Stochastic Navier-Stokes Equations for Turbulent Flows

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Abstract

This paper concerns the fluid dynamics modelled by the stochastic flow

$$\begin{cases} \dot{\boldsymbol{\eta}}\left(t,x\right)=\mathbf{u}\left(t,\boldsymbol{\eta}\left(t,x\right)\right)+\boldsymbol{\sigma}\left(t,\boldsymbol{\eta}\left(t,x\right)\right)\circ\dot{W} \\ \\ \boldsymbol{\eta}(0,x)=x \end{cases}$$

where the turbulent term is driven by the white noise $\dot{W}$. The motivation for this setting is to understand the motion of fluid parcels in turbulent and randomly forced fluid flows. Stochastic Euler equations for the undetermined components $\mathbf{u}(t,x)$ and $\boldsymbol{\sigma}(t,x)$ of the spatial velocity field is derived from the first principles. The resulting equations include as particular cases the deterministic and randomly forced counterparts of these equations.

In the second part of the paper we prove the existence and uniqueness of a strong local solution of the stochastic Navier-Stokes equation in $W_p^1\left(\mathbf{R}^d\right), d>1, p>d$. In the 2D case, the existence and uniqueness of a global strong solution is shown.

In the third part, we deal with the propagation of Wiener chaos by the stochastic Navier-Stokes equation and its relation to statistical moments of the solution.

To appear in SIAM J. Math. Analysis

Key words: Stochastic Navier-Stokes, turbulence, Kraichnan's turbulence, Wiener chaos, moments.

AMS subject classifications: 60H15, 35R60, 76M35.

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[1] This work was partially supported by NSF Grant DMS-98-02423, ONR Grant N00014-97-1-0229, and ARO Grant DAAG55-98-1-0418.

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1. INTRODUCTION

The relation of the Navier-Stokes equation to the phenomenon of hydrodynamic turbulence is widely regarded as one of the most fascinating problems of fluid mechanics. The onset of turbulence is often related to the randomness of background movement. One way to model this is to consider a randomly forced Navier-Stokes equation. Bensoussan and Temam [3] have pioneered the analytical study of a Navier-Stokes equation driven by white noise type random force. Later, this approach was substantially developed and extended by many authors (see, e.g. [4], [5], [7], [13], [15], [21], [35], [41], [51], [52], etc.).

These papers postulated some form of randomly forced Navier-Stokes equation at the inception point. A somewhat different approach was taken in the recent paper [40]. This paper assumed that the dynamics of the fluid particle was given by the stochastic diffeomorphism

$$\dot{\boldsymbol{\eta}}\,(t,x)=\mathbf{u}\,(t,\boldsymbol{\eta}\,(t,x))+\boldsymbol{\sigma}\,(t,\boldsymbol{\eta}\,(t,x))\circ\dot{W},\quad \boldsymbol{\eta}(0,x)=x \tag{1.1}$$

with undetermined local characteristics $\mathbf{u}\,(t,x)$, and $\boldsymbol{\sigma}\,(t,x)$. [2] In this setting, $\dot{W}$ is a time derivative of a Hilbert space valued Brownian motion (e.g. space-time white noise) and the stochastic integral is understood in the Stratonovich sense. The generalized random field $\sigma\,(t,x)\circ\dot{W}$ models the turbulent part of the velocity field, while $\mathbf{u}\,(t,x)$ models its regular component.

The idea of splitting up the velocity field into a sum of slow oscillating (deterministic) and fast oscillating (stochastic) components has been often entertained in fluid mechanics; important developments along these lines may be traced to work of Reynolds in the 1880es. Our interest in stochastic flows of the form (1.1) stems in part from recent developments in modelling a turbulent velocity field by a generalized Gaussian field $\mathbf{V}\,(t,x)$ with zero mean and covariance $C(x-y,t-s)=K(x-y)\delta(t-s)$ such that the spatial part is of the form

$$K^{ij}(x-y)=A^{ij}+D^{ij}|x-y|^{\kappa}\text{ for }\;|x-y|<<1$$

where $\kappa\in(0,2)$ and decays rapidly as $|x-y|\longrightarrow\infty$. This model was pioneered by Kraichnan in his work on turbulent transport [26] and substantially developed later in a series of work by Gawedzki et al. [16], [17] and other authors. The velocity field $\mathbf{V}\,(t,x)$ can be realized by way of its identification with a random field of the form $\boldsymbol{\sigma}\,(x)\cdot\dot{W}\,(t)$ (see [2], [31], and Section 2.2).

Relating the Kraichnan velocity field to classic fluid mechanics might naturally lead us to ask: “Can we compensate $\mathbf{V}\,(t,x)$ by a field $\mathbf{u}\,(t,x)$ that is more regular with respect to time variable, so that there is a balance of momentum for the resulting field $\mathbf{U}\,(t,x)=\mathbf{u}\,(t,x)+\boldsymbol{\sigma}\,(x)\circ\dot{W}\,(t)$ or, equivalently, that the motion of a fluid particle modelled by (1.1) satisfies the 2^{nd} Newton law?”[3]

The answer to this question is positive. Moreover, it turns out that following the classic scheme of Newtonian fluid mechanics (i.e. coupling (1.1) with Newton's second law), a quite general stochastic Navier-Stokes equation

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[2] Here and throughout the rest of the paper, vector fields on $\mathbf{R}^d$ are denoted by boldface letters. This convention also applies if the entries of the vector field are taking values in a Hilbert space.

[3] A priori, it is not clear in what sense the motion described by Kraichnan's velocity might fit into the paradigm of Newtonian mechanics.

$$
\begin{aligned}
\partial_t \mathbf{u} =& \Delta \mathbf{u} - (\mathbf{u}, \boldsymbol{\nabla})\mathbf{u} - \nabla p + \mathbf{f}(\mathbf{u}) \\
& [(\boldsymbol{\sigma}, \boldsymbol{\nabla})\mathbf{u} - \nabla \tilde{p} + \mathbf{g}(\mathbf{u})] \circ \dot{W}
\end{aligned} \tag{1.2}
$$

may be derived for $\mathbf{u}(t, x)$ (see [40] and Section 2.2). Special cases of this equation include the standard deterministic Navier-Stokes and Euler equation as well as many other variations of the stochastic Navier-Stokes equation considered in the literature. A more detailed treatment of this subject is given in Section 2. To emphasize the relation of equation (1.2) to the flow (1.1) involving the (short time) turbulent component $\boldsymbol{\sigma}(x) \circ \dot{W}(t)$, we will often refer to it as a turbulent stochastic Navier-Stokes equation.

Section 3 deals with the analytical theory of the stochastic Navier-Stokes equation (1.2) and some generalizations of this equation. One technically challenging feature of the SNS equation (1.2) is that it involves multiplicative noise with the diffusion coefficient depending on $\boldsymbol{\nabla}\mathbf{u}$.The existence and uniqueness of a maximal local solution in the Sobolev space $W_p^1(\mathbf{R}^d)$ for arbitrary $d > 1$and $p > d$ is shown (Theorem 1). Note that owing to the embedding $C^{1-d/p}(\mathbf{R}^d) \subset W_p^1(\mathbf{R}^d)$, the solution is Hölder continuous. The maximal solution is understood in the (probabilistic) strong sense, e.g. pathwise rather than as a solution of a martingale problem. For the latter, see e.g. [4], [15], [41], [51]. In the case of $d = 2$, it is proved in Theorem 2 that there exists a unique global solution of equation (1.2).

We remark that the results of Section 3 do not cover the case of only Hölder-continuous $\boldsymbol{\sigma}(x)$, that assumption being important for Kraichnan's turbulent velocity model. This case is addressed in the forthcoming paper [43].

The L_p- theory of strong solutions of SNS equations was studied in [5] (see also references therein). In this paper, the local (global in 2D) existence and uniqueness were proved for a randomly forced Navier-Stokes equation

$$
\partial_t \mathbf{u} = \Delta \mathbf{u} - (\mathbf{u}, \boldsymbol{\nabla})\mathbf{u} - \nabla p + \mathbf{f}(\mathbf{u}) + \mathbf{g}(\mathbf{u}) \circ \dot{W} \tag{1.3}
$$

in a smooth bounded domain of $\mathbf{R}^d$ ($d = 2$ or 3). In this equation, the noise influences the motion of the fluid only by the velocity, rather than by the velocity and its gradient, as is the case in [4], [15], [39], and the present paper. Consequently, it does not cover the case of turbulent stochastic flow.

For a substantial body of related work on L_p-solutions of deterministic Navier-Stokes equations, see, e.g. [18], [24], [25], etc.

Section 4 deals with the propagation of Wiener chaos and moment theory for SNS equations. In this Section, we derive a deterministic parabolic system for the Hermite-Fourier coefficients in a Wiener chaos expansion of $\mathbf{u}(t, x)$, which we refer to as "propagator". We show that the statistical moments of the velocity field $\mathbf{u}(t, x)$ can be directly expressed via the solution of the propagator. While still an infinite-dimensional system, the propagator for the SNS equation is a much more simple object than the related Kolmogorov equation. On the other hand, it is quite sufficient for dealing with the basic statistical properties of solutions to the SNS equation.[4]

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[4] The main results of the paper were announced in [39].

2. PHENOMENOLOGY OF STOCHASTIC NAVIER-STOKES AND EULER EQUATIONS

2.1. **Preliminaries.** Classic fluid mechanics deals with two essentially equivalent approaches to modelling the motion of fluid, namely Euclidean and Lagrangian formalisms. The centerpiece of the former is the Navier-Stokes equation for the fluid velocity $\mathbf{u}(t,x)$. This equation is expressed in Euclidean coordinates as

$$\begin{cases} \partial_t \mathbf{u} + u^l \mathbf{u}_{x_l} - \nu\Delta\mathbf{u} + \frac{1}{\rho}\boldsymbol{\nabla} p = \mathbf{f}, \text{ in } [0,\,\infty)\times\mathbf{R}^d, \\ \\ \mathbf{u}(0) = \mathbf{u}_0. \end{cases} \tag{2.1}$$

In the case of ideal fluid, (2.1) reduces to Euler equation

$$\begin{cases} \partial_t \mathbf{u} + u^l \mathbf{u}_{x_l} + \frac{1}{\rho}\boldsymbol{\nabla} p = \mathbf{f}, \text{ in } [0,\,\infty)\times\mathbf{R}^d, \\ \\ \mathbf{u}(0) = \mathbf{u}_0. \end{cases} \tag{2.2}$$

In the case of incompressible fluid, both equations have to be complemented by the equation

$$div\mathbf{u}(t,x) = 0.$$

The Lagrangian formalism emphasizes the dynamics of fluid particles. Let us write $\boldsymbol{\eta}(t,x)$ for the trajectory followed by the fluid particle that is at point x at time $t=0$. Obviously, $\boldsymbol{\eta}(t) = (\eta^i(t,x), i=1,...,d)$ verifies the equation

$$\partial_t \eta^i\left(t,x\right) = u^i\left(t,\boldsymbol{\eta}\left(t,x\right)\right), \eta^i\left(0,x\right) = x^i. \tag{2.3}$$

The function $\boldsymbol{\eta}(t,x)$ is usually referred to as a fluid flow or fluid flow map. Equation (2.3) yields that the fluid flow is defined by $\mathbf{u}(t,x)$, a solution of the Navier-Stokes (Euler) equation. On the other hand, one could argue that fluid flow $\boldsymbol{\eta}$ is an equally or even more basic notion than velocity $\mathbf{u}$. Indeed, the classic Euclidean approach postulates that the fluid particle motion is given by (2.3) with unknown smooth velocity field $\mathbf{u}$; it then shows that this equation, together with Newton's second law, yield (2.1) (see e.g. [30], [8]). A more recent approach to fluid mechanics pioneered by Arnold, Marsden and Ebin (see [1], [14]) treats the fluid flow as an intrinsically defined infinite-dimensional dynamical system.

In this paper we consider a flow similar to (2.3) but make the fluid particle subject to turbulent diffusion. The motion for this setting is to understand the motion of fluid parcels in turbulent and randomly forced fluid flows.

