathcal{W}}(\mathcal{D})$, which is compact, there must be a subsequence of $(\gamma_i^{(k)})_k$ which converges to some $\gamma' \neq \gamma_i$. Since $\gamma' \in \bar{\mathcal{W}}(\mathcal{D})$, this would contradict (o^D). The proof is finished.

Proposition 5.7 *The distribution of* $\bar{\mathcal{W}}^D(\mathcal{D})$ *as an* $(\mathcal{H}^D, \mathcal{F}_{\mathcal{H}^D})$*-valued random variable (where* $\mathcal{F}_{\mathcal{H}^D} = \mathcal{F}_{\mathcal{H}} \times \mathcal{F}_{\mathcal{H}^b}$*), does not depend on* $\mathcal{D}$*. Furthermore,*

(ii^D) *for any deterministic dense* $\mathcal{D}'$*, almost surely*

$$\bar{\mathcal{W}}^D(\mathcal{D}) = \overline{\{W_{x,t} : (x,t) \in \mathcal{D}'\}} \times \overline{\{W^b_{x,t} : (x,t) \in \mathcal{D}'\}},$$

where $W_{x,t}, W^b_{x,t}$ *are respectively the forward and backward paths in* $\bar{\mathcal{W}}^D(\mathcal{D})$ *starting from* (x,t)*, and the closures in* (ii^D) *are in* Π *for the first factor and in* Π^b *for the second one.*

Proof The proof is essentially the same as that of Proposition 3.5, using Proposition 5.6 in place of Proposition 3.1 and the natural (and similarly proved) analogue of Proposition B.8.

We now give some characterization theorems for the distribution of the double Brownian web $\bar{\mathcal{W}}^D$.

Theorem 5.8 *The double Brownian web is characterized (in distribution, on* $(\mathcal{H}^D, \mathcal{F}_{\mathcal{H}^D})$*) by conditions* $(o^D), (i^D)$ *and* (ii^D)*.*

Proof The proof is basically the same as for Theorem 2.1. We include it for completeness. Take any $(\mathcal{H}^D, \mathcal{F}_{\mathcal{H}^D})$-valued random variable $\mathcal{X}^D$ with properties $(o^D), (i^D)$ and (ii^D), and fix a deterministic countable dense subset $\mathcal{D} = \{(x_1,t_1),(x_2,t_2),\ldots\}$ of $\mathbb{R}^2$. Let $\bar{\mathcal{W}}^D = \bar{\mathcal{W}}^D(\mathcal{D})$ be the version of the double Brownian web constructed earlier in this section of the paper, using $\mathcal{D}$ (and recall that the distribution does not depend on $\mathcal{D}$, by Proposition 5.7). From (i^D) for $\mathcal{X}^D$ and Proposition 5.6, $\mathcal{X}^D_n := \{X_{x_i,t_i}, X^b_{x_i,t_i}; i = 1,\ldots,n\}$, where $X_{x,t}, X^b_{x,t}$ are the forward and backward paths of $\mathcal{X}^D$ starting at (x,t), respectively (almost surely a unique pair by (o^D)), is equidistributed with $\mathcal{W}^D_n = \{W_{x_i,t_i}, W^b_{x_i,t_i}; i = 1,\ldots,n\}$, where $W_{x,t}, W^b_{x,t}$ are the forward and backward paths of $\bar{\mathcal{W}}^D$ starting at (x,t), for all $n \geq 1$. Now (ii^D) for $\mathcal{X}^D$ and Remark 5.5 imply or say that $\mathcal{X}^D = \lim_{n\to\infty} \mathcal{X}^D_n$ and $\bar{\mathcal{W}}^D = \lim_{n\to\infty} \mathcal{W}^D_n$ a.s. . So, they have the same distribution and the proof is complete.

Definition 5.1 *(Stochastic ordering) We define a partial order in* $\mathcal{H}^D$ *as follows: given* (K_1, K^b_1)*,* $(K_2, K^b_2) \in \mathcal{H}^D$*, we say that* $(K_1, K^b_1) \leq (K_2, K^b_2)$ *if* $K_1 \subseteq K_2$ *and* $K^b_1 \subseteq K^b_2$*. For two measures* μ^D_1*,* μ^D_2 *in* $(\mathcal{H}^D, \mathcal{F}_{\mathcal{H}^D})$*, we say that* $\mu^D_1 << \mu^D_2$ *if* $\int f\, d\mu^D_1 \leq \int f\, d\mu^D_2$ *for every bounded* $f \in \mathcal{F}_{\mathcal{H}^D}$ *that is increasing in the partial order* $\leq$*.*

Theorem 5.9 *The distribution* $\mu_{\bar{\mathcal{W}}^D}$ *of the double Brownian web* $\bar{\mathcal{W}}^D$ *is characterized by conditions* $(o^D), (i^D)$ *and*

(ii'^D) *any* $(\mathcal{H}^D, \mathcal{F}_{\mathcal{H}^D})$*-valued random variable,* $\mathcal{X}^D$*, satisfying* (o^D) *and* (i^D) *has* $\mu_{\bar{\mathcal{W}}^D} << \mu_{\mathcal{X}^D}$.

Proof The proof is analogous to the one of Theorem 3.6. Suppose $\bar{\mathcal{W}}^D$ is an $(\mathcal{H}^D, \mathcal{F}_{\mathcal{H}^D})$-valued random variable satisfying (o^D), (i^D) and (ii'^D). Fix a deterministic countable dense subset $\mathcal{D} = \{(x_1, t_1), (x_2, t_2), \ldots\}$ of $\mathbb{R}^2$ and let $\mathcal{X}^D$ be the double Brownian web constructed from $\mathcal{D}$ (denoted earlier by $\bar{\mathcal{W}}^D(\mathcal{D})$). Since $\bar{\mathcal{W}}^D \supset \overline{\{W_{x,t}, W^b_{x,t};\, (x,t) \in \mathcal{D}\}}$, where $W_{x,t}, W^b_{x,t}$ are the forward and backward paths of $\bar{\mathcal{W}}^D$ starting at (x,t), and, from (o^D) and (i^D) for $\bar{\mathcal{W}}^D$, $\mathcal{W}^D_n = \{W_{x_i,t_i}, W^b_{x_i,t_i};\, i = 1, \ldots, n\}$ has the same distribution as $\mathcal{X}^D_n := \{X_{x_i,t_i}, X^b_{x_i,t_i};\, i = 1, \ldots, n\}$ for all $n \geq 1$, where $X_{x,t}, X^b_{x,t}$ are the forward and backward paths of $\mathcal{X}^D$ starting at (x,t), we have that $\mu_{\mathcal{X}^D} << \mu_{\bar{\mathcal{W}}^D}$. This and (ii'^D) together imply that $\mu_{\mathcal{X}^D} = \mu_{\bar{\mathcal{W}}^D}$, and the proof is complete.

Definition 5.2 *For* $t > 0$, $t_0, a, b \in \mathbb{R}$, $a < b$, *let* $\eta^b(t_0, t; a, b)$ *be the number of* distinct *points in* $\mathbb{R} \times \{t_0 - t\}$ *that are touched by paths in* $\bar{W}^b$ *which also touch some point in* $[a, b] \times \{t_0\}$*. Let also* $\hat{\eta}^b(t_0, t; a, b) = \eta^b(t_0, t; a, b) - 1$.

Theorem 5.10 *Let* $\mathcal{W}'^D = (\mathcal{W}', \mathcal{W}'^b)$ *be an* $(\mathcal{H}^D, \mathcal{F}_{\mathcal{H}^D})$*-valued random variable, let* $\mathcal{D}$ *be a countable dense deterministic subset of* $\mathbb{R}^2$*, and for each* $y \in \mathcal{D}$ *let* $\theta^y \in \mathcal{W}'$ *and* $\theta^{yb} \in \mathcal{W}'^b$ *be single paths starting at* y*.* $\mathcal{W}'^D$ *is equidistributed with the double Brownian web* $\mathcal{W}^D = (\bar{\mathcal{W}}, \bar{\mathcal{W}}^b)$ *if*

(i'^D) *the* θ^y*'s and* θ^{yb}*'s are distributed as coalescing/reflecting forward and backward Brownian motions, and*

(ii'^D) *for all* t_0, t, a, b, $\hat{\eta}_{\mathcal{W}'} << \hat{\eta}_{\bar{\mathcal{W}}}$ *and* $\hat{\eta}_{\mathcal{W}'^b} << \hat{\eta}_{\bar{\mathcal{W}}^b}$.

Proof The proof is analogous to the one of Theorem 4.6. We need to show that the above conditions together imply that μ', the distribution of $\mathcal{W}'^D$, equals the distribution μ of the previously constructed double Brownian web $\bar{\mathcal{W}}^D$. Let η', η'^b denote the forward and backward counting random variables appearing in condition (ii$'''$) for μ'. Choose some deterministic dense countable subset $\mathcal{D}$ and consider the countable collection $\mathcal{W}_*{}^D$ of forward and backward paths of $\mathcal{W}'^D$ starting from $\mathcal{D}$. By (i'^D), $\mathcal{W}_*{}^D$ is equidistributed with our constructed double Brownian web skeleton $\mathcal{W}^D$ (based on the same $\mathcal{D}$) and hence the closure $\bar{\mathcal{W}}^D_*$ of $\mathcal{W}_*{}^D$ is a subset of $\mathcal{W}'^D$ that is equidistributed with our constructed double Brownian web $\bar{\mathcal{W}}^D$. To complete the proof, we will use condition (ii'^D) to show that $\mathcal{W}'^D \setminus \bar{\mathcal{W}}_*^D$ is almost surely empty, by using the fact that the forward and backward counting random variables $\eta_*, {\eta_*}^b$ for $\bar{\mathcal{W}}_*^D$ already satisfy condition (ii'^D), since $\bar{\mathcal{W}}_*^D$ is distributed as a double Brownian web. If $\mathcal{W}'^D \setminus \bar{\mathcal{W}}_*^D$ were nonempty (with strictly positive probability), then there would have to be some rational t_0, t, a, b for which either $\eta' > \eta_*$ or ${\eta'}^b > {\eta_*}^b$. But then

$$\mathbb{P}\left(\eta'(t_0, t; a, b) > \eta_*(t_0, t; a, b) \text{ or } {\eta'}^b(t_0, t; a, b) > {\eta_*}^b(t_0, t; a, b)\right) > 0 \quad (5.14)$$

for some rational t_0, t, a, b, and this together with the fact that $\mathbb{P}(\eta' \geq \eta_*,\ {\eta'}^b \geq {\eta_*}^b) = 1$ (which follows from $\bar{\mathcal{W}}_*^D \subset \mathcal{W}'^D$) would violate condition (ii'^D) with those t_0, t, a, b. The proof is complete.

We now discuss "types" of points $(x, t) \in \mathbb{R}^2$, whether deterministic or not. For the (forward) Brownian web, we define

$$\begin{aligned} m_{\text{in}}(x_0, t_0) &= \lim_{\epsilon \downarrow 0}\{\text{number of paths in } \mathcal{W} \text{ starting at some } t_0 - \epsilon \text{ that pass} \\ &\qquad \text{through } (x_0, t_0) \text{ and are disjoint for } t_0 - \epsilon < t < t_0\}; \quad (5.15) \\ m_{\text{out}}(x_0, t_0) &= \lim_{\epsilon \downarrow 0}\{\text{number of paths in } \mathcal{W} \text{ starting at } (x_0, t_0) \text{ that are} \\ &\qquad \text{disjoint for } t_0 < t < t_0 + \epsilon\}. \quad (5.16) \end{aligned}$$

For $\mathcal{W}^b$, we similarly define $m_{\text{in}}^b(x_0, t_0)$ and $m_{\text{out}}^b(x_0, t_0)$.

Definition 5.3 *The type of (x_0, t_0) is the pair $(m_{\text{in}}, m_{\text{out}})$—see Figure 3. We denote by $S_{i,j}$ the set of points of $\mathbb{R}^2$ that are of type (i, j), and by $\bar{S}_{i,j}$ the set of points of $\mathbb{R}^2$ that are of type (k, l) with $k \geq i$, $l \geq j$.*

Remark 5.11 *Using the translation and scale invariance properties of the Brownian web distribution, it can be shown that for any i, j, whenever $S_{i,j}$ is nonempty, it must be dense in $\mathbb{R}^2$. The same can be said of $S_{i,j} \cap \mathbb{R} \times \{t\}$ for deterministic t. These denseness properties can also be shown for each i, j by more direct arguments.*

Proposition 5.12 *For the double Brownian web, almost surely for* every *$(x_0, t_0) \in \mathbb{R}^2$, $m_{\text{in}}^b(x_0, t_0) = m_{\text{out}}(x_0, t_0) - 1$ and $m_{\text{out}}^b(x_0, t_0) = m_{\text{in}}(x_0, t_0) + 1$. See Figure 3.*

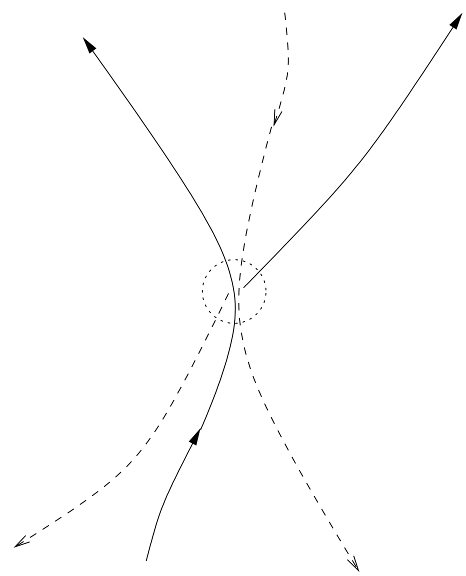

Figure 3: A schematic diagram of a point (x_0, t_0) of type $(m_{in}, m_{out}) = (1, 2)$, with necessarily also $(m^b_{in}, m^b_{out}) = (1, 2)$. In this example the incoming forward path connects to the leftmost outgoing path (with a corresponding dual connectivity for the backward paths); at some of the other points of type $(1, 2)$ it will connect to the rightmost path.

Proof It is enough to prove (i) that for every incoming forward path to a point (x, t), there are two locally disjoint backward paths starting at that point with one on either side of the forward path; and (ii) that for every two locally disjoint backward paths starting at a point (x, t), there is an incoming forward path to (x, t) between the two backward paths. (Note that by a $t \longleftrightarrow -t$ time reflection argument, one would then get a similar result for incoming backward paths and pairs of outgoing forward paths.)

Let us start with the first assertion. Let γ be an incoming forward path to (x, t). This means that the starting time s of γ is such that $s < t$. By Proposition 4.3, the portion of γ above time $s + \epsilon$ is in the forward skeleton for every $\epsilon > 0$. Now consider a sequence of pairs of backward paths (γ_k, γ'_k) starting at $((x_k, t_k), (y_k, s_k)) \in \mathcal{D} \times \mathcal{D}$ with $((x_k, t_k), (y_k, s_k)) \to ((x, t), (x, t))$ as $k \to \infty$, $s + \epsilon < s_k, t_k < t$, $x_k < \gamma(t_k)$ and $y_k > \gamma(s_k)$. From the reflection of the forward and backward skeletons off each other and the fact that two backward paths in the skeleton must coalesce once they meet, it follows that $\gamma_k(t') < \gamma'_k(t')$ for all $t' \in [\max\{s_k, t_k\}, t']$. We then conclude from compactness that there are two locally disjoint limit paths, one for (γ_k) and one for (γ'_k), both starting from (x,t).

We argue (ii) similarly. Given two locally disjoint backward paths γ, γ' starting at (x,t), there exists $s < t$ such that either $\gamma(t') < \gamma'(t')$ for $s < t' < t$ or $\gamma'(t') < \gamma(t')$ for $s < t' < t$. Suppose it is the first case; otherwise, switch labels. Then choose a point $(x', s') \in \mathcal{D}$ with $s < s' < t$ and $\gamma(s') < x' < \gamma'(s')$. The fact that the portions of γ and γ' below time $t - \epsilon$ is in the backward skeleton for every $\epsilon > 0$ and the reflection of the forward and backward skeletons off each other now implies that the forward path starting at (x', s') is squeezed between γ and γ' and goes to (x,t).

Theorem 5.13 *For the (double) Brownian web, almost surely, every* (x,t) *has one of the following types, all of which occur:* $(0,1)$, $(0,2)$, $(0,3)$, $(1,1)$, $(1,2)$, $(2,1)$.

Remark 5.14 *Points of type* $(1,2)$ *are particularly interesting in that the single incident path continues along exactly one of the two outward paths — with the choice determined intrinsically rather than by some convention. See Figure 3 for a schematic diagram of a "left-handed" continuation. An* (x_0, t_0) *is of type* $(1,2)$ *precisely if both a forward and a backward path pass through* (x_0, t_0)*. It is either left-handed or right-handed according to whether the forward path is to the left or the right of the backward path near* (x_0, t_0)*. Both varieties occur and the proof of Theorem 5.15 below shows that the Hausdorff dimension of* 1 *applies separately to each of the two varieties.*

Tóth and Werner [3] gave a definition of types of points of $\mathbb{R}^2$ similar to ours, but for a somewhat different process and proved the above theorem with that definition and for that process (see definition at page 385, paragraph of equation (2.28) and Proposition 2.4 in [3]). One way then to establish Theorem 5.13 is to show the equivalence of ours and Tóth and Werner's definition and that their arguments hold for our process. We prefer, for the sake of simplicity and completeness, to give a direct argument, out of which the following complementary results also follow.

Theorem 5.15 *Almost surely,* $S_{0,1}$ *has full Lebesgue measure in* $\mathbb{R}^2$*,* $S_{1,1}$ *and* $S_{0,2}$ *have Hausdorff dimension* $3/2$ *each,* $S_{1,2}$ *has Hausdorff dimension* 1*, and* $S_{2,1}$ *and* $S_{0,3}$ *are both countable and dense in* $\mathbb{R}^2$*.*

Theorem 5.16 *Almost surely: for every t*

a) $S_{0,1} \cap \mathbb{R} \times \{t\}$ *has full Lebesgue measure in* $\mathbb{R} \times \{t\}$*;*

b) $S_{1,1} \cap \mathbb{R} \times \{t\}$ *and* $S_{0,2} \cap \mathbb{R} \times \{t\}$ *are both countable and dense in* $\mathbb{R} \times \{t\}$*;*

c) $S_{1,2} \cap \mathbb{R} \times \{t\}$*,* $S_{2,1} \cap \mathbb{R} \times \{t\}$ *and* $S_{0,3} \cap \mathbb{R} \times \{t\}$ *have all cardinality at most* 1*.*

For every deterministic t, $S_{1,2} \cap \mathbb{R} \times \{t\}$*,* $S_{2,1} \cap \mathbb{R} \times \{t\}$ *and* $S_{0,3} \cap \mathbb{R} \times \{t\}$ *are almost surely empty.*

Proof of Theorems 5.13 and 5.15 We start by ruling out the cases that do not occur almost surely. For $i, j \geq 0$, $S_{i,j} = \emptyset$ almost surely if $j = 0$ or $i + j \geq 4$. The first case is trivial. We only need to consider $\bar{S}_{i,j}$ for the cases $i = 3, j = 1$ and $i = 2, j = 2$, since the other ones are either contained or dual to these. By Proposition 4.3, $\bar{S}_{3,1}$ consists of points which are almost surely in the skeleton and where three paths coalesce. But the event that three coalescing Brownian paths starting at distinct points coalesce at the same time is almost surely empty. By Proposition 4.3, $\bar{S}_{2,2}$ consists of points (almost surely in the double skeleton) where two different forward paths coalesce and a backward path passes. Since for any two forward and one backward Brownian paths in the double skeleton, the event that this happens is almost surely empty, by the perfectly coalescing/reflecting property of the paths in the double skeleton (see Subsubsection 3.1.1 and Theorem 8 of [14]) the conclusion follows.

Now, for the types that do occur.

Type (2,1) By the above, $S_{2,1} = \bar{S}_{2,1}$ almost surely, and $\bar{S}_{2,1}$ consists almost surely of *points of coalescence*, that is all points where two paths coalesce. By Proposition 4.3, it is almost surely a subset of the skeleton, and thus is countable (since there is at most one coalescence point for each pair of paths starting from $\mathcal{D}$ in the skeleton). It is easy to see that it is dense since the paths from a pair of nearby points in $\mathcal{D}$ also coalesce nearby with probability close to one.

Type (1,2) By the above, $S_{1,2} = \bar{S}_{1,2}$ almost surely, and $\bar{S}_{1,2}$ consists almost surely of points where forward paths meet backward paths. Thus, it is a subset of the (union of the traces of all the paths in the) skeleton. It is easy to see that it is almost surely nonempty (and also dense). We need only consider two such paths, say W and W^b, the former a forward one starting at $(0,0)$ (without loss of generality, by the translation invariance of the law of $\mathcal{W}^D$), and the latter a backward one starting at an arbitrary deterministic (x_0,t_0), with $t_0>0$ to avoid a trivial case. It is clear that the random set Λ of space-time points $(t,W(t))$ for times $t\in[0,t_0]$ when $W(t)=W^b(t)$ has a positive, less than one probability of being empty. We will argue next the following claim.

Claim Λ *has Hausdorff dimension* 1 *for almost every pair of trajectories* (W,W^b) *for which it is nonempty.*

By Proposition 5.6, the distribution of $\{(W(t),W^b(t)) : 0\le t\le t_0\}$ (which is all that matters for this) can be described in terms of two (forward) independent standard Brownian motions B,B^b as follows (see equations (5.1)-(5.6)). Let $W^b(t)=x_0+B^b(t_0-t)$, $t\le t_0$, and $\tau=\inf\{t\in[0,t_0] : B(t)=W^b(t)\}$, with $\inf\emptyset=\infty$. If $\tau=\infty$, then $W=B$; otherwise, $W(t)=B(t)$ for $0\le t\le\tau$, and for $\tau\le t\le t_0$,

$$W(t)=\begin{cases}B(t)+\sup_{0\le s\le t}(W^b(s)-B(s)), & \text{if } W^b(0)<0;\\ B(t)-\sup_{0\le s\le t}(B(s)-W^b(s)), & \text{if } W^b(0)>0.\end{cases}$$

Rewriting in terms of $W'(t):=W^b(t)-W^b(0)$, $0\le t\le t_0$, which is a standard Brownian motion independent of B, we have (for $0\le t\le t_0$)

$$W(t)=\begin{cases}B(t)+\sup_{0\le s\le t}\{W'(s)-B(s)\}-W'(t_0)+x_0, & \text{if } W'(t_0)>x_0;\\ B(t)+\inf_{0\le s\le t}\{W'(s)-B(s)\}-W'(t_0)+x_0, & \text{if } W'(t_0)<x_0,\end{cases}$$

if $\tau\le t\le t_0$, with $\tau=\inf\{t\in[0,t_0] : B(t)=W'(t)-W'(t_0)+x_0\}$; otherwise, $W(t)=B(t)$.