More specifically, we postulate that the fluid particles motion is given by the equation

$$\dot{\boldsymbol{\eta}}\left(t,x\right) = \mathbf{u}\left(t,\boldsymbol{\eta}\left(t,x\right)\right) + \boldsymbol{\sigma}\left(t,\boldsymbol{\eta}\left(t,x\right)\right)\circ\dot{W}, \boldsymbol{\eta}(0,x) = x \tag{2.4}$$

where $W(t)$ is a cylindrical Brownian motion in some Hilbert space Y (see [41]), $\dot{W} = \partial W(t)/\partial t$, and $\mathbf{u}(t,x)$ and $\boldsymbol{\sigma}(t,x)$ are unknown random fields.

The fluid flow map (2.4) corresponds to the velocity field

$$\mathbf{U}\left(t,x\right) = \boldsymbol{\sigma}\left(t,x\right)\circ\dot{W} + \mathbf{u}\left(t,x\right). \tag{2.5}$$

The singular term of this field, $\boldsymbol{\sigma}\left(t,x\right)\circ\dot{W}$, is referred to as the "turbulent component". If W and σ are statistically independent, e.g. if σ is non-random, $\boldsymbol{\sigma}\left(t,x\right)\circ\dot{W} := \boldsymbol{\sigma}(t,x)\cdot\dot{W} + \frac{1}{2}\boldsymbol{\sigma}_{x_p}\left(t,x\right)\sigma^p\left(t,x\right).$

We remark that Kraichnan's turbulence model is an interesting example of the turbulent component in (2.5) .

In the generalization of Kraichnan's model introduced in [16] (see also [17]), the turbulent component $\mathbf{V}(t,x)$ is modelled by a homogeneous, isotropic, and stationary Gaussian random field with zero mean and covariance

$$EV^i(t,x)\,V^j(s,x) = K^{ij}(x-y)\,\delta(t-s)$$

where $K^{ij}(x-y) = C_0^{ij}\delta_{ij} - D^{ij}(x-y)$ .The following asymptotic properties were assumed:

(i) The spatial covariance $K^{ij}(x-y)$ decays fast for $|x-y|>1$;

(ii) For $|x-y| << 1$,

$$D^{ij}(x-y) = D\left((d+\kappa-1)\,\delta_{ij} - \kappa x_i x_j / |x|^2\right)|x|^{\kappa}.$$

As illustrated in [31] , one possible construction of a homogeneous Gaussian random field with the Kraichnan type covariance is given by

$$\boldsymbol{\sigma}(x)\cdot\dot{W}(t) = \frac{d}{dt}\left(\sum_{i=1}^{\infty}\boldsymbol{\sigma}_k(x)\,w^k(t)\right)$$

where $w^k(t)$ are independent one-dimensional Brownian motions and $\boldsymbol{\sigma}_k(x)$ are Hölder continuous with an exponent $\kappa/2$ and so that $div\boldsymbol{\sigma}_k(x) = 0$ and $\sum_{ik} \mid \sigma^{ik}(x)\mid^2 \le K < \infty$.

In this section we will derive equations for $\mathbf{u}(t,x)$ and $\boldsymbol{\sigma}(t,x)$. This will be done by coupling (2.4) with Newton's second law, in much the same way as it is done in classic macroscopic fluid dynamics.

The obtained equations include as particular cases the deterministic Navier-Stokes equation (2.1), as well as Navier-Stokes equation with stochastic forcing (see [3], [4], [5], [7], [15], [41], [51] etc.).

2.2. **Balance of Momentum.** Let $(\Omega,\mathcal{F},\mathbf{P})$ be a complete probability space and Y be a separable Hilbert space. Let W be an Y-valued cylindrical Brownian motions on $(\Omega,\mathcal{F},\mathbf{P})$. Write $\mathcal{F}_t^W = \sigma(W(s), s\le t)$.

Consider the equation

$$d\boldsymbol{\eta}(t,x) = \mathbf{u}(t,\boldsymbol{\eta}(t,x))dt + \boldsymbol{\sigma}(t,\boldsymbol{\eta}(t,x))\circ dW(t), \boldsymbol{\eta}(0,x) = x \tag{2.6}$$

where $\circ$ indicates the Stratonovich version of the stochastic integral .

Let us assume the following:

(H1) $\mathbf{u}$ is continuous semimartingale given by

$$d\mathbf{u}(t,x) = \boldsymbol{\alpha}(t,x)\,dt + \boldsymbol{\beta}(t,x)\circ dW(t) \tag{2.7}$$

where $\boldsymbol{\alpha} : \Omega\times[0,\infty)\times R^d \mapsto R^d$ and $\boldsymbol{\beta} : \Omega\times[0,\infty)\times R^d \mapsto Y^d$ are measurable and $\mathcal{F}_t^W$-adapted functions for every x; $\sigma : [0,\infty)\times R^d \mapsto Y^d$ is a non-random measurable function.

In what follows, we shall also assume that for fixed $t, \boldsymbol{\eta}$ is an invertible mapping; $\boldsymbol{\alpha}, \boldsymbol{\beta}$ and $\boldsymbol{\sigma}$ are appropriately integrable and smooth so that the stochastic integrals are defined and the following manipulations are legitimate. In particular, to define the Stratonovich integral one needs to assume the existence and some regularity of the joint quadratic variation $\langle\boldsymbol{\beta}(\cdot,x),W\rangle_t$ (see [29]).

One fundamental postulate of fluid mechanics (see e.g. [8]) is the Newton 2nd law: "the rate of change of momentum of a fluid particle equals the force applied to it" , that is to say

$$\frac{d}{dt}\dot{\boldsymbol{\eta}}\,(t)=\frac{\mathbf{F}\,(t,\boldsymbol{\eta}\,(t))}{\rho(t,\boldsymbol{\eta}(t))} \tag{2.8}$$

where $\mathbf{F}(t,x)$ is the total force applied to the fluid particle and $\rho(t,x)$ is the mass density. For the sake of simplicity, in this paper we assume that $\rho=1$.

In our case the acceleration,

$$\frac{d}{dt}\dot{\boldsymbol{\eta}}\,(t)=\frac{d}{dt}\left(\boldsymbol{\sigma}\,(t,\boldsymbol{\eta}\,(t))\circ\dot{W}\right)+\frac{d}{dt}\mathbf{u}\,(t,\boldsymbol{\eta}\,(t))\,,$$

is highly irregular. Thus (2.8) shall be interpreted in the sense of distributions, i.e. for every $\varphi\in C_0^\infty\left(\mathbf{R}^1\right)$,

$$\int\varphi\,(t)\,\mathbf{F}\,(t,\boldsymbol{\eta}\,(t))\;dt=-\int\varphi'\,(t)\,\boldsymbol{\sigma}\,(t,\boldsymbol{\eta}\,(t))\circ dW(t)+\int\varphi\,(t)\,d\mathbf{u}\,(t,\boldsymbol{\eta}\,(t))\;. \tag{2.9}$$

Obviously, both sides of (2.8) must have the same structure. Hence, formulas (2.9) and (2.11) yield that there exist $\mathcal{F}_t^W$-adapted functions $\mathbf{f}:\Omega\times[0,\infty)\times\mathbf{R}^d\mapsto\mathbf{R}^d,\mathbf{g}:\Omega\times[0,\infty)\times\mathbf{R}^d\mapsto Y^d$, and $\mathbf{d}:\Omega\times[0,\infty)\times\mathbf{R}^d\mapsto Y^d$ so that

$$\begin{aligned}&\textstyle\int\varphi\,(t)\,\mathbf{F}\,(t,\boldsymbol{\eta}\,(t))\;dt=-\int\varphi'\,(t)\,\mathbf{d}\,(t,\boldsymbol{\eta}\,(t))\circ dW(t)+\\&\textstyle\int\varphi\,(t)\,(\mathbf{f}\,(t,\boldsymbol{\eta}\,(t))\,dt+\mathbf{g}(t,\boldsymbol{\eta}\,(t))\circ dW(t))\end{aligned} \tag{2.10}$$

By the Itô-Wentzell formula (see e.g. Theorem 3.3.2 in [29] or Theorem 1.4.9 in [47]),

$$\begin{aligned}&d\mathbf{u}\,(t,\boldsymbol{\eta}\,(t))=\boldsymbol{\alpha}\,(t,\boldsymbol{\eta}\,(t))\,dt+\boldsymbol{\beta}\,(t,\boldsymbol{\eta}\,(t))\circ dW(t)+\\&\mathbf{u}_{x_i}u^i(t,\boldsymbol{\eta}(t))dt+\mathbf{u}_{x_i}\sigma^i(t,\boldsymbol{\eta}(t))\circ dW(t)\end{aligned} \tag{2.11}$$

By matching terms in (2.9) and (2.10) and taking into account (2.11), we obtain the following equalities:

$$\mathbf{d}=\boldsymbol{\sigma},\mathbf{g}=\boldsymbol{\beta}+\mathbf{u}_{x_i}\sigma^i, \tag{2.12}$$

$$\boldsymbol{\alpha}=-\mathbf{u}_{x_j}u^j+\mathbf{f}, \tag{2.13}$$

Thus, we arrive at the following equation for the regular velocity component $\mathbf{u}$:

$$d\mathbf{u}=\left[-\mathbf{u}_{x_j}u^j+\mathbf{f}\right]dt+\left[\mathbf{g}(t,x)-\mathbf{u}_{x_p}\sigma^p\,(t,x)\right]\circ dW(t). \tag{2.14}$$

2.3. **Derivation of Stochastic Euler and Navier -Stokes Equations.**

2.3.1. *Incompressible Stochastic Fluids and Euler Equation.* A fluid characterized by flow $\boldsymbol{\eta}$ given by (2.4) is incompressible if $\boldsymbol{\eta}(t,x)$ is a volume preserving map. It can be shown that the latter holds iff

$$div\, \boldsymbol{\sigma}\,(t,x) = div\,\mathbf{u}\,(t,x) = 0 \tag{2.15}$$

Indeed, we may easily see that the Jacobian of $\boldsymbol{\eta}$ verifies the following equation

$$\begin{aligned} dJ\eta\,(t) &= J\boldsymbol{\eta}\,(t)\,\{\mathrm{div}\,\boldsymbol{\sigma}\,(t,\boldsymbol{\eta}\,(t))\cdot dW\,(t) + \mathrm{div}\,\mathbf{u}\,(t,\boldsymbol{\eta}\,(t))\;dt \\ &+ (1/2)\,[|\mathrm{div}\,\boldsymbol{\sigma}\,(t,\boldsymbol{\eta}\,(t))\,|_Y^2 + (\partial_j\,\mathrm{div}\,\boldsymbol{\sigma})(t,\boldsymbol{\eta}(t))\cdot\sigma^j(t,\boldsymbol{\eta}(t))]\,dt\}. \end{aligned} \tag{2.16}$$

The rest of the proof is similar to the case of $\boldsymbol{\sigma} = 0$ (see e.g. [8]).

Suppose that the fluid is ideal (non-viscous). Similar to that found in the classic setting, we can assume that the force acting on the fluid particle is of the form $\mathbf{F} = -\boldsymbol{\nabla} P + \bar{\mathbf{F}}$ where P is the (unknown) pressure and $\bar{\mathbf{F}}$ is the given body force. More specifically, we assume that $\mathbf{f} = -\boldsymbol{\nabla} P^a + \bar{\mathbf{f}}$, $\mathbf{g} = -\boldsymbol{\nabla} P^d + \bar{\mathbf{g}}$, and $\mathbf{d} = -\boldsymbol{\nabla} P^t + \bar{\mathbf{d}}$. The body force components are considered to be given, while those of the pressure are subject to determination.

$$\begin{cases} d\mathbf{u} = [-\mathbf{u}_{x_i}u^i - \boldsymbol{\nabla} P^a + \bar{\mathbf{f}}]dt + \left(\bar{\mathbf{g}} - \boldsymbol{\nabla} P^d - \mathbf{u}_{x_i}\sigma^i\right)\circ dW; \\ \boldsymbol{\sigma}\,(t,x) = -\boldsymbol{\nabla} P^t\,(t,x) + \bar{\mathbf{d}}\,(t,x)\,,\ \mathrm{div}\,\mathbf{u} = 0, \mathrm{div}\,\boldsymbol{\sigma} = 0; \\ \mathbf{u}(0,x) = \mathbf{u}_0(x) \end{cases} \tag{2.17}$$

Since div$\mathbf{u}$=div$\boldsymbol{\sigma} = 0$, we have $\Delta P^t =$div$\bar{\mathbf{d}}$, $\Delta P^d =$div$\bar{\mathbf{g}}$, and

$$\Delta P^a = \mathrm{div}\,[\bar{\mathbf{f}} - \mathbf{u}_{x_i}u^i].$$

The number of equations equals the number of unknown functions and so mathematically this is a reasonable system.

Write $2a^{ij} = \sigma^i\cdot\sigma^j$. Since div$\boldsymbol{\sigma} = 0$, the first equation in (2.17) can be rewritten in the Itô form as follows

$$\begin{aligned} d\mathbf{u} \quad &= \quad \left[(a^{ij}\mathbf{u}_{x_i})_{x_j} - \mathbf{u}_{x_j}u^j - \frac{1}{2}\left(\mathbf{g}_{x_p}\sigma^p - \mathbf{G}\right) + \mathbf{f}\right]dt \\ &+ \left[\mathbf{g} - \mathbf{u}_{x_p}\sigma^p\right]\cdot dW. \end{aligned}$$

In spite of the presence of the “effective viscosity” term $(a^{ij}\mathbf{u}_{x_i}\,(t,x))_{x_j}$, which is induced by the turbulent term, we shall still regard the equation (2.17) as *stochastic Euler* equation. First, it was derived for the ideal fluid. Second, (2.17) passes the ultimate test for Euler type equations, namely, it conserves the energy. Specifically, it can be easily shown that if there are no free forces, $\bar{\mathbf{f}} = \bar{\mathbf{g}} = \bar{\mathbf{h}} = 0$, then

$$\int |\mathbf{u}(t,x)|^2dx = \int |\mathbf{u}(0,x)|^2dx \quad P-a.s. \tag{2.18}$$

Besides, in "appearance" the equation (2.17) bears more of a resemblance to the deterministic Euler equation, since it does not contain the second order term. A special case of equation (2.17) was derived (very informally) in [21] using the variational formulation of Euler equation. In this paper it was assumed that $\boldsymbol{\sigma} = const$, $\bar{g} = \bar{h} = 0$, and W was a one-dimensional Brownian motion.

Now let us consider some comparatively straightforward generalization of the setup considered above. Let V be a Y-valued cylindrical Brownian motion independent of W . Write $\mathcal{F}_t^{W,V} = \sigma(W(s), V(s), s \le t)$.

Assume

(H1') $d\mathbf{u}\,(t,x) = \boldsymbol{\alpha}\,(t,x)\,dt + \boldsymbol{\beta}\,(t,x) \cdot dW(t) + \boldsymbol{\gamma}(t,x) \cdot dV(t)$

where $\boldsymbol{\alpha} : \Omega \times [0,\infty) \times R^d \mapsto R^d, \boldsymbol{\beta} : \Omega \times [0,\infty) \times R^d \mapsto Y^d$, and $\boldsymbol{\gamma} : \Omega \times [0,\infty) \times R^d \mapsto Y^d$ are $\mathcal{F}_t^{W,V}$-adapted functions.

The stochastic integrals in (H1') are understood in the Itô sense.

Remark 1. *The Itô setting has its advantages and disadvantages. One advantage of the Itô formulation is that it does not require the existence of joint quadratic variations* $\langle \mathbf{g}\,(\cdot,x)\,, W\rangle_t\,, \langle \boldsymbol{\beta}\,(\cdot,x)\,, W\rangle_t\,,$ *and* $\langle \boldsymbol{\gamma}\,(\cdot,x)\,, V\rangle_t$ *which is a necessary assumption for the existence of the related Stratonovich integrals. On the other hand, if the fluid particles motion is given by the Itô equation*

$$d\boldsymbol{\eta}\,(t,x) = \mathbf{u}(t,\boldsymbol{\eta}(t,x))dt + \boldsymbol{\sigma}(t,\boldsymbol{\eta}(t,x)) \cdot dW(t), \tag{2.19}$$

the equations (2.15) *would not anymore guarantee that* $\boldsymbol{\eta}(t,x)$ *is a volume preserving map. Instead, a more cumbersome condition would be needed. Therefore, we will continue with the Stratonovich form* (2.6)

Of course, owing to (H1'), the balance of momentum considerations yield that the force must be of the form

$$\mathbf{F} = \frac{d}{dt}\left(\mathbf{d} \circ \dot{W}\right) + \mathbf{f} + \mathbf{g} \cdot \dot{W}(t) + \mathbf{h} \cdot \dot{V}(t)$$

The interested reader could prove that in the new setting we have

$$\begin{cases} d\mathbf{u}\,(t,x) = (\left(a^{ij}\mathbf{u}_{x_i}\right)_{x_j} - \mathbf{u}_{x_i}u^i - \mathbf{g}_{x_i} \cdot \sigma^i - \boldsymbol{\nabla} P^a + \bar{\mathbf{f}})dt + \\ \left(\bar{\mathbf{g}} - \boldsymbol{\nabla} P^d - \mathbf{u}_{x_i}\sigma^i\right) \cdot dW(t) + \left(\bar{\mathbf{h}} - \boldsymbol{\nabla}\tilde{P}^d\right) \cdot dV(t); \\ \boldsymbol{\sigma}\,(t,x) = -\boldsymbol{\nabla} P^t\,(t,x) + \bar{\mathbf{d}}\,(t,x)\,,\ \operatorname{div}\mathbf{u} = 0, \operatorname{div}\boldsymbol{\sigma} = 0; \\ \mathbf{u}(0,x) = \mathbf{u}_0(x) \end{cases}$$

where $\mathbf{h} = -\boldsymbol{\nabla}\tilde{P}^d + \bar{\mathbf{h}}$ (for detail see [40]) .

2.3.2. *Stochastic Navier-Stokes Equation.* Let us drop now assumption (2.17) and assume that the fluid we are dealing with is viscous. This requires the appropriate modification of the structure of forces acting on the fluid particle. Because of the molecular motion of particles, the force exerted per unit area on an arbitrary surface S in the fluid has a component of the form

$$\nu\nabla\mathbf{U}\,(x,t)\,\vec{n} = \nu[\nabla\boldsymbol{\sigma}\,(t,x) \circ \dot{W} + \nabla\mathbf{u}\,(t,x)]\vec{n},$$

where $\vec{n}$ is the unit normal to S (see [8]). By divergence theorem, this implies the following structure of forces: $\mathbf{f} = -\boldsymbol{\nabla} P^a + \nu\Delta\mathbf{u} + \frac{1}{2}\Delta\boldsymbol{\sigma}_{x_p}\,(t,x)\,\Delta\sigma^p\,(t,x) + \bar{\mathbf{f}},$

$\mathbf{g} = -\boldsymbol{\nabla} P^d + \Delta\sigma + \bar{\mathbf{g}}$, $\mathbf{h} = -\boldsymbol{\nabla}\tilde{P}^d + \bar{\mathbf{h}}$. The resulting stochastic Navier-Stokes equation for the components of the velocity field (2.5) is as follows:

$$\begin{cases} d\mathbf{u} = [\nu\Delta\mathbf{u} + (a^{ij}\mathbf{u}_{x_i})_{x_j} - \mathbf{u}_{x_i}u^i - \left(\bar{\mathbf{g}}_{x_i} - \boldsymbol{\nabla} P^d_{x_i}\right)\cdot\sigma^i - \\ \boldsymbol{\nabla} P^a + \bar{\mathbf{f}} + \frac{1}{2}\Delta\boldsymbol{\sigma}_{x_p}(t,x)\cdot\Delta\sigma^p(t,x)]dt + \\ \left(\bar{\mathbf{h}} - \boldsymbol{\nabla}\tilde{P}^d\right)\cdot dV(t) + \left(\bar{\mathbf{g}} + \nu\Delta\sigma - \boldsymbol{\nabla} P^d - \mathbf{u}_{x_i}\sigma^i\right)\cdot dW; \\ \boldsymbol{\sigma}(t,x) = -\boldsymbol{\nabla} P^t(t,x) + \bar{\mathbf{d}}(t,x),\ \operatorname{div}\mathbf{u} = 0, \operatorname{div}\boldsymbol{\sigma} = 0; \mathbf{u}(0,x) = \mathbf{u}_0(x) \end{cases} \tag{2.20}$$

where ν is the viscosity coefficient.

2.3.3. *Special cases.* Now let us review several important particular cases.

1. Assume that the stochastic components of the force $\mathbf{g} = \mathbf{h} = \mathbf{d} = 0$. Then, by (2.12), $\boldsymbol{\sigma} = \boldsymbol{\gamma} = \boldsymbol{\beta} = 0$, and we arrive at the standard deterministic Euler equation.

2. Assume that there is no turbulent component in (2.4) and the force has no turbulent component, i.e. $\boldsymbol{\sigma} = \mathbf{g} = 0$. Then, by (2.12), $\mathbf{d} = \boldsymbol{\beta} = 0$, and the stochastic Navier-Stokes equation reduces to the following Navier-Stokes equation with random forcing:

$$\begin{cases} d\mathbf{u} = [\nu\Delta\mathbf{u} - \mathbf{u}_{x_i}u^i - \boldsymbol{\nabla} P^a + \bar{\mathbf{f}}]dt + \mathbf{h}\cdot dV(t); \\ \operatorname{div}\mathbf{u} = 0, \mathbf{u}(0,x) = \mathbf{u}_0(x). \end{cases}$$

3. Assume that $\nu = 0$, $\boldsymbol{\beta} = 0$ and $\boldsymbol{\alpha}, \boldsymbol{\gamma}$ are $\mathcal{F}_t^V$−adapted. Then, by (2.12), we have

$$\boldsymbol{\alpha} = -\mathbf{u}_{x_i}u^j + \mathbf{f} - \left(\mathbf{u}_{x_i}a^{ij}\right)_{x_j}, \mathbf{g} = \mathbf{u}_{x_p}\sigma^p. \tag{2.21}$$

We then arrive at the following equation for $\mathbf{u}$:

$$\begin{cases} d\mathbf{u} = [-\mathbf{u}_{x_i}u^i - \boldsymbol{\nabla} P^a + \bar{\mathbf{f}} - \\ \left(\bar{\mathbf{g}} - \boldsymbol{\nabla} P^d\right)_{x_j}\cdot\left(\bar{d}^j - P^t_{x_j}\right)]dt + \left(\bar{\mathbf{h}} - \boldsymbol{\nabla}\tilde{P}^d\right)\cdot dV(t); \\ div\,\mathbf{u} = 0, \mathbf{u}(0,x) = \mathbf{u}_0(x). \end{cases} \tag{2.22}$$

In addition, (2.21) yields

$$\Delta P^d = u^i_{x_p}\sigma^p_{x_i} - div\bar{\mathbf{g}}. \tag{2.23}$$

Obviously, if $\mathbf{h} = \mathbf{0}$, the dynamics of the non-turbulent component $\mathbf{u}$ of the velocity field $\mathbf{U} = \boldsymbol{\sigma}\circ\dot{W} + \mathbf{u}$ is given by a deterministic Euler type equation.

We remark that the above results rectify the statement in [40] that equation (2.17) is ill-posed if $\boldsymbol{\beta} = 0$.

3. Analytical Theory of Turbulent Stochastic Navier-Stokes Equations

3.1. **Preliminaries.**

3.1.1. *Notation.* We begin by outlining some of the notation that will be used in the paper.

$\mathbf{R}^d$ denotes a d-dimensional Euclidean space with elements $x = (x_1, \ldots, x_d)$; if $x, y \in \mathbf{R}^d$, we write

$$(x, y) = \sum_{i=1}^{d} x_i y_i,\ |x| = \sqrt{(x, x)}.$$

Let us fix a separable Hilbert space Y. The scalar product of $x, y \in Y$ will be denoted by $x \cdot y$.

If u is a function on $\mathbf{R}^d$, the following notational conventions will be used for its partial derivatives: $\partial_i u = \partial u/\partial x_i, \partial^2_{ij} = \partial^2 u/\partial x_i \partial x_j$, $\partial_t u = \partial u/\partial t$, and $\nabla u = \partial u = (\partial_1 u, \ldots, \partial_d u)$, and $\partial^2 u = (\partial^2_{ij} u)$ denotes the Hessian matrix of second derivatives. Let $\alpha = (\alpha_1, ..., \alpha_d)$ be a multi-index, then $\partial^\alpha_x = \Pi^d_{i=1} \partial^{\alpha_i}_{x_i}$.

Let $C_0^\infty = C_0^\infty(\mathbf{R}^d)$ be the set of all infinitely differentiable functions on $\mathbf{R}^d$ with compact support.