From the above discussion, we conclude that Λ has the same distribution as the random set $\mathcal{G}$ obtained as follows. Let $\mathcal{T}^+$ and $\mathcal{T}^-$ be the sets of positive and negative record times of the standard Brownian motion $X(t):=(W'(t)-B(t))/\sqrt{2}$, respectively, i.e., $\mathcal{T}^+$ is the set of $t\ge0$ such that $X(t)=\sup_{0\le s\le t}X(s)$ and $\mathcal{T}^-$ is the same except with inf in place of sup. Consider also the standard Brownian motion $Y(t):=(W'(t)+B(t))/\sqrt{2}$, which is independent of X. If $W'(t_0)>x_0$, then $\mathcal{G}=\{([(X(t)+Y(t))/\sqrt{2}]-[(X(t_0)+Y(t_0))\sqrt{2}]+x_0,t) : t\in\mathcal{T}^+\cap[\tau,t_0]\}$; if $W'(t_0)<x_0$, then $\mathcal{G}=\{([(X(t)+Y(t))/\sqrt{2}]-[(X(t_0)+Y(t_0))\sqrt{2}]+x_0,t) : t\in\mathcal{T}^-\cap[\tau,t_0]\}$.

It follows from Proposition C.1 in Appendix C that the sets $\mathcal{G}^\pm:=\{(X(t)+Y(t),t) : t\in\mathcal{T}^\pm\cap[0,t_0]\}$ (one for each sign, respectively) both have Hausdorff dimension 1 almost surely. Since the events $\{W'(t_0) > x_0\}$, $\{W'(t_0) < x_0\}$ and $\{\tau < t_0\}$ all have positive probability, the claim follows.

Type (1,1) $\bar{S}_{1,1}$ almost surely consists of *points of continuation* of paths, that is, all points (x, t) such that there is a path starting earlier than t that touches (x, t). By Proposition 4.3, $\bar{S}_{1,1}$ is almost surely a subset of the skeleton. Since the trace of any single path has Hausdorff dimension 3/2 [23] and the countable union of such sets has the same dimension, it follows that $\bar{S}_{1,1}$ has Hausdorff dimension 3/2 almost surely. By the previous parts of the proof, $\bar{S}_{1,1} \setminus S_{1,1}$ has lower dimension and so $S_{1,1}$ has the same Hausdorff dimension of 3/2.

Type (0,1) We claim that any deterministic point is a.s. of this type, hence (by applying Fubini's Theorem) $S_{0,1}$ is a.s. of full Lebesgue measure in the plane. That $m_{\text{in}}(x_0, t_0) = 0$ a.s. for every deterministic (x_0, t_0) follows from Proposition 4.3, since, if $m_{\text{in}}(x_0, t_0) \geq 1$, then there would be a path in the skeleton passing through (x_0, t_0), but this event clearly has probability zero. The assertion that $m_{\text{out}}(x_0, t_0) = 1$ a.s. for every deterministic (x_0, t_0) is property (o) of Theorem 2.1.

By Proposition 5.12, the remaining types $(0, 2)$ and $(0, 3)$ are dual respectively to $(1, 1)$ and $(2, 1)$, since the other types are dual to these. Since $\bar{\mathcal{W}}^b$ is distributed like the standard Brownian web (modulo a time reflection), the claimed results for types $(0, 2)$ and $(0, 3)$ follow from those already proved for $(1, 1)$ and $(2, 1)$.

Proof of Theorem 5.16

Type (0,1) $\bar{S}_{1,1}$ is almost surely in the skeleton, thus making $\bar{S}_{1,1} \cap \mathbb{R} \times \{t\}$ countable for all t. By a duality argument, the same is true for $\bar{S}_{0,2}$. Since $\bar{S}_{0,1} = \mathbb{R}^2$ a.s. by Theorem 5.13, it follows that a.s. for all t, $S_{0,1} \cap \mathbb{R} \times \{t\}$ is of full Lebesgue measure in the line.

Again, of the remaining types, it is enough by duality to consider $(1, 1)$, $(2, 1)$ and $(1, 2)$.

Type (2,1) For any deterministic t and (x_i, t_i) with $t_i < t$, $i = 1, 2$, the probability that two independent Brownian paths starting at (x_i, t_i), $i = 1, 2$, respectively, coalesce exactly at time t is zero. Since $S_{2,1}$ is in the skeleton, $S_{2,1} \cap \mathbb{R} \times \{t\} = \emptyset$ almost surely. Now, for any t, $|S_{2,1} \cap \mathbb{R} \times \{t\}| > 1$ implies that there are four independent Brownian paths starting at different points, and such that the coalescence time of the first two and that of the last two are the same. That this has zero probability implies that a.s. for all t, $|S_{2,1} \cap \mathbb{R} \times \{t\}| \leq 1$.

Type (1,2) For any deterministic t, $S_{1,2} \cap \mathbb{R} \times \{t\} = \emptyset$ almost surely, since the probability of two fixed paths, one forward, one backward, meeting at a given deterministic time is 0. Indeed, from the analysis of type $(1,2)$ done above in the proof of Theorem 5.15, this is because the probability that a Brownian motion has a record value at a given deterministic time is 0. For any t, $|S_{1,2} \cap \mathbb{R} \times \{t\}| > 1$ implies that there exist in the double Brownian web skeleton two pairs, each consisting of one forward and one backward path, such that in both pairs the forward and backward paths meet at the same time. We claim that this has zero probability and thus that $|S_{1,2} \cap \mathbb{R} \times \{t\}| \leq 1$ almost surely. To verify the claim, we again use the analysis of type $(1,2)$ done for Theorem 5.15, which shows that it suffices to prove that there is zero probability that two independent standard Brownian motions B_1, B_2 have a common strictly positive record time. But, as noted in Appendix C, this is the same as having zero probability for B_1, B_2 to both have a zero at a common strictly positive time. This latter probability is indeed zero because of the well known fact that the two-dimensional Brownian motion (B_1, B_2) a.s. does not return to $(0,0)$.

Type (1,1) Since points with $m_{\text{in}} \geq 1$ are a.s. in the skeleton, $\bar{S}_{1,1} \cap \mathbb{R} \times \{t\}$ is a.s. countable (and easily seen to be dense) for every $t \in \mathbb{R}$. Now the previous parts of the proof imply that the same holds for $S_{1,1} \cap \mathbb{R} \times \{t\}$ for every $t \in \mathbb{R}$.

6 General convergence results

In this section, we state and prove Theorem 6.3, which is an extension of our convergence result for noncrossing paths, Theorem 2.3, to the case where paths can cross (before the scaling limit has been taken). At the end of the section, we show that the noncrossing Theorem 2.3 follows from Theorem 6.3 and other results.

Before stating our general theorem that allows crossing, we briefly discuss some systems with crossing paths, to which it should be applicable. Consider the stochastic difference equation (1.1) where the $\Delta_{i,j}$'s are i.i.d. integer-valued random variables, with zero mean and finite nonzero variance. Allowing (i,j) to be arbitrary in $\mathbb{Z}^2$, we obtain as a natural generalization of Figure 1, a collection of random piecewise linear paths that can cross each other, but that still coalesce when they meet at a lattice point in $\mathbb{Z}^2$.

With the natural choice of diffusive space-time scaling and under an irreducibility condition (to insure that the walks from any two starting points have a strictly positive probability of coalescing), the scaling limit of such a discrete time system should be the standard Brownian web. To see what happens in reducible cases, consider simple random walks ($\Delta_{i,j} = \pm 1$), where the paths on the even and odd subsets of $\mathbb{Z}^2$ are independent of each other, and so the scaling limit on all of $\mathbb{Z}^2$ consists of the union of two independent Brownian webs. For $\Delta_{i,j} = \pm 2$, the limit would be the union of four independent Brownian webs. We remark that for continuous time random walks (as discussed in the next section of this paper for $\Delta_{i,j} = \pm 1$), no irreducibility condition is needed.

We proceed with some definitions needed for our general convergence theorem. For $a, b, t_0 \in \mathbb{R}$, $a < b$, and $t > 0$, we define two real-valued measurable functions $l_{t_0,t}([a,b])$ and $r_{t_0,t}([a,b])$ on $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$ as follows. For $K \in \mathcal{H}$, $l_{t_0,t}([a,b])$ evaluated at K is defined as $\inf\{x \in [a,b] | \exists y \in \mathbb{R}$ and a path in K which touches both (x, t_0) and $(y, t_0+t)\}$ and $r_{t_0,t}([a,b])$ is defined similarly with the inf replaced by sup. We also define the following functions on $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$ whose values are subsets of $\mathbb{R}$. As before we let $K \in \mathcal{H}$ and suppress K on the left hand side of the formula for ease of notation.

$$\mathcal{N}_{t_0,t}([a,b]) = \{y \in \mathbb{R} \,|\, \exists\, x \in [a,b] \text{ and a path in } K \text{ which touches both } (x, t_0) \text{ and } (y, t_0 + t)\} \tag{6.1}$$

$$\mathcal{N}^-_{t_0,t}([a,b]) = \{y \in \mathbb{R} \,|\, \text{ there is a path in } K \text{ which touches both } (l_{t_0,t}([a,b]), t_0) \text{ and } (y, t_0 + t)\} \tag{6.2}$$

$$\mathcal{N}^+_{t_0,t}([a,b]) = \{y \in \mathbb{R} \,|\, \text{ there is a path in } K \text{ which touches both } (r_{t_0,t}([a,b]), t_0) \text{ and } (y, t_0 + t)\} \tag{6.3}$$

Remark 6.1 *We notice that* $|\mathcal{N}_{t_0,t}([a,b])| = \eta(t_0, t; a, b)$.

Let $\{\mathcal{X}_m\}$ be a sequence of $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued random variables with distributions $\{\mu_m\}$. We define conditions $(B_1'), (B_2')$ as follows.

(B_1') $\forall \beta > 0,\ \limsup_{m\to\infty} \sup_{t>\beta} \sup_{t_0, a \in \mathbb{R}} \mu_m(|\mathcal{N}_{t_0,t}([a-\epsilon, a+\epsilon])| > 1) \to 0$ as $\epsilon \to 0^+$

(B_2') $\forall \beta > 0,\ \frac{1}{\epsilon} \limsup_{m\to\infty} \sup_{t>\beta} \sup_{t_0, a \in \mathbb{R}} \mu_m(\mathcal{N}_{t_0,t}([a-\epsilon, a+\epsilon]) \neq \mathcal{N}^+_{t_0,t}([a-\epsilon, a+\epsilon]) \cup \mathcal{N}^-_{t_0,t}([a-\epsilon, a+\epsilon])) \to 0$ as $\epsilon \to 0^+$

Remark 6.2 *Note that if we consider a process with non-crossing paths then conditions* (B_1') *and* (B_2') *follow from conditions* (B_1) *and* (B_2) *respectively because of the following monotonicity property. For all* $a < b, t_0$ *and* $0 < s < t$

$$\mathbb{P}(|\eta(t_0, t; a, b)| \geq k) \leq \mathbb{P}(|\eta(t_0, s; a, b)| \geq k)$$

for all $k \in \mathbb{N}$.