For $s \in (-\infty, \infty)$, write $\Lambda^s = \Lambda^s_x = \left(1 - \sum_{i=1}^d \partial^2/\partial x_i^2\right)^{s/2}$.

For $p \in [1, \infty]$ and $s \in (-\infty, \infty)$, we define the space $H^s_p = H^s_p(\mathbf{R}^d)$ as the space of generalized functions u with the finite norm

$$|u|_{s,p} = |\Lambda^s u|_p,$$

where $|\cdot|_p$ is the L_p norm. Obviously, $H^0_p = L_p$. Note that if $s \geq 0$ is an integer, the space H^s_p coincides with the Sobolev space $W^s_p = W^s_p(\mathbf{R}^d)$.

If $p \in [1, \infty]$, and $s \in (-\infty, \infty)$, $H^s_p(Y) = H^s_p(\mathbf{R}^d, Y)$ denotes the space of $Y-$valued functions on $\mathbf{R}^d$ so that the norm $||g||_{s,p} = |\,|\Lambda^s g|_Y\,|_p < \infty$. We also write $L_p(Y) = L_p(\mathbf{R}^d, Y) = H^0_p(Y) = H^0_p(\mathbf{R}^d, Y)$. Let $C_0^\infty(Y)$ be the space of Y-valued infinitely differentiable functions on $\mathbf{R}^d$ with compact support.

Obviously, the spaces $C_0^\infty, C_0^\infty(Y)$, $H^s_p\left(\mathbf{R}^d\right)$ and $H^s_p(\mathbf{R}^d, Y)$ can be extended to vector functions (denoted by bold-faced letters). For example, the space of all vector functions $\mathbf{u} = (u^1, \ldots, u^d)$ such that $\Lambda^s u^l \in L_p$, $l = 1, \ldots, d$, with the finite norm

$$|\mathbf{u}|_{s,p} = (\sum_l |u^l|^p_{s,p})^{1/p},$$

we denote by $\mathbb{H}^s_p = \mathbb{H}^s_p(\mathbf{R}^d)$. Similarly, we denote by $\mathbb{H}^s_p(Y) = \mathbb{H}^s_p(\mathbf{R}^d, Y)$ the space of all vector functions $g = (g^l)_{1 \leq l \leq d}$, with Y-valued components g^l, $1 \leq l \leq d$, so that $||g||_{s,p} = (\sum_l |g^l|^p_{s,p})^{1/p} < \infty$. The set of all infinitely differentiable vector functions $u = (u^1, \ldots, u^d)$ on $\mathbf{R}^d$ with compact support will be denoted by $\mathbb{C}_0^\infty$. We denote $\mathbb{C}_0^\infty(Y)$ the set of all infinitely differentiable vector functions $u = (u^1, \ldots, u^d)$ on $\mathbf{R}^d$ with compact support (all u^l are Y-valued).

When $s = 0$, $\mathbb{H}^s_p(Y) = \mathbb{L}_p(Y) = \mathbb{L}_p(\mathbf{R}^d, Y)$. Also, in this case, the norm $||g||_{0,p}$ is denoted more briefly by $||g||_p$. To forcefully distinguish L_p-norms in spaces of $Y-$valued functions, we write $||\cdot||_p$, while in all other cases a norm is denoted by $|\cdot|$.

The duality $\langle\cdot,\cdot\rangle_s$ between $\mathbb{H}_q^s\left(\mathbf{R}^d\right)$, and $\mathbb{H}_p^{-s}\left(\mathbf{R}^d\right), p\geq 2, s\in(-\infty,\infty)$, and $q=p/(p-1)$ is defined by

$$\langle\boldsymbol{\phi},\boldsymbol{\psi}\rangle_s=\langle\boldsymbol{\phi},\boldsymbol{\psi}\rangle_{s,p}=\sum_{i=1}^d\int_{\mathbf{R}^d}\left[\Lambda^s\phi^i\right](x)\,\Lambda^{-s}\psi^i(x)\,dx, \boldsymbol{\phi}\in\ \mathbb{H}_q^s, \boldsymbol{\psi}\in\mathbb{H}_p^{-s}.$$

3.1.2. *Solenoidal and gradient projections of Hilbert-valued vector fields.* In this section we present some facts about solenoidal and gradient projections of vector fields mostly proved in [37].

We will use the Riesz transform for the definition of the projections. We set for $f\in L_2(\mathbf{R}^d,Y)$,

$$R_j(f)(x)=\lim_{\varepsilon\to 0}c_*\int_{|y|\geq\varepsilon}\frac{y_j}{|y|^{d+1}}f(x-y)\,dy, j=1,\ldots,d,$$

with $c_*=G(\frac{n+1}{2})/\pi^{(n+1)/2}$ (G is the Gamma function). R_j is called a Riesz transform. According to [49] (see Chapter III , formula (8), p. 58),

$$\widehat{R_jf}(x)=-i\frac{\xi_j}{|\xi|}\widehat{f},$$

where

$$\widehat{f}(\xi)=\mathcal{F}(f)=(2\pi)^{-d/2}\int e^{-i(\xi,x)}f(x)\,dx.$$

Given a function $f\in L_p(\mathbf{R}^d,Y)$, we define a vector Riesz transform $Rf=(R_1f,\ldots,R_df)$.

For $\mathbf{v}\in\mathbb{L}_2(Y)$ set (see [24], [25])

$$\mathcal{G}(\mathbf{v})=-RR_jv^j, \mathcal{S}(\mathbf{v})=\mathbf{v}-\mathcal{G}(\mathbf{v}).$$

Then (see [24], [25], Lemma 2.7 in [37]), $\mathbb{L}_2(Y)$ is a direct sum

$$\mathbb{L}_2(Y)=\mathcal{G}(\mathbb{L}_2(Y))\oplus\mathcal{S}(\mathbb{L}_2(Y)),$$

$$\mathcal{S}(\mathbb{L}_2(Y))=\{\mathbf{g}\in\mathbb{L}_2(Y):\operatorname{div}\mathbf{g}=0\},$$

and $\mathcal{G}(\mathbb{L}_2(Y))$ is a Hilbert subspace orthogonal to $\mathcal{S}(\mathbb{L}_2(Y))$.

Remark 2. *If $f\in C_0^\infty(\mathbf{R}^d)$, it is known (see e.g.* [44]*) that the classical solution to*

$$\Delta u(x)=f(x),\ x\in\mathbf{R}^d \tag{3.1}$$

is given by the formula

$$u(x)=\int\Gamma(x-y)f(y)\,dy \tag{3.2}$$

where

$$\Gamma(x-y)=\begin{cases}|x-y|^{2-d}/d(2-d)\omega_d, & d>2\\ \frac{1}{2\pi}\ln|x-y|, & d=2.\end{cases}$$

and ω_d is the volume of the unit ball in $\mathbf{R}^d$. If $\mathbf{f}\in\mathbb{C}_0^\infty(Y)$, we may easily show that

$$\mathcal{G}(\mathbf{f})=\boldsymbol{\nabla}\int\Gamma_{x_i}(x-y)f^i(y)\,dy=-RR_jf^j. \tag{3.3}$$

The functions $\mathcal{G}(\mathbf{v})$ and $\mathcal{S}(\mathbf{v})$ are usually referred to as the potential and the solenoidal, projections, respectively, of the vector field $\mathbf{v}$.

The following statement holds.

Lemma 1. *(see* [24]*,* [25]*, Lemma 2.11 and Lemma 2.12 in* [37]*)* $\mathcal{G},\mathcal{S}$ *can be extended continuously to all* $\mathbb{H}_p^s(Y), s\in(-\infty,\infty)$ *: there is a constant* C *so that for all* $\mathbf{v}\in\mathbb{H}_p^s(Y)$

$$||\mathcal{G}(\mathbf{v})||_{s,p}\leq C||\mathbf{v}||_{s,p}, ||\mathcal{S}(\mathbf{v})||_{s,p}\leq C||\mathbf{v}||_{s,p}.$$

Moreover, the space $\mathbb{H}_p^s(Y)$ *can be decomposed into the direct sum:*

$$\mathbb{H}_p^s(Y)=\mathcal{G}(\mathbb{H}_p^s(Y))\oplus\mathcal{S}(\mathbb{H}_p^s(Y)),$$

and, if $(1/p)+(1/q)=1, \mathbf{f}\in\mathcal{G}(\mathbb{H}_p^s(Y)), \mathbf{g}\in\mathcal{S}(\mathbb{H}_q^{-s}(Y))$*, then*

$$\langle\mathbf{f},\mathbf{g}\rangle_{\mathbb{H}_p^s(Y),\mathbb{H}_q^{-s}(Y)}=0. \tag{3.4}$$

Also,

$$\mathcal{S}(\mathbb{H}_p^s(Y))=\{\mathbf{v}\in\mathbb{H}_p^s(Y): div\,\mathbf{v}=\mathbf{0}\}. \tag{3.5}$$

3.2. **Strong solutions of Navier-Stokes equation in $\mathbf{R}^d$.**

3.2.1. *Main Results.* Let $(\Omega,\mathcal{F},\mathbf{P})$ be a probability space with a filtration $\mathbb{F}$ of right continuous σ-algebras $(\mathcal{F}_t)_{t\geq0}$. All the $\sigma-$algebras are assumed to be $\mathbf{P}-$completed. Let $W(t)$ be an $\mathbb{F}$-adapted cylindrical Brownian motion in Y.

For $\mathbf{v}\in\mathbb{H}_p^1$, let $\mathbf{G}(\mathbf{v},t)=\mathbf{G}(\mathbf{v},t,x)$ be a predictable $\mathbb{L}_p(Y)$-valued function and $\mathbf{F}(\mathbf{v},t)=\mathbf{F}(\mathbf{v},t,x)$ a predictable $\mathbb{L}_p$-valued function. Let us consider the following Navier-Stokes equation:

$$\begin{cases}\partial_t u^l(t,x)=\partial_i\left(a^{ij}(t,x)\,\partial_j u^l(t,x)\right)-u^k(t,x)\,\partial_k u^l(t,x)\\ -\partial_l P(t,x)+b^i(t,x)\partial_i u(t,x)+F^l(\mathbf{u}(t),t,x)+\\ +[\sigma^i(t,x)\partial_i u^l(t,x)+G^l(\mathbf{u}(t),t,x)-\partial_l\tilde{P}(t,x)]\,\dot{W}_t,\\ \operatorname{div}\mathbf{u}=0,\mathbf{u}(0,x)=\mathbf{u}_0(x),\ \ l=1,\dots d.\end{cases} \tag{3.6}$$

In the vector form, the equation would be

$$\begin{aligned}&\partial_t\mathbf{u}(t)=\partial_i\left(a^{ij}(t)\,\partial_j\mathbf{u}(t)\right)-u^k(t)\,\partial_k\mathbf{u}(t)-\boldsymbol{\nabla}P(t)+b^i(t)\partial_i\mathbf{u}(t)+\mathbf{F}(\mathbf{u}(t),t)+\\ &\mathbf{F}(\mathbf{u}(t),t)+[\sigma^i(t)\partial_i\mathbf{u}(t)+\mathbf{G}(\mathbf{u}(t),t)-\boldsymbol{\nabla}\tilde{P}(t)]\,\dot{W}_t,\\ &\mathbf{u}(0)=\mathbf{u}_0,\operatorname{div}\mathbf{u}=0.\end{aligned} \tag{3.7}$$

Of course, the unknowns in the equation (3.7) are the functions $\mathbf{u}=\left(u^l\right)_{1\leq l\leq d}$, P, and $\tilde{P}$.

Everywhere in this section it is assumed that $p\geq2$. The vector field $\mathbf{u}_0$ is always $\mathcal{F}_0$-measurable and $\operatorname{div}\mathbf{u}_0=0$.

It is assumed that a^{ij},b^i are measurable $\mathbb{F}$-adapted functions on $[0,\infty)\times\mathbf{R}^d$, and the matrix $\left(a^{ij}\right)$ is symmetric. Let us assume also that σ^i is an Y-valued measurable $\mathbb{F}$-adapted functions on $[0,\infty)\times\mathbf{R}^d$.

In addition, we will need the following assumptions.

B1. **P**-a.s.

$$\sum_{k=0}^{1}(|\partial^k a^{ij}| + |\partial^k\, b^i| + |\partial^k\, \sigma^i|_Y) \le K;$$

for all $t \ge 0,\ x, \lambda \in \mathbf{R}^d$, we have

$$K|\lambda|^2 \ge [a^{ij}(t,x) - \frac{1}{2}\sigma^i(t,x)\cdot\sigma^j(t,x)]\lambda^i\lambda^j \ge \delta|\lambda|^2,$$

where K, δ are fixed strictly positive constants (notice, that this assumption excludes Euler equation).

B2(p). For all $\mathbf{v} \in \mathbb{H}^1_p, t > 0$

$$|\mathbf{F}(\mathbf{v},t) - \mathbf{F}(\bar{\mathbf{v}},t)|_p \le C|\mathbf{v} - \bar{\mathbf{v}}|_p, ||\mathbf{G}(\mathbf{v},t) - \mathbf{G}(\bar{\mathbf{v}},t)||_p \le C|\mathbf{v} - \bar{\mathbf{v}}|_p,$$

and for all $t > 0, \mathbf{v} \in \mathbb{H}^1_p$,

$$|\,|\mathbf{G}(\mathbf{v},t)|\,|_{1,p} \le |\,|\mathbf{G}(\mathbf{0},t)|\,|_{1,p} + C|\mathbf{v}|_{1,p}, |\mathbf{F}(\mathbf{v},t)|_{1,p} \le |\mathbf{F}(\mathbf{0},t)|_{1,p} + C|\mathbf{v}|_{1,p}.$$

Suppose also that

$$\int_0^t (|\,|\mathbf{G}(\mathbf{0},r)|\,|^p_{1,p} + |\mathbf{F}(\mathbf{0},r)|^p_{1,p})\,dr < \infty$$

P-a.s. for all t.

B3(p). For each M, there is a constant C such that for all $\mathbf{v}, \bar{\mathbf{v}} \in B_{M,p} = \{\mathbf{v} \in \mathbb{H}^1_p : |\mathbf{v}|_{1,p} \le M\}, t > 0$

$$||\nabla(\mathbf{G}(\mathbf{v},t) - \mathbf{G}(\bar{\mathbf{v}},t))||_p \le C|\mathbf{v} - \bar{\mathbf{v}}|_{1,p}.$$

Since $div\,\mathbf{u} = 0$, we have

$$\begin{gathered} div\left(\sigma^i(t)\partial_i\mathbf{u}\,(t) + \mathbf{G}\,(\mathbf{u}\,(t)\,,t) - \boldsymbol{\nabla}\tilde{P}\,(t)\right) = 0 \\ \text{and} \\ div\left[-u^k\,(t)\,\partial_k\mathbf{u}(t)) + \partial_i\left(a^{\,ij}\,(t)\,\partial_j\mathbf{u}\,(t)\right) + \mathbf{F}\,(\mathbf{u}(t),t) - \boldsymbol{\nabla}P(t)\right] = 0. \end{gathered} \tag{3.8}$$

Then, if the expressions in the left hand sides of both equations in (3.8) belong to $\mathbb{H}^1_p$ for some $p > 1$, by Remark 2 we have

$$\boldsymbol{\nabla}\tilde{P}(t,x) = \mathcal{G}\left(\sigma^i(t)\partial_i\mathbf{u}\,(t) + \mathbf{G}(\mathbf{u}(t),t)\right),$$

and

$$\boldsymbol{\nabla}P(t,x) = \mathcal{G}[-u^k\,(t))\partial_k\mathbf{u}(t)) + \partial_i\left(a^{\,ij}\,(t)\,\partial_j\mathbf{u}\,(t)\right) + \mathbf{F}\,(\mathbf{u}(t),t)].$$

So, in $\mathbb{L}_p$-theory instead of equation (3.7), we can and will consider its equivalent form

$$\tag{3.9} \left\{ \begin{aligned} &\partial_t\mathbf{u}\,(t) = \mathcal{S}[\partial_i\left(a^{\,ij}\,(t)\,\partial_j\mathbf{u}\,(t)\right) - u^k\,(t)\,\partial_k\mathbf{u}(t) + b^i(t)\partial_i\mathbf{u}(t) + \mathbf{F}\,(\mathbf{u}\,(t)\,,t)] \\ &+\mathcal{S}[\sigma^i(t)\partial_i\mathbf{u}\,(t) + \mathbf{G}\,(\mathbf{u}\,(t)\,,t)]\,\dot{W}_t, \mathbf{u}\,(0) = \mathbf{u}_0. \end{aligned} \right.$$

Given a stopping time τ, we define a stochastic interval

$$[[0,\tau(\omega)]] = \left\{ \begin{array}{cc} [0,\tau(\omega)], & \text{if } \tau(\omega) < \infty, \\ [0,\infty), & \text{otherwise.} \end{array} \right.$$

Let $s \in \{0,1\}$.

Definition 1. *Given a stopping time* τ*, an* $\mathbb{H}_p^s(\mathbf{R}^d)$*-valued* $\mathbb{F}-$*adapted function* $\mathbf{u}(t)$ *on* $[0,\infty)$ *is called an* $\mathbb{H}_p^s$*-solution of equation (3.7) (or (3.9)) in* $[[0,\tau]]$ *if it is strongly continuous in* t *with probability 1,*

$$\mathbf{u}(t)=\mathbf{u}(t\wedge\tau) \text{ and } \int_0^{t\wedge\tau}|\mathbf{u}(r)|_{s+1,p}^p\,dr<\infty\ \forall t>0,\mathbf{P}-a.s., \tag{3.10}$$

and the equality

$$\begin{aligned}\mathbf{u}(t)=\mathbf{u_0}+\textstyle\int_0^{t\wedge\tau}\mathcal{S}[-u^i\,(r)\,\partial_i\mathbf{u}\,(r)+\partial_i(a^{ij}(r)\partial_j\mathbf{u}\,(r)\,)+\mathbf{F}(\mathbf{u}(r),r)]dr+\\ \textstyle\int_0^{t\wedge\tau}\mathcal{S}(\sigma^k(r)\partial_k\mathbf{u}\,(r)+\mathbf{G}(\mathbf{u}(r),r))\cdot dW\,(r)\end{aligned} \tag{3.11}$$

holds in $\mathbb{H}_p^{s-1}(\mathbf{R}^d)$ *for every* $t>0$*,* $\mathbf{P}$*-a.s.*

If $\tau=\infty$, we simply say $\mathbf{u}$ is an $\mathbb{H}_p^s$-solution of equation (**??**). The stochastic integral in (3.11) is defined in the Appendix.

Sometimes, when the context is clear, instead of "$\mathbb{H}_p^s$-solution" we will just say "solution".

If an $\mathbb{H}_p^s$-solution in $[[0,\tau]]$ is also $\mathbb{H}_q^s$-solution in $[[0,\tau]]$, we call it $\mathbb{H}_p^s\cap\mathbb{H}_q^s$-solution in $[[0,\tau]]$.

Example 1. *Let* f^l *be measurable* $\mathbb{F}$*-adapted functions on* $[0,\infty)\times\mathbf{R}^d\times\mathbf{R}^d$*. Let* $h^{l,i}$ *be* Y*-valued measurable* $\mathbb{F}$*-adapted functions on* $[0,\infty)\times\mathbf{R}^d$*, and* g^l *be* Y*-valued measurable* $\mathbb{F}$*-adapted functions on* $[0,\infty)\times\mathbf{R}^d\times\mathbf{R}^d$*. Given* $\mathbf{v}\in\mathbb{H}_p^1$*, define*

$$\begin{aligned}\mathbf{G}(\mathbf{v},t)&=(g^l(t,x,\mathbf{v}(x)))_{1\le l\le d}\\ \mathbf{F}(\mathbf{v},t)&=(f^l(t,x,\mathbf{v}(x))_l+(h^{l,j}(t,x)L_j(t,x,\mathbf{v}))_{1\le l\le d},\end{aligned} \tag{3.12}$$

where $\mathbf{L}(t,x,\mathbf{v})=(L_l(t,x,\mathbf{v}))_{1\le l\le d}=\mathcal{G}[\sigma^k(t)\partial_k\mathbf{v}+\mathbf{G}(\mathbf{v},t)]$*.*

Assume that for each $t\ge 0,x\in\mathbf{R}^d$*,* $u\in\mathbf{R}^d$*,*

$$\begin{aligned}\sum_{k=0}^1|\partial_x^k\mathbf{g}(s,x,u)|&\le G_1(s,x)+K|u|,\\ \sum_{k=0}^1|\partial_x^k\mathbf{f}(s,x,u)|&\le F_1(s,x)+K|u|,\\ |\partial_u\mathbf{g}|+|\partial_u\mathbf{f}|+\sum_{k=0}^1(|\partial_x^k h^{l,j}|_Y+|\partial_x^k\sigma^j|_Y)&\le C,\end{aligned} \tag{3.13}$$

and $\mathbf{P}$*-a.s. for all* t

$$\int_0^t(|G_1(r)|_p^p+|F_1(r)|_p^p)\ dr<\infty. \tag{3.14}$$

The assumptions (3.13), (3.14), imply the assumption ***B2****(p) for* $\mathbf{G},\mathbf{F}$ *defined by (3.12).*

The assumptions (3.13), (3.14) and the boundedness of $\partial_u^2\mathbf{g}(t,x,u)$ *imply the assumption* ***B3****(p) for* $\mathbf{G}$*.*

Now we can formulate the main theorems on local and global existence and uniqueness.

Theorem 1. *a) Let **B1**, **B2**(p), **B3**(p) be satisfied* ($p > d$) *and* $\mathbf{E}(|\mathbf{u}_0|^p_{1,p}) < \infty$.

Then there is a unique predictable stopping time ζ, $\mathbf{P}(\zeta > 0) = 1$, *such that for each stopping time* $S, [0, S] \subseteq [0, \zeta)$ *if and only if there is a* $\mathbb{H}^1_p$*-valued continuous* $\mathbb{L}_p$*-solution to (3.7) in* $[[0, S]]$;

Also, there is a unique $\mathbb{H}^1_p$*-valued continuous process* $\mathbf{u}(t)$ *on* $[0, \zeta)$, *such that* $\limsup_{t\uparrow\zeta} |\mathbf{u}(t)|_{1,p} = \infty$ *on* $\{\zeta < \infty\}$, *and* $\mathbf{u}(t \wedge S)$ *is* $\mathbb{L}_p$*-solution of (3.7) in* $[[0, S]]$ *for each* S, *so that* $[0, S] \subseteq [0, \zeta)$.

Moreover, if $\mathbf{E}(|\mathbf{u}_0|^p_{2-2/p,p}) < \infty$, *then* $\mathbf{u}(t)$ *is also* $\mathbb{H}^1_p$*-solution of (3.7) in* $[[0, S]]$ *for all stopping times* S, *such that* $[0, S] \subseteq [0, \zeta)$ *and* $\lim_{t\uparrow\zeta} |\mathbf{u}(t)|_{1,p} = \infty$ *on* $\{\zeta < \infty\}$ $\mathbf{P}$*-a.s.*

*b) Let **B1**, **B2**(p), **B3**(p), **B2**(2), **B3**(2) be satisfied, and*

$$\mathbf{E}(|\mathbf{u}_0|^p_{1,p} + |\mathbf{u}_0|^p_{1,2}) < \infty,$$

$$\int_0^t (||\mathbf{G}(\mathbf{0}, r)||^p_{1,2} + |\mathbf{F}(\mathbf{0}, r)|^p_{1,2})\, dr < \infty$$

$\mathbf{P}$*-a.s. for all* t. *Then there is a unique predictable stopping time* ζ, $\mathbf{P}(\zeta > 0) = 1$, *such that for each stopping time* $S, [0, S] \subseteq [0, \zeta)$ *if and only if there is a* $\mathbb{H}^1_p \cap \mathbb{H}^1_2$*-valued continuous* $\mathbb{L}_p \cap \mathbb{L}_2$*-solution to (3.7) in* $[[0, S]]$;

Also, there is a unique $\mathbb{H}^1_p \cap \mathbb{H}^1_2$*-valued continuous process* $\mathbf{u}(t)$ *on* $[0, \zeta)$, *such that* $\limsup_{t\uparrow\zeta}(|\mathbf{u}(t)|_{1,p} + |\mathbf{u}(t)|_{1,2}) = \infty$ *on* $\{\zeta < \infty\}$, *and* $\mathbf{u}(t \wedge S)$ *is an* $\mathbb{L}_p \cap \mathbb{L}_2$*-solution to (3.7) on* $[0, S]$ *for each* S, *so that* $[0, S] \subseteq [0, \zeta)$.