Theorem 6.3 *Suppose that* $\{\mu_m\}$ *is tight. If Conditions* $(I_1), (B_1')$ *and* (B_2') *hold, then* $\{\mathcal{X}_m\}$ *converges in distribution to the Brownian web* $\bar{\mathcal{W}}$.

Remark 6.4 *There is a natural analogue to (and corollary of) this theorem for the double Brownian web, in which* (I_1) *is replaced by Condition* (I_1^D) *of Theorem 7.2 below and* (B_1'), (B_2') *and their backward analogues are separately valid. The proof of this corollary is like that of Theorem 7.2, except based on the general convergence theorem for the (forward) Brownian web rather than the special convergence theorem for the noncrossing case.*

Theorem 6.3 is proved through a series of lemmas.

Lemma 6.1 *Let* μ *be a subsequential limit of* $\{\mu_m\}$ *and suppose that* μ *satisfies condition* (i') *of Theorem 4.6 and*

$$(B_1'') \quad \forall \beta > 0, \ \sup_{t>\beta} \sup_{t_0,a} \mu(|\mathcal{N}_{t_0,t}([a-\epsilon, a+\epsilon])| > 1) \to 0 \ \textit{as} \ \epsilon \to 0^+$$

$$(B_2'') \quad \forall \beta > 0, \ \frac{1}{\epsilon} \sup_{t>\beta} \sup_{t_0,a} \mu(\mathcal{N}_{t_0,t}([a-\epsilon, a+\epsilon]) \neq \mathcal{N}_{t_0,t}^{+}([a-\epsilon, a+\epsilon]) \cup \mathcal{N}_{t_0,t}^{-}([a-\epsilon, a+\epsilon])) \to 0 \ \textit{as} \ \epsilon \to 0^+.$$

Then μ *is the distribution of the Brownian web.*

Proof It follows from condition (i') and (B_1'') that the limiting random variable $\mathcal{X}$ satisfies condition (i) of the characterization theorem 2.1. That is (μ) almost surely there is exactly one path starting from each point of $\mathcal{D}$ and these paths are distributed as coalescing Brownian motions. Let us define a $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued random variable $\mathcal{X}'$ on the same probability space as the one on which $\mathcal{X}$ is defined to be the closure in (Π, d) of the paths of $\mathcal{X}$ starting from $\mathcal{D}$. We will denote probabilities in the common probability space by $\mathbb{P}$. $\mathcal{X}'$ has the distribution of $\bar{\mathcal{W}}$. We need to show that it also satisfies condition

(ii''') of Theorem 4.6. Let $a < b, t_0 \in \mathbb{R}$ and $t > 0$ be given. For the random variable $\mathcal{X}$ we will denote the counting random variable $\eta(t_0, t; a, b)$ by η and the corresponding variable for $\mathcal{X}'$ by η'. Let $z_j = (a + j(b-a)/M, t_0)$ for $j = 0, 1, \ldots, M$, be $M+1$ equally spaced points in the interval $[a, b]$.

Now define $\eta_M = |\{x \in \mathbb{R} | \exists$ a path in $\mathcal{X}$ which touches both a point in $\{z_0, \ldots, z_M\}$ and $(x, t+t_0)\}|$, where $|\cdot|$ stands for cardinality. Let η'_M be the corresponding random variable for $\mathcal{X}'$. Clearly $\eta \geq \eta_M$ and $\eta' \geq \eta'_M$. From (B''_1) it follows that $\eta_M = \eta'_M$ almost surely. Now let $\epsilon = \frac{(b-a)}{M}$. By condition (B''_2), letting $M \to \infty$ $(\epsilon \to 0)$, we obtain

$$\mathbb{P}(\eta > \eta'_M) = \mathbb{P}(\eta > \eta_M) \to 0 \text{ as } M \to \infty.$$

Thus, $\mathbb{P}(\eta > \eta') = 0$, showing that η is stochastically dominated by η'. This completes the proof of the lemma.

For $t > 0$, $\epsilon > 0$, $0 < \epsilon' < \frac{\epsilon}{8}$, $0 \leq \delta < \frac{t}{2}$, consider the following event.

$$\begin{gathered} O(a, t_0, t, \epsilon, \epsilon', \delta) = \\ \{K \in \mathcal{H} | \text{ there are three paths } (x_1(t), t_1), (x_2(t), t_2), (x_3(t), t_3) \text{ in } K \\ \text{with } t_1, t_2, t_3 < t_0 + \delta, x_1(t_0+\delta) \in (a - \epsilon - \epsilon', a - \epsilon + \epsilon'), \\ x_2(t_0+\delta) \in (a - \epsilon + 2\epsilon', a + \epsilon - 2\epsilon'), x_3(t_0+\delta) \in (a + \epsilon - \epsilon', a + \epsilon + \epsilon') \\ \text{and } x_2(t_0+t) \neq x_1(t_0+t), x_2(t_0+t) \neq x_3(t_0+t)\} \end{gathered}$$

Lemma 6.2 (B''_2) *in Lemma 6.1 can be replaced by:*

$$(B'''_2)\ \forall \beta > 0,\ \frac{1}{\epsilon} \limsup_{\epsilon' \to 0} \sup_{t > \beta} \sup_{t_0, a} \limsup_{\delta \to 0} \mu(O(a, t_0, t, \epsilon, \epsilon', \delta)) \to 0 \text{ as } \epsilon \to 0^+.$$

Proof We prove the lemma by showing that conditions (i') and (B''_1) together with (B'''_2) imply condition B''_2. Let $\beta > 0$. Define $C_1(b, t_0, \epsilon', \delta)$ as

$$\begin{aligned} \{K \in \mathcal{H} | & \text{ there is a path in } K \text{ which touches both } (b, t_0) \\ & \text{and } \{b - \epsilon'\} \times [t_0, t_0+\delta] \cup \{b + \epsilon'\} \times [t_0, t_0+\delta]\}, \end{aligned}$$

and $C_2(a, t_0, \epsilon, \epsilon', \delta)$ as

$$\begin{aligned} \{K \in \mathcal{H} | & \text{ there is a path in } K \text{ which touches both } [a-\epsilon, a+\epsilon] \times \{t_0\} \\ & \text{and } \{a - \epsilon - \epsilon'\} \times [t_0, t_0+\delta] \cup \{a + \epsilon + \epsilon'\} \times [t_0, t_0+\delta]\}. \end{aligned}$$

Now observe that (modulo sets of zero μ measure)

$$\begin{aligned}
&\{\mathcal{N}_{t_0,t}([a-\epsilon,a+\epsilon]) \neq \mathcal{N}^+_{t_0,t}([a-\epsilon,a+\epsilon]) \cup \mathcal{N}^-_{t_0,t}([a-\epsilon,a+\epsilon])\} \\
&\cap C_1^c(a+\epsilon,t_0,\epsilon',\delta) \cap C_1^c(a-\epsilon,t_0,\epsilon',\delta) \cap C_2^c(a,t_0,\epsilon,\epsilon',\delta) \\
&\cap\{|\mathcal{N}_{t_0+\delta,t-\delta}([a-\epsilon-2\epsilon',a-\epsilon+2\epsilon'])|=1\} \\
&\cap\{|\mathcal{N}_{t_0+\delta,t-\delta}([a+\epsilon-2\epsilon',a+\epsilon+2\epsilon'])|=1\} \\
&\subseteq O(a,t_0,t,\epsilon,\epsilon',\delta).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\mu(\mathcal{N}_{t_0,t}([a-\epsilon,a+\epsilon]) \neq \mathcal{N}^+_{t_0,t}([a-\epsilon,a+\epsilon]) \cup \mathcal{N}^-_{t_0,t}([a-\epsilon,a+\epsilon])) \\
&\leq \mu(O(a,t_0,t,\epsilon,\epsilon',\delta)) + \mu(C_2(a,t_0,\epsilon,\epsilon',\delta)) + \mu(C_1(a+\epsilon,t_0,\epsilon',\delta)) \\
&+ \mu(C_1(a-\epsilon,t_0,\epsilon',\delta)) + \mu(|\mathcal{N}_{t_0+\delta,t-\delta}([a-\epsilon-2\epsilon',a-\epsilon+2\epsilon'])|>1) \\
&+ \mu(|\mathcal{N}_{t_0+\delta,t-\delta}([a+\epsilon-2\epsilon',a+\epsilon+2\epsilon'])|>1).
\end{aligned}$$

Letting $\delta \to 0$, we obtain

$$\begin{aligned}
&\mu(\mathcal{N}_{t_0,t}([a-\epsilon,a+\epsilon]) \neq \mathcal{N}^+_{t_0,t}([a-\epsilon,a+\epsilon]) \cup \mathcal{N}^-_{t_0,t}([a-\epsilon,a+\epsilon])) \leq \\
&\limsup_{\delta\to 0}\{\mu(O(a,t_0,t,\epsilon,\epsilon',\delta)) + \mu(C_2(a,t_0,\epsilon,\epsilon',\delta)) + \mu(C_1(a+\epsilon,t_0,\epsilon',\delta)) \\
&+ \mu(C_1(a-\epsilon,t_0,\epsilon',\delta)) + \mu(|\mathcal{N}_{t_0+\delta,t-\delta}([a-\epsilon-2\epsilon',a-\epsilon+2\epsilon'])|>1) \\
&+ \mu(|\mathcal{N}_{t_0+\delta,t-\delta}([a+\epsilon-2\epsilon',a+\epsilon+2\epsilon'])|>1)\}. \qquad (6.4)
\end{aligned}$$

Now,

$$\begin{aligned}
\lim_{\delta\to 0}\mu(C_1(a+\epsilon,t_0,\epsilon',\delta)) &= \lim_{\delta\to 0}\mu(C_1(a-\epsilon,t_0,\epsilon',\delta)) \\
&= \lim_{\delta\to 0}\mu(C_2(a,t_0,\epsilon,\epsilon',\delta)) = 0,
\end{aligned}$$

since elements of $\mathcal{H}$ are compact subsets of Π, and paths in Π cannot have completely flat segments.