Moreover, if $\mathbf{E}(|\mathbf{u}_0|^p_{2-2/p,p}) < \infty$, *then* $\mathbf{u}(t)$ *is also* $\mathbb{H}^1_p \cap \mathbb{H}^1_2$*-solution of (3.7) in* $[[0, S]]$ *for all stopping times* S, *such that* $[0, S] \subseteq [0, \zeta)$.

In both cases, $(u(t), \zeta)$ is called a maximal solution to (3.7), and ζ is called its explosion time.

If $d = 2$, a stronger result holds. Specifically, there is a unique global solution.

Theorem 2. *Let **B1**, **B2**(p), **B3**(p), **B2**(2), **B3**(2) be satisfied,* $p > d = 2$, *and*

$$\mathbf{E}(|\mathbf{u}_0|^p_{2-2/p,p} + |\mathbf{u}_0|^p_{1,2}) < \infty,$$

$$\int_0^t (||\mathbf{G}(\mathbf{0}, r)||^p_{1,2} + |\mathbf{F}(\mathbf{0}, r)|^p_{1,2})\, dr < \infty$$

$\mathbf{P}$*-a.s. for all* t.

Then there is a maximal unique $\mathbb{H}^1_p \cap \mathbb{H}^1_2$*-solution* $(\mathbf{u}(t), \zeta)$ *of (3.7) and* $\mathbf{P}(\zeta = \infty) = 1$.

Moreover, for each $T > 0$ *there is a constant* C, *such that for all stopping times* $\tau \le T$,

$$\mathbf{E}\sup_{s\le\tau}(|\mathbf{u}(t)|^p_{1,p} + |\mathbf{u}(t)|^p_{1,2}) \le C[\mathbf{E}(|\mathbf{u}_0|^p_{1,p} + |\mathbf{u}_0|^p_{1,2})$$

$$+ \textstyle\int_0^\tau (||\mathbf{G}(\mathbf{0}, r)||^p_{1,p} + |\mathbf{F}(\mathbf{0}, r)|^p_{1,p} + ||\mathbf{G}(\mathbf{0}, r)||^p_{1,2} + |\mathbf{F}(\mathbf{0}, r)|^p_{1,2})\, dr].$$

Proof of these theorems will be given in Sections 3.3-3.6.

3.3. **Mollified Navier-Stokes equation.** In this section we consider an auxiliary equation obtained from (3.7) by applying the standard mollifier to the first term of the Navier-Stokes nonlinearity $(\mathbf{u}\cdot\boldsymbol{\nabla})\,\mathbf{u}$.

Let $\psi(x)\in C_0^\infty(\mathbf{R}^d),\psi\geq 0,\ \int\psi\,dx=1$. Given a scalar function v on $\mathbf{R}^d$, we define

$$\Psi^\varepsilon(v)(x)=\begin{cases}\int v(x-y)\psi_\varepsilon(y)\,dy, & \varepsilon>0,\\ v, & \varepsilon=0,\end{cases}$$

where $\psi_\varepsilon(x)=\varepsilon^{-d}\psi(x/\varepsilon)$, $\varepsilon>0$. Similarly, for a vector function $\mathbf{v}$:

$$\Psi^\varepsilon(\mathbf{v})(x)=\begin{cases}(\int v^l(x-y)\psi_\varepsilon(y)\,dy)_l, & \varepsilon>0,\\ \mathbf{v}(x), & \varepsilon=0.\end{cases}$$

For a fixed $\varepsilon\geq 0$, we consider the equation for $\mathbf{u}=\left(u^l\right)_{1\leq l\leq d}$, P, $\tilde{P}$

$$\partial_t\mathbf{u}(t,x)=\partial_i(a^{ij}(t,x)\partial_j\mathbf{u})-\Psi^\varepsilon(u^k(t))\partial_k\mathbf{u}(t)+\mathbf{D}(\mathbf{u}(t),t,x)-\boldsymbol{\nabla}P(t,x)$$

$$+[\sigma^k(t,x)\partial_k\mathbf{u}(t,x)+\mathbf{G}(\mathbf{u}(t),t,x)-\boldsymbol{\nabla}\tilde{P}(t,x)]\cdot\dot{W}, \tag{3.15}$$

$$\mathbf{u}(0,x)=\mathbf{u}_0(x),\ \operatorname{div}\mathbf{u}=0,$$

where $\mathbf{u}(t)=\mathbf{u}(t,x)=(u^k(t,x))_{1\leq k\leq d}$ and $\mathbf{D}(\mathbf{v},t)=b^i(t)\partial_i\mathbf{v}+\mathbf{F}(\mathbf{v},t)$.

Obviously, if $\varepsilon=0$, (3.15) coincides with (3.7).

Similarly to (3.7), the equation (3.15) is equivalent to

$$\partial_t\mathbf{u}(t)=\mathcal{S}[\partial_i(a^{ij}(t)\partial_j\mathbf{u}(t))-\Psi^\varepsilon(u^k(t))\partial_k\mathbf{u}(t)+\mathbf{D}(\mathbf{u}(t),t))]+\mathcal{S}[\sigma^k(t)\partial_k\mathbf{u}(t)+\mathbf{G}(\mathbf{u}(t),t)]\cdot\dot{W}, \tag{3.16}$$

$$\mathbf{u}(0)=\mathbf{u}_0.$$

For $\varepsilon>0$ we will solve (3.16) in $\mathbb{H}_p^s, s\in(-\infty,\infty), p\geq 2$.

Definition 2. *Given a stopping time τ, an $\mathbb{H}_p^s(\mathbf{R}^d)$-valued $\mathbb{F}-$adapted function $\mathbf{u}(t)$ on $[0,\infty)$ is called an $\mathbb{H}_p^s$-solution of equation (3.15) (or (3.16)) in $[[0,\tau]]$ if it is strongly continuous in t with probability 1,*

$$\mathbf{u}(t)=\mathbf{u}(t\wedge\tau),\ \int_0^{t\wedge\tau}|\mathbf{u}(r)|_{s+1,p}^p\,dr<\infty\ \forall t>0,\mathbf{P}-a.s., \tag{3.17}$$

and the equality

$$\mathbf{u}(t)=\mathbf{u_0}+\int_0^{t\wedge\tau}\mathcal{S}[-\Psi^\varepsilon(u^i)\partial_i\mathbf{u}+\partial_i(a^{ij}(r)\partial_j\mathbf{u})+\mathbf{D}(\mathbf{u})]dr+\int_0^{t\wedge\tau}\mathcal{S}(\sigma^k\partial_k\mathbf{u}+\mathbf{G}(\mathbf{u}))\cdot dW\,(r) \tag{3.18}$$

holds in $\mathbb{H}_p^{s-1}(\mathbf{R}^d)$ for every $t>0$, $\mathbf{P}$-a.s.

If $\tau=\infty$, we simply say $\mathbf{u}$ is an $\mathbb{H}_p^s$-solution of equation (3.15).

If an $\mathbb{H}_p^s$-solution in $[[0,\tau]]$ is also $\mathbb{H}_q^s$-solution in $[[0,\tau]]$, we call it $\mathbb{H}_p^s\cap\mathbb{H}_q^s$-solution in $[[0,\tau]]$.

In this subsection, we fix $\varepsilon>0$ and consider the corresponding equation (3.15) (equivalently (3.16)).

For an integer $s > 0$, we denote

$$\mathcal{C}^s(Y) = \left\{ u \in C^{s-1} : ||u||_{\mathcal{C}^s} = \sum_{|a| \le s-1} ||\partial^\alpha u||_\infty + \sum_{|\alpha| = s-1} \sup_{x \ne y} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|_Y}{|x-y|} < \infty \right\}.$$

Define

$$B^s(Y) = \begin{cases} H^s_\infty(Y), & \text{if } s > 0 \text{ is not an integer,} \\ \mathcal{C}^s(Y), & \text{if } s > 0 \text{ is an integer,} \\ L_\infty(Y), & \text{if } s = 0, \end{cases}$$

and denote the corresponding norms by $|\cdot|_{B^s}$.

The following assumptions will be often used in the future.

A. For all $t \ge 0$, $x, \lambda \in R^d$,

$$K|\lambda|^2 \ge [a^{ij}(t,x) - \frac{1}{2}\sigma^i(t,x) \cdot \sigma^j(t,x)]\lambda^i \lambda^j \ge \delta |\lambda|^2,$$

where K, δ are fixed strictly positive constants.

A1(s,p). For all t, x, y, **P**-a.s.

$$|a^{ij}(t,x) - a^{ij}(t,y)| + |\sigma^i(t,x) - \sigma^i(t,y)|_Y \le K|x-y|$$

and

$$\begin{cases} |a^{ij}(t)|_{B^s} \le K, & \text{if } s \ge 1, \\ \\ |a(t,x)| \le K, & \text{if } -1 < s < 1, \\ \\ |a^{ij}(t)|_{B^{-s+\varepsilon}} \le K, & \text{if } s \le -1. \end{cases}$$

where $\varepsilon \in (0,1)$.

For all i, t, x

$$\begin{cases} ||\sigma^i(t)||_{B^s} \le K, & \text{if } s \ge 1, \\ |\sigma^i(t,x)|_Y \le K, & \text{if } s \in (-1,1), \\ ||\sigma^i(t)||_{B^{-s+\varepsilon}} \le K, & \text{if } s \le -1, \end{cases}$$

where $\varepsilon \in (0,1)$.

A2(s,p) For $\mathbf{v} \in \mathbb{H}^{s+1}_p$, $\mathbf{G}(\mathbf{v}, t) = \mathbf{G}(\mathbf{v}, t, x)$ is a predictable $\mathbb{H}^s_p(Y)$-valued function and $\mathbf{D}(\mathbf{v}, t) = \mathbf{D}(\mathbf{v}, t, x)$ is a predictable $\mathbb{H}^{s-1}_p$-valued function, and **P**-a.s. for each t

$$\int_0^t (|\mathbf{D}(\mathbf{0}, r)|^p_{s-1,p} + ||\mathbf{G}(\mathbf{0}, r)||^p_{s,p})\, dr < \infty \ \forall t > 0, \mathbf{P} - a.s.$$

where $\mathbf{0} = (0, \ldots, 0)$.

A3(s,p). For every $\varepsilon > 0$, there exists a constant K_ε such that for any $\mathbf{u}, \mathbf{v} \in \mathbb{H}^{s+1}_p$,

$$|\mathbf{D}(\mathbf{u}, t, x) - \mathbf{D}(\mathbf{v}, t, x)|_{s-1,p} + ||\mathbf{G}(\mathbf{u}, t, x) - \mathbf{G}(\mathbf{v}, t, x)||_{s,p} \le$$

$$\varepsilon |\mathbf{u} - \mathbf{v}|_{s+1,p} + K_\varepsilon |\mathbf{u} - \mathbf{v}|_{s-1,p} \ \ \mathbf{P} - a.s.$$

We start with the following statement.

Proposition 1. *Let* $s \in (-\infty, \infty), p \in [2, \infty)$. *Assume* ***A***, ***A1***(s,p)-***A3***(s,p) *are satisfied and* $\mathbf{E}(|\mathbf{u}_0|^p_{s+1-2/p,p}) < \infty$. *Then there is a unique predictable stopping time* ζ, $\mathbf{P}(\zeta > 0) = 1$, *such that, for each stopping time* S, $[0, S] \subseteq [0, \zeta)$ *if and only if there is a unique* $\mathbb{H}^s_p$*-solution to (3.15) in* $[[0, S]]$;

Also, there is a unique $\mathbb{H}^s_p$*-valued continuous process* $\mathbf{u}(t)$ *on* $[0, \zeta)$ *such that* **P**-*a.s.* $\limsup_{t \uparrow \zeta} |\mathbf{u}(t)|_{s,p} = \infty$ *on* $\{\zeta < \infty\}$, *and* $\mathbf{u}(t \wedge S)$ *is a solution to (3.15) in*

$[[0,S]]$ for each S so that $[0,S]\subseteq[0,\zeta)$. Moreover, for each $T>0, M>1$ there is a constant C, such that for each stopping time $\tau\le T\wedge\tau_M$

$$\mathbf{E}[\sup_{r\le\tau}|\mathbf{u}(r)|^p_{s,p}+\int_0^\tau|\partial^2\mathbf{u}(r)|^p_{s-1,p}\,dr]\le C\mathbf{E}[|\mathbf{u}_0|^p_{s+1-2/p,p}+\int_0^\tau(|\mathbf{D}(\mathbf{0},r)|^p_{s-1,p}$$
(3.19)
$$+|\mathbf{G}(\mathbf{0},r)|^p_{s,p})\,dr],$$

where $\tau_M=\inf(t:|\mathbf{u}(t)|_{s,p}\ge M)$.