Now since, $t-\delta > \frac{t}{2} > \frac{\beta}{2}$, it follows from (B_1'') that for all $\gamma > 0$,

$$\begin{aligned}
&\sup_{t>\beta}\sup_{a,t_0}\sup_{0<\delta<\frac{t}{2}} \mu(|\mathcal{N}_{t_0+\delta,t-\delta}([a-\gamma,a+\gamma])|>1|) \\
\leq\ &\sup_{t>\frac{\beta}{2}}\sup_{a,t_0}\mu(|\mathcal{N}_{t_0,t}([a-\gamma,a+\gamma])|>1|) \to 0 \text{ as } \gamma\to 0.
\end{aligned}$$

This implies that for all $\epsilon > 0$,

$$\lim_{\epsilon' \to 0} \sup_{t>\beta} \sup_{a,t_0} \limsup_{\delta \to 0} \mu(|\mathcal{N}_{t_0+\delta,t-\delta}([a \pm \epsilon - 2\epsilon', a \pm \epsilon + 2\epsilon'])| > 1) = 0. \quad (6.5)$$

Together with (6.4), this gives us

$$\begin{aligned} &\sup_{t>\beta} \sup_{t_0,a} \mu(\mathcal{N}_{t_0,t}([a-\epsilon, a+\epsilon]) \neq \mathcal{N}^+_{t_0,t}([a-\epsilon, a+\epsilon]) \cup \mathcal{N}^-_{t_0,t}([a-\epsilon, a+\epsilon])) \\ &\qquad \leq \lim_{\epsilon' \to 0} \sup_{t>\beta} \sup_{a,t_0,} \limsup_{\delta \to 0} \mu(O(a, t_0, t, \epsilon, \epsilon', \delta)) \qquad (6.6) \end{aligned}$$

Now, using (B_2'''), we obtain

$$\frac{1}{\epsilon} \sup_{t>\beta} \sup_{t_0,a} \mu(\mathcal{N}_{t_0,t}([a-\epsilon, a+\epsilon]) \neq \mathcal{N}^+_{t_0,t}([a-\epsilon, a+\epsilon]) \cup \mathcal{N}^-_{t_0,t}([a-\epsilon, a+\epsilon])) \to 0$$

as $\epsilon \to 0^+$, proving the lemma.

Proof of Theorem 6.3 Tightness implies that every subsequence of $\{\mu_m\}$ has a subsequence converging to some μ. Let us denote the corresponding limiting random variable by $\mathcal{X}$. We prove the theorem by showing that every such $\mu = \mu_{\bar{\mathcal{W}}}$. From Lemmas 6.1 and 6.2 it follows that it is sufficient to prove condition (i') of Theorem 4.6, condition (B_1'') and condition (B_2''').

Let $\beta > 0$ and define for all $0 \leq \delta < t/2$, $\mathcal{N}'^{\delta}_{t_0,t}([a,b]) = \{y \in \mathbb{R} | \exists$ a path $(x(s), s_0), s_0 < t_0+\delta$ in K such that $x(t_0+\delta) \in (a,b)$ and $x(t_0+t) = y\}$. We note that the set $\{|\mathcal{N}'^{\delta}_{t_0,t}([a,b])| > 1\}$ is an open subset of $\mathcal{H}$ for all $\delta \geq 0$. Then we have

$$\begin{aligned} &\sup_{t>\beta} \sup_{t_0,a} \mu(|\mathcal{N}_{t_0,t}([a-\epsilon, a+\epsilon])| > 1) \\ \leq\; &\sup_{t>\beta} \sup_{t_0,a} \limsup_{\delta \to 0} \{\mu(|\mathcal{N}'^{\delta}_{t_0,t}([a-2\epsilon, a+2\epsilon])| > 1) \\ +\; &\mu(C_2(a, t_0, \epsilon, \epsilon, \delta))\} \\ \leq\; &\sup_{t>\frac{\beta}{2}} \sup_{t_0,a} \mu(|\mathcal{N}'^{0}_{t_0,t}([a-2\epsilon, a+2\epsilon])| > 1) \\ +\; &\sup_{t_0,a} \limsup_{\delta \to 0} \mu(C_2(a, t_0, \epsilon, \epsilon, \delta)) \end{aligned}$$

Now, $\limsup_{\delta \to 0} \mu(C_2(a, t_0, \epsilon, \epsilon, \delta)) = 0$, since elements of $\mathcal{H}$ are compact subsets of Π. This together with the fact that $\{|\mathcal{N}'^{\delta}_{t_0,t}([a,b])| > 1\}$ is an open subset of $\mathcal{H}$ leads to

$$
\begin{aligned}
& \sup_{t>\beta}\sup_{t_0,a}\mu(|\mathcal{N}_{t_0,t}([a-\epsilon,a+\epsilon])|>1) \\
\le\ & \sup_{t>\frac{\beta}{2}}\sup_{t_0,a}\mu(|\mathcal{N}'^0_{t_0,t}([a-2\epsilon,a+2\epsilon])|>1) \\
\le\ & \sup_{t>\frac{\beta}{2}}\sup_{t_0,a}\limsup_{m}\mu_m(|\mathcal{N}'^0_{t_0,t}([a-2\epsilon,a+2\epsilon])|>1) \\
\le\ & \sup_{t>\frac{\beta}{2}}\sup_{t_0,a}\limsup_{m}\mu_m(|\mathcal{N}_{t_0,t}([a-2\epsilon,a+2\epsilon])|>1) \\
\le\ & \limsup_{m}\sup_{t>\frac{\beta}{2}}\sup_{t_0,a}\mu_m(|\mathcal{N}_{t_0,t}([a-2\epsilon,a+2\epsilon])|>1).
\end{aligned}
$$

It follows from (B_1') that

$$
\limsup_{m}\sup_{t>\frac{\beta}{2}}\sup_{t_0,a}\mu_m(|\mathcal{N}_{t_0,t}([a-2\epsilon,a+2\epsilon])|>1)\to 0 \text{ as } \epsilon\to 0^+.
$$

This proves (B_1''), which implies that

- (o) starting from any deterministic point, there is μ-almost surely only a single path in $\mathcal{X}$.

Combining this with (I_1), we readily obtain that

- (i) the finite-dimensional distributions of $\mathcal{X}$ are those of coalescing Brownian motions with unit diffusion constant.

Condition (i') of Theorem 4.6 follows immediately from (o) and (i). Now we proceed to verify condition (B_2''').

We have

$$
\begin{aligned}
& \sup_{t>\beta}\sup_{t_0,a}\limsup_{\delta\to 0}\mu(O(a,t_0,t,\epsilon,\epsilon',\delta)) \\
\le\ & \sup_{t>\frac{\beta}{2}}\sup_{a,t_0}\mu(O(a,t_0,t,\epsilon,\epsilon',0)) \\
\le\ & \limsup_{m}\sup_{t>\frac{\beta}{2}}\sup_{a,t_0}\mu_m(O(a,t_0,t,\epsilon,\epsilon',0)) \\
\le\ & \limsup_{m}\sup_{t>\frac{\beta}{2}}\sup_{a,t_0}\mu_m(\mathcal{N}_{t_0,t}([a-\epsilon-\epsilon',a+\epsilon+\epsilon']) \\
& \neq \mathcal{N}^+_{t_0,t}([a-\epsilon-\epsilon',a+\epsilon+\epsilon'])\cup\mathcal{N}^-_{t_0,t}([a-\epsilon-\epsilon',a+\epsilon+\epsilon'])),
\end{aligned}
$$

where the second inequality follows from the fact that $O(a, t_0, t, \epsilon, \epsilon', 0)$ is an open subset of $\mathcal{H}$. For the third inequality to hold we need to insure that there is no more than one path touching either $(a-\epsilon-\epsilon', t_0)$ or $(a+\epsilon+\epsilon', t_0)$; this follows from (B_1'). Since $\epsilon' < \frac{\epsilon}{8}$, $\epsilon + \epsilon' \to 0$ as $\epsilon \to 0$, using condition (B_2'), we obtain

$$\frac{1}{\epsilon} \limsup_{\epsilon' \to 0} \sup_{t_0, a} \limsup_{\delta \to 0} \mu(O(a, t_0, t, \epsilon, \epsilon', \delta)) \to 0 \text{ as } \epsilon \to 0^+,$$

proving condition (B_2'''). This completes the proof of the theorem.

We now suppose that $\mathcal{X}_1, \mathcal{X}_2, \ldots$ is a sequence of $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued random variables so that each $\mathcal{X}_i$ consists of *noncrossing* paths. The noncrossing condition produces a considerable simplification of Theorem 5.1; namely Theorem 2.3.

Proof of Theorem 2.3 This is an immediate consequence of Remark 6.2, Theorem 6.3 and Proposition B.3 of the appendix.

7 Convergence for coalescing random walks

We now apply Theorem 2.3 to coalescing random walks. For that, we begin by precisely defining Y (resp., $\tilde{Y}$), the set of all discrete (resp., continuous) time coalescing random walks on $\mathbb{Z}$. For δ an arbitrary positive real number, we obtain sets of rescaled walks, $Y^{(\delta)}$ and $\tilde{Y}^{(\delta)}$, by the usual rescaling of space by δ and time by δ^2. The (main) paths of Y are the discrete-time random walks Y_{y_0,s_0}, as described in the Introduction and shown in Figure 1, with $(y_0, s_0) = (i_0, j_0) \in \mathbb{Z} \times \mathbb{Z}$ arbitrary except that $i_0 + j_0$ must be even. Each random walk path goes from (i, j) to $(i \pm 1, j + 1)$ linearly. In addition to these, we add some boundary paths so that Y will be a compact subset of Π. These are all the paths of the form (f, s_0) with $s_0 \in \mathbb{Z} \cup \{-\infty, \infty\}$ and $f \equiv \infty$ or $f \equiv -\infty$. Note that for $s_0 = -\infty$ there are two different paths starting from the single point at $s_0 = -\infty$ in $\bar{\mathbb{R}}^2$.

The continuous time $\tilde{Y}$ can be defined similarly, except that here y_0 is any $i_0 \in \mathbb{Z}$ and s_0 is arbitrary in $\mathbb{R}$. Continuous time walks are normally seen as jumping from i to $i \pm 1$ at the times $T_k^{(i)} \in (-\infty, \infty)$ of a rate one Poisson process. If the jump is, say, to $i + 1$, then our polygonal path will have a linear segment between $(i, T_k^{(i)})$ and $(i + 1, T_{k'}^{(i+1)})$, where $T_{k'}^{(i+1)}$ is the first Poisson event at $i + 1$ after $T_k^{(i)}$. Furthermore, if $T_k^{(i_0)} < s_0 < T_{k+1}^{(i_0)}$, then there will be a constant segment in the path before the first nonconstant linear segment. If $s_0 = T_k^{(i_0)}$, then we take two paths: one with an initial constant segment and one without.

Theorem 7.1 *Each of the collections of rescaled coalescing random walk paths, $Y^{(\delta)}$ (in discrete time) and $\tilde{Y}^{(\delta)}$ (in continuous time) converges in distribution to the standard Brownian web as $\delta \to 0$.*

Proof By Theorem 2.3, it suffices to verify conditions I_1, B_1 and B_2.

Condition I_1 is basically a consequence of the Donsker invariance principle, as already noted in the Introduction. Conditions B_1 and B_2 follow from the coalescing walks version of the inequality of (4.3), which is

$$\mu_\delta(\eta(t_0, t; a, a+\epsilon) \geq k) \ \leq \ [\mu_\delta(\eta(t_0, t; a, a+\epsilon) \geq 2)]^{k-1}. \qquad (7.1)$$

Taking the sup over (a, t_0) and the lim sup over δ and using standard random walk arguments produces an upper bound of the from $C_k(\epsilon/\sqrt{t})^{k-1}$ which yields B_1 and B_2 as desired.

We next state extensions of Theorems 2.3 and 7.1 to the Double Brownian web.

Theorem 7.2 *Suppose $\mathcal{X}_1^D = (\mathcal{X}_1, \mathcal{X}_1^b), \mathcal{X}_2^D = (\mathcal{X}_2, \mathcal{X}_2^b), \ldots$ are $(\mathcal{H}^D, \mathcal{F}_{\mathcal{H}^D})$-valued random variables such that each $\mathcal{X}_n$ consists only of noncrossing paths and similarly for each $\mathcal{X}_n^b$. Suppose further that there is a dense countable deterministic subset $\mathcal{D}$ of $\mathbb{R}^2$ such that*

(I_1^D) *there exists $\theta_n^y \in \mathcal{X}_n$ and $\theta_n^{y\,b} \in \mathcal{X}_n^b$ such that for any deterministic $y_1, \ldots, y_m \in \mathcal{D}, \{\theta_n^{y_1}, \theta_n^{y_1\,b}, \ldots, \theta_n^{y_m}, \theta_n^{y_m\,b}\}$ converge in distribution to coalescing/reflecting forward/backward Brownian motions.*

If in addition conditions (B_1) and (B_2) of Theorem 2.3 above are valid for $\hat{\eta}$ and $\hat{\eta}^b$ (separately), then $\mathcal{X}_n^D$ converges in distribution to the double Brownian web.

Proof This could be proved as a corollary of a general (i.e., allowing paths that cross) convergence to the double Brownian web result (see Remark 6.4); instead we prove it as a corollary to Theorem 2.3 as follows. By Prop. B.3, (I_1^D) implies tightness separately for the forward and backward processes; this, in turn, implies tightness of the joint process. This and (I_1^D) imply that any subsequential limit $\mathcal{X}^D = (\mathcal{X}, \mathcal{X}^b)$ of $(\mathcal{X}_n^D)$ contains a version of the DBW, say $\mathcal{Y}^D = (\mathcal{Y}, \mathcal{Y}^b)$. But, from Theorem 2.3, separate convergence of

$\mathcal{X}_n$ and $\mathcal{X}_n^b$ to the forward and backward BW's, respectively, implies that $\mathcal{X}$ and $\mathcal{X}^b$ are versions of the forward and backward BW's, respectively. Now, Theorem 3.6 implies that $\mathcal{X} = \mathcal{Y}$ and $\mathcal{X}^b = \mathcal{Y}^b$ (almost surely), and the argument is complete.

We now consider the joint convergence of forward coalescing random walks and the dual backward coalescing random walks. The backward (dual) $Y^{b,(\delta)}$ and $\tilde{Y}^{b,(\delta)}$ processes for discrete and continuous time random walks can be defined in a straightforward way. In discrete time, they are polygonal paths on the dual lattice of all (i, j) with $i + j$ *odd*; in continuous time, they are the paths of walkers on the dual lattice $\mathbb{Z} + \frac{1}{2}$ (see [14] and [4, 5] for more details).

Theorem 7.3 *$(Y^{(\delta)}, \tilde{Y}^{b,(\delta)})$ and $(\tilde{Y}^{(\delta)}, \tilde{Y}^{b,(\delta)})$ converge in distribution as $\delta \to 0$ to the double Brownian web.*

Proof The validity of (I_1^D) is shown in Subsubsection 3.3.6 of [14]; (B_1) and (B_2) for the forward and backward processes separately have already been shown.

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A Some measurability issues

Let $(\mathcal{H}, d_{\mathcal{H}})$ denote the Hausdorff metric space induced by (Π, d). $\mathcal{F}_{\mathcal{H}}$ denotes the σ–field generated by the open sets of $\mathcal{H}$. We will consider now *cylinders* of $\mathcal{H}$. Let us fix nonempty horizontal segments $I_1, \ldots, I_n$ in $\mathbb{R}^2$ (i.e., $I_k =$

$I'_k \times \{t_k\}$), where each I'_k is an interval (which need not be finite and can be open, closed or neither) and $t_k \in \mathbb{R}$. Define

$$
\begin{aligned}
C^{t_0}_{I_1,\dots,I_n} &:= \{K \in \mathcal{H} : \text{ there exists } (f,t) \in K \text{ such that} \\
&\qquad t > t_0 \text{ and } (f,t) \text{ goes through } I_1,\dots,I_n\} \qquad \text{(A.1)} \\
\bar{C}^{t_0}_{I_1,\dots,I_n} &:= \{K \in \mathcal{H} : \text{ there exists } (f,t) \in K \text{ such that} \\
&\qquad t \geq t_0 \text{ and } (f,t) \text{ goes through } I_1,\dots,I_n\} \qquad \text{(A.2)} \\
C_{I_1,\dots,I_n} &:= \{K \in \mathcal{H} : \text{ there exists } (f,t) \in K \text{ such that} \\
&\qquad (f,t) \text{ goes through } I_1,\dots,I_n\}. \qquad \text{(A.3)}
\end{aligned}
$$

We will call sets of the form (A.1) *open cylinders* if each I_k is open, and sets of the of the form (A.2) *closed cylinders* if each I_k is closed.

Remark A.1 *It is easy to see that sets of the form (A.1-A.3) for arbitrary* $I_1,\dots,I_n$ *can be generated by open cylinders.*

Let now $\mathfrak{C}$ be the σ–field generated by the open cylinders.

Proposition A.2 $\mathcal{F}_{\mathcal{H}} = \mathfrak{C}$

The proposition is a consequence of the following two lemmas.

Lemma A.1 $\mathcal{F}_{\mathcal{H}} \supset \mathfrak{C}$

Proof It is enough to observe that the open cylinders are open sets of $\mathcal{H}$. Indeed, take an open cylinder, an element K in that cylinder, and $(f,t) \in K$ such that $t > t_0$ and $a_i < f(t_i) < b_i$ for all $i = 1,\dots,n$. All points of $B_{\mathcal{H}}(K,\epsilon)$, the open ball in $\mathcal{H}$ around K with radius ϵ, contains a path (f',t') in a ball in Π around (f,t) of radius ϵ. Thus by choosing ϵ small enough, (f',t') will satisfy $t_0 < t' < t_i$ and $a_i < f'(t_i) < b_i$ for all $i = 1,\dots,n$.

Lemma A.2 $\mathcal{F}_{\mathcal{H}} \subset \mathfrak{C}$

Proof It is enough to generate the ϵ-balls in $\mathcal{H}$ with cylinders. We will start with ϵ-balls around points of $\mathcal{H}$ consisting of finitely many paths of Π.

We will use the concept of a *cone* in $\mathbb{R}^2$ around (f,t). Let $r^- = r^-(t,\epsilon)$ and $r^+ = r^+(t,\epsilon)$ be the two solutions of

$$|\tanh(r) - \tanh(t)| = \epsilon, \qquad \text{(A.4)}$$

with $r^- \le r^+$. For s fixed, let $x^-(s) = x^-(s,\epsilon)$ and $x^+(s) = x^+(s,\epsilon)$ be the solutions for small ϵ of

$$\frac{\tanh(x) - \tanh(\hat{f}(s))}{|s|+1} = \pm\epsilon, \tag{A.5}$$

with $x^-(s) \le x^+(s)$. The cone around (f,t) is defined as

$$\mathbb{C} := \{(x,y) \in \mathbb{R}^2 :\ x^-(y) \le x \le x^+(y),\ y \ge r^-\} \tag{A.6}$$

Now let $K_0 = \{(f_1,t_1),\ldots,(f_n,t_n)\}$. Let $\mathbb{C}_1,\ldots,\mathbb{C}_n$ be the respective cones of $(f_1,t_1),\ldots,(f_n,t_n)$. For $i = 1,\ldots,n$, let $r_i^+ = r^+(t_i,\epsilon)$, $r_i^- = r^-(t_i,\epsilon)$.

Consider now a family of horizontal lines $\{L_1,L_2,\ldots\} = \mathbb{R}\times\mathcal{S}$, where $\mathcal{S} = \{s_1,s_2,\ldots\}$, with the s_k's distinct and such that $\cup_{k\ge1}L_k$ is dense in $\mathbb{R}^2$. For fixed k consider the segments (of nonzero length) I_k^i into which L_k is divided by all the points of the form $x_i^-(s_k)$ and $x_i^+(s_k)$ for $i = 1,\ldots,n$ (number of such segments $\le 2n+1$). Segments with interior points in some cone are closed; otherwise, they are open.

Let $\mathcal{I} = \{I_{k_1}^{i_1},\ldots,I_{k_m}^{i_m}\}$ be any finite sequence of the intervals defined above, with $k_1,\ldots,k_m$ distinct. For $1 \le i \le n$, we will say that $\mathcal{I}$ is *i-good* if $\mathbb{C}_i$ contains all the intervals in $\mathcal{I}$. If $\mathcal{I}$ is not i-good for any $1 \le i \le n$, then $\mathcal{I}$ is *bad*. Let

$$\begin{aligned}\hat{C}_i \ &:= \ \{K \in \mathcal{H} :\ \text{there exists } (f,t) \in K \text{ such that} \\ &\qquad\qquad t \in [r_i^-, r_i^+] \text{ and } \{f(s)\}\times\{s\} \in \mathbb{C}_i \text{ for all } s \ge t\} &\text{(A.7)}\\ &= \ \{K \in \mathcal{H} :\ \text{there exists } (f,t) \in K \text{ such that} \\ &\qquad\qquad d((f,t),(f_i,t_i)) \le \epsilon\}. &\text{(A.8)}\end{aligned}$$

It is not hard to see that $\hat{C}_i$ belongs to $\mathfrak{C}$ by writing $[r_i^-, r_i^+]$ as a finite union of small subintervals and then approximating $\hat{C}_i$ by a finite union of sets of the form $\bar{C}_{\mathcal{I}}^s$, where: $\mathcal{I} = \{I_1,\ldots,I_m\}$; $I_\ell = [x_i^-(s'_\ell), x_i^+(s'_\ell)] \times \{s'_\ell\}$; $s \le s'_1 \le s'_2 \le \ldots$; $s, s'_1 \in [r_i^-, r_i^+]$; and each $s'_\ell \in \mathcal{S}$. Note that in the definition (A.2) of such a $\bar{C}_{\mathcal{I}}^s$, the starting time t of the path (f,t) must be in $[s,s'_1]$. Define $\hat{C} := \cap_{i=1}^n \hat{C}_i$, $i = 1,\ldots,n$. Then,

$$\overline{B_{\mathcal{H}}(K_0,\epsilon)} = \left[\left(\bigcup_{m,\mathcal{I}:\mathcal{I}\text{ is bad}} C_{\mathcal{I}}\right)\bigcup\left(\bigcup_{i=1}^{n}\ \bigcup_{\substack{m,\mathcal{I},k:\ \mathcal{I}\text{ is } i\text{-good and}\\ s_k > \sup_j(r_j^+:\ \mathcal{I}\text{ is } j\text{-good})}} C_{\mathcal{I}}^{s_k}\right)\right]^c \bigcap \hat{C}. \tag{A.9}$$

We point out that the expression inside the square brackets on the right hand side is the set of all points $K \in \mathcal{H}$ such that for some $i = 1, \ldots, n$, there is a path in K at a distance greater than ϵ from (f_i, t_i). Bad $\mathcal{I}$'s, in the first term, ensure that some path in K is at distance greater than ϵ of (f_i, t_i) for some $i = 1, \ldots, n$ spatially. In the second term, some path in K starts at a distance greater than ϵ from the starting time of some (f_i, t_i). $\hat{C}$ is the set of all $K \in \mathcal{H}$ with the property, that for all $1 \le i \le n$, K contains a path which is within ϵ of (f_i, t_i).