(The pair $(\mathbf{u}(t),\zeta)$ is called a maximal $\mathbb{H}^s_p$-solution of (3.15))

Proof. For each $M>0$, we define a function on $\mathbb{H}^s_p$

$$\varphi_M(\mathbf{u})=\begin{cases}\mathbf{u}, & \text{if } |\mathbf{u}|_{s,p}\le M,\\ M|\mathbf{u}|^{-1}_{s,p}\mathbf{u}, & \text{otherwise.}\end{cases}$$

For every $\mathbf{u},\bar{\mathbf{u}}\in\mathbb{H}^s_p$, we have obviously $|\varphi_M(\mathbf{u})|_{s,p}\le M$ and

$$|\varphi_M(\mathbf{u})-\varphi_M(\bar{\mathbf{u}})|_{s,p}\le 2|\mathbf{u}-\bar{\mathbf{u}}|_{s,p}.$$

Define a function $\mathbf{B}^k_M(\mathbf{u})=u^k_\varepsilon\varphi_M(\mathbf{u}),\mathbf{u}\in\mathbb{H}^s_p$, where $u^k_\varepsilon=\Psi^\varepsilon(u^k)$. There is a constant C, so that for each $\mathbf{u},\mathbf{v}\in\mathbb{H}^s_p$

$$|\mathbf{B}^k_M(\mathbf{u})-\mathbf{B}^k_M(\mathbf{v})|_{s,p}\le CM|\mathbf{u}-\mathbf{v}|_{s,p}. \tag{3.20}$$

Indeed, if $|\mathbf{u}|_{s,p}\le M,|\mathbf{v}|_{s,p}\le M$, then

$$\mathbf{B}^k_M(\mathbf{u})-\mathbf{B}^k_M(\mathbf{v})=u^k_\varepsilon\mathbf{u}-v^k_\varepsilon\mathbf{v}=(u^k_\varepsilon-v^k_\varepsilon)\mathbf{u}+v^k_\varepsilon(\mathbf{u}-\mathbf{v}),$$

and, by Lemma 7 in [38],

$$|\mathbf{B}^k_M(\mathbf{u})-\mathbf{B}^k_M(\mathbf{v})|_{s,p}\le|u^k_\varepsilon-v^k_\varepsilon|_{B^{|s|}}|\mathbf{u}|_{s,p}+|v^k_\varepsilon|_{B^{|s|}}|\mathbf{u}-\mathbf{v}|_{s,p}\le CM|\mathbf{u}-\mathbf{v}|_{s,p}.$$

If $|\mathbf{u}|_{s,p}\le M,|\mathbf{v}|_{s,p}>M$, then

$$\mathbf{B}^k_M(\mathbf{u})-\mathbf{B}^k_M(\mathbf{v})=u^k_\varepsilon\mathbf{u}-v^k_\varepsilon\mathbf{v}M|\mathbf{v}|^{-1}_{s,p}$$

$$=|\mathbf{v}|^{-1}_{s,p}[(u^k_\varepsilon-v^k_\varepsilon)\mathbf{v}M+Mu^k_\varepsilon(\mathbf{u}-\mathbf{v})+u^k_\varepsilon\mathbf{u}(|\mathbf{v}|_{s,p}-M)],$$

and, by Lemma 7 in [38],

$$|\mathbf{B}^k_M(\mathbf{u})-\mathbf{B}^k_M(\mathbf{v})|_{s,p}\le C[|u^k_\varepsilon-v^k_\varepsilon|_{B^{|s|}}M+M|u^k_\varepsilon|_{B^{|s|}}|\mathbf{u}-\mathbf{v}|_{s,p}$$

$$+|u^k_\varepsilon|_{B^{|s|}}|\mathbf{u}-\mathbf{v}|_{s,p}]\le CM|\mathbf{u}-\mathbf{v}|_{s,p}.$$

Similarly, if $|\mathbf{u}|_{s,p}>M,|\mathbf{v}|_{s,p}>M$, then

$$\mathbf{B}^k_M(\mathbf{u})-\mathbf{B}^k_M(\mathbf{v})=u^k_\varepsilon\mathbf{u}M|\mathbf{u}|^{-1}_{s,p}-v^k_\varepsilon\mathbf{v}M|\mathbf{v}|^{-1}_{s,p}=$$

$$M|\mathbf{u}|^{-1}_{s,p}|\mathbf{v}|^{-1}_{s,p}[(u^k_\varepsilon-v^k_\varepsilon)\mathbf{u}|\mathbf{v}|_{s,p}+v^k_\varepsilon(\mathbf{u}-\mathbf{v})|\mathbf{u}|_{s,p}+v^k_\varepsilon\mathbf{u}(|\mathbf{v}|_{s,p}-|\mathbf{u}|_{s,p})],$$

and

$$|\mathbf{B}^k_M(\mathbf{u})-\mathbf{B}^k_M(\mathbf{v})|_{s,p}\le C[|u^k_\varepsilon-v^k_\varepsilon|_{B^{|s|}}M+M|v^k_\varepsilon|_{B^{|s|}}|\mathbf{u}-\mathbf{v}|_{s,p}|\mathbf{v}|^{-1}_{s,p}$$

$$+|v^k_\varepsilon|_{B^{|s|}}|\mathbf{u}-\mathbf{v}|_{s,p}|\mathbf{v}|^{-1}_{s,p}]\le CM|\mathbf{u}-\mathbf{v}|_{s,p}.$$

So, (3.20) holds and therefore

$$|\mathbf{B}^k_M(\mathbf{u})|_{s,p} \le CM|\mathbf{u}|_{s,p}. \tag{3.21}$$

Since (3.20), (3.21) hold, then, according to Theorem 3.3 in [37] and Remark 5.5 in [28], for each M, there is a unique $\mathbb{H}^s_p$-solution $\mathbf{u} = \mathbf{u}_M$ of the equation

$$\partial_t \mathbf{u}(t) = \partial_i(a^{ij}(t)\partial_j \mathbf{u}(t)) - \partial_k \mathbf{B}^k_M(\mathbf{u}(t)) + \mathbf{D}(\mathbf{u}(t),t) + \nabla p(t)$$

$$+ [\sigma^k(t)\partial_k \mathbf{u}(t) + \mathbf{G}(\mathbf{u}(t),t) + \nabla\tilde{p}(t)] \cdot \dot{W},$$

$$\mathbf{u}(0) = \mathbf{u}_0, \operatorname{div}\mathbf{u} = 0.$$

Let $\tau_M = \inf\{t : |\mathbf{u}_M(t)|_{s,p} \ge M\}$. By Corollary 3.6 in [37], $\mathbf{P}$-a.s.

$$\mathbf{u}_M(t \wedge \tau_M) = \mathbf{u}_{M'}(t \wedge \tau_M) \text{ for all } t, \tag{3.22}$$

if $M' > M$. Following the proof of Theorem 14.21 in [22], we consider the set $\mathcal{S}$ of all stopping times S, such that a $\mathbb{H}^s_p$-solution to (3.15) exists on $[0, S]$. Obviously, $\mathcal{S}$ is not empty ($\tau_M \in \mathcal{S}$ for all M). It is closed with respect to the finite minimum and finite maximum operations. Let ζ be the essential upper bound of the set $\mathcal{S}$. So, there is a sequence $T_n \in \mathcal{S}$ increasing to ζ. Let $\mathbf{U}_n$ be a corresponding sequence of solutions in $[0, T_n]$. The sequence $\mathbf{U}_n$ defines a solution $\mathbf{u}$ on $\cup_n[0, T_n]$. Let $y_t = |\mathbf{u}(t)|^p_{s,p}$, $R_m = \zeta \wedge \inf(t : y_t \ge m)$. Then $\mathbf{u}(\,\cdot \wedge T_q \wedge R_m)$ is a solution in $[0, T_q \wedge R_m]$. Passing to a limit as $q \to \infty$, we obtain that $\mathbf{u}(\cdot \wedge R_m)$ is a solution in $[0, R_m]$. If $\mathbf{P}(R_m = \zeta < \infty) > 0$, then (3.22) would imply that there is a stopping time $S \in \mathcal{S}$ such that $S \ge R_m$ and $\mathbf{P}(R_m = \zeta < S) > 0$. This would contradict the definition of ζ. Thus $\mathbf{P}$-a.s. $R_m < \zeta$ on $\{\zeta < \infty\}$, and $\limsup_{t\uparrow\zeta} y_t = \infty$ on $\{\zeta < \infty\}$. So the sequence (R_m) "announces" ζ and ζ is a predictable stopping time. Let S be a stopping time such that $\mathbf{P}$-a.s. $S < \zeta$. Then $\mathbf{u}(\cdot \wedge S)$ is a solution in $[0, S]$: it is enough to notice that $\mathbf{u}(\cdot \wedge R_q \wedge S)$ is a solution in $[0, R_q \wedge S]$ and pass to the limit as $q \to \infty$.

Let $\tau_M = \inf(t : y_t \ge m)$. Since $\mathbf{u}(\cdot \wedge \tau_M) = \mathbf{u}_M(\cdot \wedge \tau_M)$, it follows by Theorem 3.3 in [37], that for each T and M there is a constant C, so that for each stopping time $\tau \le \tau_M \wedge T$ and all t,

$$\mathbf{E}[\sup_{r\le\tau} |\mathbf{u}(r \wedge \tau)|^p_{s,p} + \int_0^{t\wedge\tau} |\partial^2 \mathbf{u}(r)|^p_{s-1,p}\, dr]$$

$$\le C\mathbf{E}[|\mathbf{u}_0|^p_{s+1,p} + \int_0^{t\wedge\tau} (|\mathbf{D}(\mathbf{0},r)|^p_{s-1,p} + |u^k_\varepsilon(r)\,\mathbf{u}(r)|^p_{s,p} + |\mathbf{G}(\mathbf{0},r)|^p_{s,p})\, dr]$$

$$\le C\mathbf{E}[|\mathbf{u}_0|^p_{s+1,p} + \int_0^{t\wedge\tau} (|\mathbf{D}(\mathbf{0},r)|^p_{s-1,p} + |\mathbf{u}(r)|^p_{s,p} + |\mathbf{G}(\mathbf{0},r)|^p_{s,p})\, dr].$$

So, the inequality (3.19) follows by Gronwall Lemma. ∎

Corollary 1. *Let* $s \in (-\infty, \infty), p \in [2, \infty)$. *Assume* ***A***, ***A1***(s,p)-***A3***(s,p). *Assume further* ***A1***(s,q)-***A3***(s,q) *for* $q \ge 2$, *and suppose that* $\mathbf{E}(|\mathbf{u}_0|^p_{s+1-2/p,p} + |\mathbf{u}_0|^q_{s+1-2/q,q}) < \infty$. *Then the maximal unique* $\mathbb{H}^s_p$*−solution* $(\mathbf{u}, \boldsymbol{\zeta})$ *of equation*

(3.15) defined in Proposition 1 is also a maximal unique $\mathbb{H}^s_q-$*solution of the equation. Moreover, for each* $T>0, M>1$, *there is a constant* C, *such that for each stopping time* $\tau \le T\wedge\tau_M$,

$$\mathbf{E}[\sup_{r\le\tau}|\mathbf{u}(r)|^l_{s,l}+\int_0^\tau|\partial^2\mathbf{u}(r)|^l_{s-1,l}\,dr]\le C\mathbf{E}[|\mathbf{u}_0|^l_{s+1,l}+\int_0^\tau(|\mathbf{D}(\mathbf{0},r)|^l_{s-1,l}$$

$$+|\mathbf{G}(\mathbf{0},r)|^l_{s,l})\,dr],\ l=p,q$$

where $\tau_M=\inf(t:|\mathbf{u}(t)|_{s,p}\ge M)$.