To generate $B_{\mathcal{H}}(K, \epsilon)$ for arbitrary $K \in \mathcal{H}$, we approximate $B_{\mathcal{H}}(K, \epsilon)$ by an increasing sequence of balls around $\tilde{K}$'s consisting of finitely many paths. For that, we note that, by compactness of K, for every integer $j > 1$, there exists $K_j \in \mathcal{H}$ consisting of finitely many paths such that $K_j \subset K$ (as subsets of Π) and $d_{\mathcal{H}}(K, K_j) < \epsilon/j$ for all $j > 1$. We then have

$$B_{\mathcal{H}}(K, (1 - 2/j)\epsilon) \subset B_{\mathcal{H}}(K_j, (1 - 1/j)\epsilon) \subset B_{\mathcal{H}}(K, \epsilon). \tag{A.10}$$

The first inclusion is justified as follows. Let $K' \in B_{\mathcal{H}}(K, (1 - 1/j)\epsilon)$. Then $d_{\mathcal{H}}(K, K') < (1 - 1/j)\epsilon$ and, by the triangle inequality,

$$d_{\mathcal{H}}(K_j, K') \le d_{\mathcal{H}}(K_j, K) + d_{\mathcal{H}}(K, K') < \epsilon/j + (1-2/j)\epsilon = (1-1/j)\epsilon. \tag{A.11}$$

Thus $K' \in B_{\mathcal{H}}(K_j, \epsilon)$. The second inclusion is justified similarly. It is clear now that $\cup_{j>1} B_{\mathcal{H}}(K, (1 - 2/j)\epsilon) = \cup_{j>1} B_{\mathcal{H}}(K_j, (1 - 1/j)\epsilon) = B_{\mathcal{H}}(K, \epsilon)$.

B Compactness and tightness

Let $\Lambda_{L,T} = [-L, L] \times [-T, T]$, and let $\{\mu_m\}$ be a sequence of probability measures on $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$. For $x_0, t_0 \in \mathbb{R}$ and $u, t > 0$, let $R(x_0, t_0; u, t)$ denote the rectangle $[x_0 - u/2, x_0 + u/2] \times [t_0, t_0 + t]$ in $\mathbb{R}^2$. Define $A_{t,u}(x_0, t_0)$ to be the event (in $\mathcal{F}_{\mathcal{H}}$) that K (in $\mathcal{H}$) contains a path touching both $R(x_0, t_0; \frac{u}{2}, t)$ and (at a later time) the left or right boundary of the bigger rectangle $R(x_0, t_0; u, 2t)$. See Figure 4.

Our tightness condition is:

$$(T_1) \quad \tilde{g}(t, u; L, T) \equiv t^{-1} \limsup_{m} \sup_{(x_0,t_0)\in\Lambda_{L,T}} \mu_m(A_{t,u}(x_0, t_0)) \to 0 \text{ as } t \to 0+$$

Proposition B.1 *Condition T_1 implies tightness of $\{\mu_m\}$.*

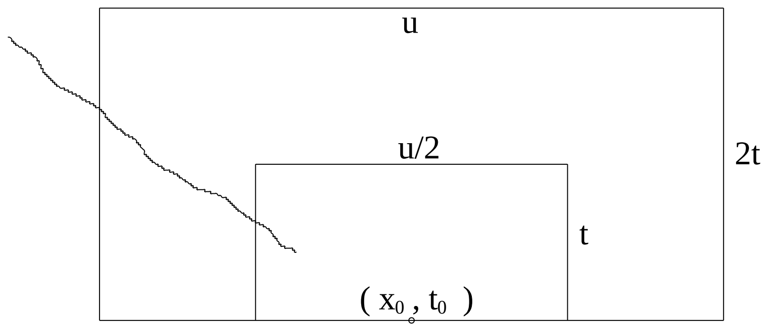

Figure 4: Schematic diagram of a path causing the unlikely event $A_{t,u}(x_0, t_0)$ to occur.

Proof Let

$$g_m(t, u; L, T) = \sup_{(x_0,t_0)\in\Lambda_{L,T}} \mu_m(A_{t,u}(x_0, t_0)).$$

Now define $B_{t,u}(x_0, t_0)$ as the event (in $\mathcal{F}_{\mathcal{H}}$) that K (in $\mathcal{H}$) contains a path which touches a point $(x', t') = (f(t'), t') \in R(x_0, t_0; u/2, t)$ and for some $t'' \in [t', t' + t], |f(t'') - f(t')| \geq u$. We observe that $B_{t,u}(x_0, t_0)) \subseteq A_{t,u}(x_0, t_0)$.

We now cover $\Lambda_{L,T}$ with $\frac{u}{2} \times t$ rectangular boxes. Let $L_D = L_D(u) = \{-L + k\frac{u}{2} : k \in \mathbb{Z}, 0 \leq k \leq \lceil \frac{2L}{u/2} \rceil\}$ and $T_D = T_D(t) = \{-T + mt : m \in \mathbb{Z}, 0 \leq m \leq \lceil \frac{2T}{t} \rceil\}$. Then,

$$
\begin{aligned}
& \mu_m(\cup_{(x_0,t_0)\in\Lambda_{L,T}} B_{t,u}(x_0, t_0)) && \text{(B.1)}\\
\leq\ & \mu_m(\cup_{(x_0,t_0)\in L_D\times T_D} B_{t,u}(x_0, t_0)) && \text{(B.2)}\\
\leq\ & \mu_m(\cup_{(x_0,t_0)\in L_D\times T_D} A_{t,u}(x_0, t_0)) && \text{(B.3)}\\
\leq\ & \left\lceil \frac{2L+1}{u/2} \right\rceil \left\lceil \frac{2T+1}{t} \right\rceil g_m(t, u; L, T) && \text{(B.4)}\\
\leq\ & C' \frac{LT}{tu} g_m(t, u; L, T) && \text{(B.5)}\\
\leq\ & C' \frac{LT}{u} (\tilde{g}(t, u; L, T) + \delta). && \text{(B.6)}
\end{aligned}
$$

for any $\delta > 0$, where in (B.6) m is larger than some $M(t, u; L, T; \delta)$. The first inequality follows from the observation that if K (in $\mathcal{H}$) is an outcome in $B_{t,u}(x, t)$ for some $(x, t) \in \Lambda_{L,T}$ then K is an outcome in $B_{t,u}(x', t')$ for some $(x', t') \in L_D \times T_D$.

Now let $\{\phi_n\}$ be a sequence of positive real numbers with $\lim_{n\to\infty} \phi_n = 0$. Now choose $L_n \to \infty$ and $T_n \to \infty$ such that $|\Phi(x,t)| < \frac{\phi_n}{3}$ if $|x| \geq L_n$ or $t \geq T_n$. Let $\{u_n\}$ be the sequence of real numbers where $u_n = \frac{\phi_n}{3}$. Now choose sequences of positive real numbers $\{t'_n\}, \{\delta_n\} \to 0$ such that $C' \frac{L_n T_n}{u_n}(\tilde{g}(t'_n, u_n, L_n, T_n) + \delta_n) \leq \frac{1}{2^n}$. For all $n \in \mathbb{N}$, let

$$C_n(t) = C(t, u_n; L_n, T_n) \equiv \cup_{(x_0,t_0)\in\Lambda_{L_n,T_n}} B_{t,u_n}(x_0, t_0).$$

Then, from (B.2) and (B.6) we have that if $m \geq M(t'_n, u_n; L_n, T_n; \delta_n)$, then $\mu_m(C_n(t'_n)) \leq \frac{1}{2^n}$. Since $(\mathcal{H}, d_{\mathcal{H}})$ is a complete separable metric space, any single measure on $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$ is tight. Therefore there exists D^ϵ_M, a compact set in $\mathcal{H}$, such that $\mu_m(D^\epsilon_M) \geq 1 - \frac{\epsilon}{2}$, for all $m \leq M$.

Let $K \in C^c_n$ be a compact set of paths. Let $\psi_n = \tanh(T_n + t_n) - \tanh(T_n)$. Suppose $\{(f(t), t) : t \geq t_0\} \in K$. Now, if $t_0 \leq t_1 \leq t_2$ are times such that $|\Psi(t_2) - \Psi(t_1)| \leq \psi_n$, then $|\Phi(f(t_1), t_1) - \Phi(f(t_2), t_2)| \leq \phi_n$. Let $G_n = \cap^\infty_{i=n+1} C^c_i$. Then, for any $m \geq M$,

$$\mu_m(G_n) = 1 - \mu_m(\cup^\infty_{i=n+1} C_i) \geq 1 - \sum_{i=n+1}^{\infty} \frac{1}{2^i} = 1 - \frac{1}{2^n}. \qquad \text{(B.7)}$$

Let $D_n = \cup_{K\in G_n} K$. Then D_n is a family of equicontinuous functions. By the Arzelà-Ascoli theorem, D_n is a compact subset of Π. Since G_n is a collection of compact subsets of D_n, G_n is a compact subset of $\mathcal{H}$. Let $\epsilon > 0$. Choose $n \in \mathbb{N}$ such that $\frac{1}{2^n} < \frac{\epsilon}{2}$. Let $K_\epsilon = D^\epsilon_M \cap G_n$. Then we have

$$\sup_m \mu_m(K_\epsilon) \geq 1 - \epsilon$$

where K_ϵ is a compact subset of $\mathcal{H}$. This proves that the family of measures $\{\mu_m\}$ is tight.

Remark B.2 *An argument similar to that for Proposition B.1 can be made to show that, if instead of* (T_1)*, one has the condition*

(T'_1) $\quad \sum_{t:t=2^{-k},k\in\mathbb{N}} t^{-(1+\alpha)} \sup_m \sup_{x_0,t_0} \mu_m\left(A_{t,t^\alpha}(x_0, t_0)\right) < \infty$

for some $\alpha > 0$*, then each* μ_m *as well as any subsequential limit* μ *of* (μ_m) *is supported on paths which are Hölder continuous with index* α*.*

Proposition B.3 *Suppose $\{\mathcal{X}_m\}$ is a sequence of $(\mathcal{H}, d_{\mathcal{H}})$-valued random variables whose paths are non-crossing. Suppose in addition,*

(I_1') For each $y \in \mathcal{D}$, there exist (measurable) path-valued random variables $\theta_m^y \in \mathcal{X}_m$ such that θ_m^y converges in distribution to a Brownian motion Z_y starting at y.
Then the distributions $\{\mu_m\}$ of $\{\mathcal{X}_m\}$ are tight.

Proof From the proof of the Proposition B.1, it is sufficient to show that for each $u > 0$,

$$\limsup_m \mu_m(\cup_{(x_0,t_0)\in L_D(u)\times T_D(t)} B_{t,u}(x_0,t_0)) \to 0 \text{ as } t \to 0.$$

For $u > 0, t > 0, (x_0, t_0) \in \mathbb{R}^2$, choose two points $y_1, y_2 \in \mathcal{D}$ from the two rectangles $R(x_0 \mp \frac{3}{8}u, t_0 - \frac{t}{2}; \frac{u}{8}, \frac{t}{4})$ respectively. Let

$$B_1^m(x_0,t_0,t,u) = \left\{ K \in \mathcal{H} \,\middle|\, \max_{s\le t_0+2t} |\theta_m^{y_1}(s) - y_1| < u/16 \right\}$$

$$B_2^m(x_0,t_0,t,u) = \left\{ K \in \mathcal{H} \,\middle|\, \max_{s\le t_0+2t} |\theta_m^{y_2}(s) - y_2| < u/16 \right\}$$

and $D_{t,u}^m(x_0,t_0) = B_1^m \cap B_2^m$. Now observe that $D_{t,u}^m(x_0,t_0) \subseteq B_{t,u}^c(x_0,t_0)$ for large enough m. Therefore we have

$$\limsup_m \mu_m(\cup_{(x_0,t_0)\in L_D\times T_D} B_{t,u}(x_0,t_0)) \tag{B.8}$$

$$\le \sum_{(x_0,t_0)\in L_D\times T_D} [1 - \liminf_m \mu_m(D_{t,u}^m(x_0,t_0))] \tag{B.9}$$

Since θ_m^y converges in distribution to a Brownian motion Z_y starting at y, we have

$$\liminf_m(\mu_m(B_1^m)) = \mathbb{P}\left(\max_{s\le t_0+2t} |X_{y_1}(s) - y_1| < u/16\right) \tag{B.10}$$

$$\ge 1 - Ct^2/u^4 \tag{B.11}$$

and

$$\liminf_m(\mu_m(B_2^m)) = \mathbb{P}\left(\max_{s\le t_0+2t} |X_{y_2}(s) - y_2| < u/16\right) \tag{B.12}$$

$$\ge 1 - Ct^2/u^4 \tag{B.13}$$

Therefore we have

$$\liminf_m \mu_m(D^m_{t,u}(x_0,t_0)) \geq 1 - 2Ct^2/u^4$$

which gives us

$$\limsup_m \mu_m(\cup_{(x_0,t_0)\in L_D(u)\times T_D(t)} B_{t,u}(x_0,t_0)) \leq 2C \sum_{(x_0,t_0)\in L_D(u)\times T_D(t)} t^2/u^4$$

Since $|L_D(u) \times T_D(t)| \sim \frac{1}{ut}$ we have shown that

$$\limsup_m \mu_m(\cup_{(x_0,t_0)\in L_D(u)\times T_D(t)} B_{t,u}(x_0,t_0)) \to 0 \text{ as } t \to 0,$$

and the proof is complete.

Remark B.4 *The proof of Proposition B.3 shows that the limiting processes* Z_y *starting at* $y = (\bar{x}, \bar{t})$ *need not be Brownian motions. It is sufficient that they be continuous processes such that for each fixed* $u > 0$,

$$\frac{1}{t}\sup_y \mathbb{P}(\sup_{\bar{t}\leq s\leq \bar{t}+t} |Z_y(s) - Z_y(\bar{t})| \geq u) \to 0 \text{ as } t \to 0^+. \tag{B.14}$$

Proposition B.5 *Let* $\mathcal{D}$ *be a countable dense subset of* $\mathbb{R}^2$ *and let* μ_k *be the distribution of the* $(\mathcal{H}, \mathcal{F}_\mathcal{H})$*-valued random variable* $\mathcal{W}_k = \mathcal{W}_k(\mathcal{D}) = \{\tilde{\mathcal{W}}_1, \ldots, \tilde{\mathcal{W}}_k\}$. *Then the family of measures* $\{\mu_k\}$ *is tight.*

Proof This is an immediate consequence of Proposition(B.3).

Proposition B.6 *If* $\mathcal{W}_n$ *is an a.s. increasing sequence of* $(\mathcal{H}, d_\mathcal{H})$*- valued random variables and the family of distributions* $\{\mu_n\}$ *of* $\mathcal{W}_n$ *is tight, then* $\overline{\cup_n \mathcal{W}_n}$ *is almost surely compact (in* (Π, d)*).*

Proof Let $\tilde{\mathcal{W}}_k$ be an increasing sequence of points (subsets of Π) in $(\mathcal{H}, d_\mathcal{H})$, which converge in $d_\mathcal{H}$ metric to some point $\tilde{\mathcal{W}}$ in $(\mathcal{H}, d_\mathcal{H})$. If for some k, $\tilde{\mathcal{W}}_k$ is not a subset of $\tilde{\mathcal{W}}$, then there exists an $\epsilon > 0$ such that $d_\mathcal{H}(\tilde{\mathcal{W}}, \tilde{\mathcal{W}}_n) > \epsilon$ for all $n \geq k$ contradicting the claim that $\tilde{\mathcal{W}}_k$ converges to $\tilde{\mathcal{W}}$. Therefore $\tilde{\mathcal{W}}_k \subseteq \tilde{\mathcal{W}}$ for all k. This implies $\overline{\cup_k \tilde{\mathcal{W}}_k} \subseteq \tilde{\mathcal{W}}$ and therefore is a compact subset of Π since it is a closed subset of the compact set $\tilde{\mathcal{W}}$. Since $\{\mu_n\}$ is tight, given an $\epsilon > 0$, there exists a compact subset K of $\mathcal{H}$

such that $\mathbb{P}(\mathcal{W}_n \in K) \geq 1 - \epsilon$ for all n, so by monotonicity, $P(\mathcal{W}_n \in K$ for all $n) \geq 1 - \epsilon$. But if $\mathcal{W}_n \in K$ for all n, then since K is compact, there exists a subsequence $\mathcal{W}_{n_j}$ which converges to a point in K and thus in $\mathcal{H}$. This implies by the first part of this proof that $\overline{\cup_{n_j}\mathcal{W}_{n_j}}(= \overline{\cup_n \mathcal{W}_n}$ because $\mathcal{W}_n$ is increasing in n) is a compact subset of Π. Thus we have shown that $\mathbb{P}(\overline{\cup_n \mathcal{W}_n}$ is a compact subset of $\Pi) \geq 1 - \epsilon$. Since the claim is true for all $\epsilon > 0$ we have proved the proposition.

Proposition B.7 *Let $\hat{\mathcal{D}} = \{(\hat{x}_i, \hat{t}_i) : i = 1, 2, \ldots\}$ be a (deterministic) dense countable subset of $\mathbb{R}^2$ and let $\{\hat{\mathcal{W}}_i : i = 1, 2, \ldots\}$ be (Π, d)-valued random variables starting from $(\hat{x}_i, \hat{t}_i)$. Suppose that the joint distribution of each finite subset of the $\hat{\mathcal{W}}_i$'s is that of coalescing Brownian motions. Then $\overline{\cup_{n=1}^{\infty}\{\hat{\mathcal{W}}_1, \hat{\mathcal{W}}_2 \ldots \hat{\mathcal{W}}_n\}}$ is almost surely compact. In particular for $\mathcal{W}_n$ defined in Proposition (B.5), $\bar{\mathcal{W}} = \overline{\cup_n \mathcal{W}_n}$ is almost surely compact.*

Proof The proof follows immediately from Propositions B.5 and B.6.

Proposition B.8 *Let $\hat{\mathcal{D}}$ and $\{\hat{\mathcal{W}}_i : i = 1, 2, \ldots\}$ be as in Proposition B.7 and let $\{\hat{\mathcal{W}}'_i : i = 1, 2, \ldots\}$ (on some other probability space) be equidistributed with $\{\hat{\mathcal{W}}_i : i = 1, 2, \ldots\}$. Then $\hat{\mathcal{W}} \equiv \overline{\{\hat{\mathcal{W}}_i : i = 1, 2, \ldots\}}$ and $\hat{\mathcal{W}}' \equiv \overline{\{\hat{\mathcal{W}}'_i : i = 1, 2, \ldots\}}$ are equidistributed $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued random variables.*

Proof It is an easy consequence of B.7 that $\{\hat{\mathcal{W}}_i : i = 1, \ldots, n\}$ (respectively $\{\hat{\mathcal{W}}'_i : i = 1, \ldots, n\}$) converges a.s. as $n \to \infty$ in $(\mathcal{H}, d_{\mathcal{H}})$ to $\hat{\mathcal{W}}$ (respectively $\hat{\mathcal{W}}'$). But then the identical distributions of $\{\hat{\mathcal{W}}_i : i = 1, \ldots, n\}$ and $\{\hat{\mathcal{W}}'_i : i = 1, \ldots, n\}$ converge respectively to the distributions of $\hat{\mathcal{W}}$ and $\hat{\mathcal{W}}'$, which thus must be identical.

C Hausdorff dimension of the graph of the sum of two Brownian motions

Proposition C.1 *Let X, Y be two independent standard Brownian motions and let $\mathcal{T}^+$ denote the set of record times of X, i.e., $\mathcal{T}^+ = \{t \geq 0 : X(t) = M(t)\}$, where $M(t) := \sup_{0 \leq s \leq t} X(s)$ is the maximum of X up to time t. Then, for $t_0 > 0$ and $a, b \in \mathbb{R}$ with $|a| + |b| > 0$, the set $\mathcal{G}^+ := \{(aX(t) + bY(t), t) : t \in \mathcal{T}^+ \cap [0, t_0]\}$, the projection of $\mathcal{T}^+ \cap [0, t_0]$ onto the graph of $aX + bY$, has Hausdorff dimension 1 almost surely.*

Proof An upper bound of 1 for the Hausdorff dimension follows readily from the fact that $\mathcal{G}^+$ is the image of a set, $\mathcal{T}^+ \cap [0, t_0]$, of Hausdorff dimension 1/2 a.s. (since $\mathcal{T}^+ \cap [0, t_0]$ has the same distribution as the set of zeros of X, $\{t \in [0, t_0] : X(t) = 0\}$; this follows from $(M(t) - X(t) : 0 \leq t \leq t_0)$ having the same distribution as $(|X(t)| : 0 \leq t \leq t_0)$ [22]) — see [23] — under a map which is a.s. (uniformly) Hölder continuous of exponent α for every $\alpha < 1/2$, namely, the map $t \to (aX(t) + bY(t), t)$, where we use the well known Hölder continuity properties of Brownian motion.

The desired lower bound is obtained by noting that the Hausdorff dimension of $\mathcal{G}^+$ is bounded below by the Hausdorff dimension of the image of $\mathcal{T}^+ \cap [0, t_0]$ under $aX + bY$, or equivalently under $aM + bY$, namely $\{aM(t) + bY(t) : t \in \mathcal{T}^+ \cap [0, t_0]\}$. Notice that the latter set equals $\{as + bY(T(s)) : s \in [0, M(t_0)]\}$, where T is the hitting time process associated to X, defined as $T(x) := \inf\{t \geq 0 : X(t) = x\}$. It suffices to show that the Hausdorff dimension of $\{as + bY(T(s)) : s \in [0, L]\}$ is a.s. greater than or equal to 1 for every deterministic $L > 0$. But that follows from known results as well. $Z(t) := at + bY(T(t))$ is a self similar process of exponent 1 with stationary increments and satisfies also the following condition of Theorem 3.3 in [24], from which the dimension bound follows. The condition is that there exists a constant K such that $\mathbb{P}(|Z(1)| \leq x) \leq Kx$ for every $x \geq 0$. This property is readily obtained from the distributions of Y and the hitting time variable $T(1)$.

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