(The pair $(\mathbf{u}(t),\zeta)$ *is called a maximal* $\mathbb{H}^1_p\cap\mathbb{H}^1_q$*-solution.)*

Proof. Let $(\mathbf{u}(t),\zeta)$ be the maximal $\mathbb{H}^s_p-$solution $\mathbf{u}$ of equation (3.15), $\tau_M=\inf\{t:|\mathbf{u}(t)|_{s,p}\ge M\}$. Consider the equation for $\boldsymbol{\xi}$:

$$\partial_t\boldsymbol{\xi}(t)=\mathcal{S}[\partial_i(a^{ij}(t)\partial_j\boldsymbol{\xi}(t))-\partial_k(\Psi^\varepsilon(u^k(t\wedge\tau_M))\boldsymbol{\xi}(t))+\mathbf{D}(\boldsymbol{\xi}(t),t)]$$

$$+\mathcal{S}[\sigma^k(t)\partial_k\boldsymbol{\xi}(t)+\mathbf{G}(\boldsymbol{\xi}(t),t)]\cdot\dot{W},\quad \boldsymbol{\xi}(0)=\mathbf{u}_0(x).$$

By Theorem 3.3, Corollary 3.7, and Corollary 3.6 in [37], $\boldsymbol{\xi}(t)=\mathbf{u}(t)$ is also a unique $\mathbb{H}^s_q$-solution of (3.15) in $[[0,\tau_M]]$, and the statement obviously follows. ∎

Proposition 2. *Assume that for each* $\mathbf{v}\in\mathbb{H}^{s+1}_p$, $\mathbf{G}(\mathbf{v},t)$ *is a predictable* $\mathbb{H}^{s+1}_p$*-valued process and* $\mathbf{D}(\mathbf{v},t)$ *is a predictable* $\mathbb{H}^s_p$*-valued process. Let* ***A, A1***(s,p)*-**A3***(s,p), ***A1***$(s+1,p)$, ***A2***$(s+1,p)$ *be satisfied,* $\mathbf{E}|\mathbf{u}_0|^p_{s+2-2/p,p}<\infty$, *and for all* $t>0,\mathbf{v}\in\mathbb{H}^{s+1}_p$,

$$|\,|\mathbf{G}(\mathbf{v},t)|\,|_{s+1,p}\le|\,|\mathbf{G}(\mathbf{0},t)|\,|_{s+1,p}+C|\mathbf{v}|_{s+1,p}.$$

$$|\mathbf{D}(\mathbf{v},t)|_{s,p}\le|\mathbf{D}(\mathbf{0},t)|_{s,p}+C|\mathbf{v}|_{s+1,p}.$$

Suppose also that

$$\int_0^t(|\,|\mathbf{G}(\mathbf{0},r)|\,|^p_{s+1,p}+|\mathbf{D}(\mathbf{0},r)|^p_{s,p})\,dr<\infty$$

$\mathbf{P}$*-a.s. for all* t.

Then the unique maximal $\mathbb{H}^s_p$*-solution of (3.15) is also a unique maximal* $\mathbb{H}^{s+1}_p$*-solution.*

Moreover, for each $T>0,M>1$, *there is a constant* C, *such that for each stopping time* $\tau\le T\wedge\tau_M$,

$$\mathbf{E}[\sup_{r\le\tau}|\mathbf{u}(r)|^p_{s+1,p}+\int_0^\tau|\partial^2\mathbf{u}(r)|^p_{s,p}\,dr]\le C\mathbf{E}[|\mathbf{u}_0|^p_{s+2-2/p,p}$$

$$+\int_0^\tau(|\mathbf{D}(\mathbf{0},r)|^p_{s,p}+|\,|\mathbf{G}(\mathbf{0},r)|\,|^p_{s+1,p})\,dr],$$

where $\tau_M=\inf(t:|\mathbf{u}(t)|_{s,p}\ge M)$.

Proof. Since the assumptions **A, A1**(s,p)-**A3**(s,p) are satisfied, the existence and uniqueness of maximal $\mathbb{H}_p^s$- solution is guaranteed by Proposition 1. Let $\tau_M = \inf\{t : |\mathbf{u}(t)|_{s,p} \geq M\}$. Consider a linear equation

$$\partial_t \boldsymbol{\xi}(t) = \mathcal{S}(\partial_i(a^{ij}(t)\partial_j \boldsymbol{\xi}(t)) - \Psi^\varepsilon(u^k\,(t \wedge \tau_M))\partial_k u^l(t) + \mathbf{D}(\mathbf{u}(t),t) +$$

$$\mathcal{S}[\sigma^k(t)\partial_k \boldsymbol{\xi}(t) + \mathbf{G}(\mathbf{u}(t),t)] \cdot \dot{W}, \quad \boldsymbol{\xi}(0) = \mathbf{u}_0.$$

By Proposition 3.8 in [37], the linear equation has a unique $\mathbb{H}_p^{s+1}$-solution in $[[0,\tau_M]]$, which is also a unique $\mathbb{H}_p^s$-solution. Thus, $\boldsymbol{\xi} = \mathbf{u}$ **P**-a.s. on $[[0,\tau_M]]$. Moreover, for each T, there is a constant C, such that for all stopping times $\tau \leq T \wedge \tau_M$,

$$\mathbf{E}[\sup_{r \leq t\wedge\tau} |\mathbf{u}(r)|_{s+1,p}^p + \int_0^{t\wedge\tau} |\partial^2 \mathbf{u}(r)|_{s,p}^p\,dr] \leq C\mathbf{E}[|\mathbf{u}_0|_{s+2-2/p,p}^p + \int_0^{t\wedge\tau} (|\mathbf{u}(r)|_{s+1,p}^p$$

$$+ |\Psi^\varepsilon(u^k\,(r))\mathbf{u}(r)|_{s+1,p}^p + |\mathbf{D}(\mathbf{0},r)|_{s,p}^p + ||\mathbf{G}\,(\mathbf{0},r)||_{s+1,p}^p)\,dr]$$

for all t. Since

$$|\Psi^\varepsilon(u^k\,(r\wedge\tau_M))\mathbf{u}(r)|_{s+1,p}^p \leq CM|\mathbf{u}(r)|_{s+1,p}^p,$$

we have

$$\mathbf{E}[\sup_{r \leq t\wedge\tau} |\mathbf{u}(r)|_{s+1,p}^p + \int_0^{t\wedge\tau} |\partial^2 \mathbf{u}(r)|_{s,p}^p\,dr] \leq C\mathbf{E}[|\mathbf{u}_0|_{s+2-2/p,p}^p + \int_0^{t\wedge\tau} (|\mathbf{u}(r)|_{s+1,p}^p$$

$$+ |\mathbf{D}(\mathbf{0},r)|_{s,p}^p + ||\mathbf{G}\,(\mathbf{0},r)||_{s+1,p}^p)\,dr].$$

Now the estimate of the statement follows by Gronwall lemma. ∎

Corollary 2. *Assume* ***A, A1***$(s,2)$*-**A3***$(s,2), p \geq 2$. *Suppose* $\mathbf{E}|\mathbf{u}_0|_{s,2}^p < \infty$. *Assume further that*

$$\int_0^t (|\mathbf{D}(\mathbf{0},r)|_{s-1,2}^p + |\,|\mathbf{G}(\mathbf{0},r)|\,|_{s,2}^p)\,dr < \infty$$

P-*a.s. for all* t. *Let* $(\mathbf{u},\zeta)$ *be the maximal* $\mathbb{H}_2^s$*-solution to (3.15).*

Then for each $T > 0, M > 1$, *there is a constant* C, *such that for each stopping time* $\tau \leq T \wedge \tau_M$,

$$\mathbf{E}[\sup_{r\leq\tau} |\mathbf{u}(r)|_{s,2}^p + \int_0^\tau |\mathbf{u}(r)|_{s,2}^{p-2} |\nabla \mathbf{u}(r)|_{s,2}^2\,dr \leq C\mathbf{E}[|\mathbf{u}_0|_{s,2}^p + \int_0^\tau (|\mathbf{D}(\mathbf{0},r)|_{s-1,2}^p$$

$$+ |\,|\mathbf{G}(\mathbf{0},r)|\,|_{s,2}^p)\,dr],$$

where $\tau_M = \inf(t : |\mathbf{u}(t)|_{s,p} \geq M)$.

Proof. Let $M > 0, \tau_M = \inf\{t : |\mathbf{u}(t)|_{s,2} \geq M\}$. We easily obtain the statement by Proposition 2 in [38] applied to $\mathbb{H}^s_2$-solution $\boldsymbol{\xi}$ of the equation

$$\partial_t \boldsymbol{\xi}(t) = \partial_i(a^{ij}(t)\partial_j \boldsymbol{\xi}(t)) - \Psi^\varepsilon(u^k\,(t \wedge \tau_M))\partial_k \boldsymbol{\xi}(t) + \mathbf{D}(\boldsymbol{\xi}(t), t)$$

$$- \mathcal{G}[\partial_i(a^{ij}(t)\partial_j \boldsymbol{\xi}(t)) - \Psi^\varepsilon(u^k\,(t \wedge \tau_M))\partial_k \boldsymbol{\xi}(t) + \mathbf{D}(\boldsymbol{\xi}(t), t)]$$

$$[\sigma^k(t)\partial_k \boldsymbol{\xi}(t) + \mathbf{G}(\boldsymbol{\xi}(t), t) - \mathcal{G}(\sigma^k(t)\partial_k \boldsymbol{\xi}(t) + \mathbf{G}(\boldsymbol{\xi}(t), t))] \cdot \dot{W},$$

$$\boldsymbol{\xi}(0) = \mathbf{u}_0,\ \operatorname{div} \boldsymbol{\xi} = 0.$$

According to Proposition 2 in [38],

$$\mathbf{E} \sup_{r \leq \tau} |\mathbf{u}(r)|^p_{s,2} \leq C\mathbf{E}[|\mathbf{u}_0|^p_{s,2} + \int_0^\tau (|\mathbf{D}(\mathbf{0}, r)|^p_{s-1,2} +$$

$$+ |\Psi^\varepsilon(u^k(r))\mathbf{u}(r)|^p_{s,2} + |\,|\mathbf{G}(\mathbf{0}, r)|\,|^p_{s,2})\, dr].$$

Since

$$|\Psi^\varepsilon(u^k(r))\mathbf{u}(r)|^p_{s,2} \leq CM|\mathbf{u}(r)|^p_{s,2},$$

the statement follows by Gronwall lemma. ▮

Corollary 3. *Let* $s \in \{0, 1, \ldots\}, q \geq 2, \mathbf{E}|\mathbf{u}_0|^q_{s+1-2/q,q} < \infty$, *and* ***A**, **A1**(s, q)-**A3**(s, q) hold. Assume further that* $a^{ij} \in B^{s\vee 2}$, *if* $s \geq 1$, *and*

$$|\,|\mathbf{G}(\mathbf{v}, t)|\,|_{s,q} \leq |\,|\mathbf{G}(\mathbf{0}, t)|\,|_{s,q} + C|\mathbf{v}|_{s,q}, \quad |\mathbf{D}(\mathbf{v}, t)|_{s-1,q} \leq |\mathbf{D}(\mathbf{0}, t)|_{s-1,q} + C|\mathbf{v}|_{s,q},$$

$$\textstyle\int_0^t (|\mathbf{D}(\mathbf{0}, r)|^p_{s-1,q} + |\,|\mathbf{G}(\mathbf{0}, r)|\,|^p_{s,q})\, dr < \infty$$

$\mathbf{P}$*-a.s. for all* t. *Let* $(\mathbf{u}, \zeta)$ *be a maximal* $\mathbb{H}^s_q$*-solution to (3.15).*

Then for each $T > 0, M > 1$, *there is a constant* C, *such that for each stopping time* $\tau \leq T \wedge \tau_M$,

$$\mathbf{E} \sup_{r \leq \tau} |\mathbf{u}(r)|^p_{s,q} \leq C\mathbf{E}[|\mathbf{u}_0|^p_{s,q} + \int_0^\tau (|\mathbf{D}(\mathbf{0}, r)|^p_{s-1,q} + |\,|\mathbf{G}(\mathbf{0}, r)|\,|^p_{s,q})\, dr],$$

where $\tau_M = \inf\{t : |\mathbf{u}(t)|_{s,q} \geq M\}$.

Proof. Let $M > 1, \tau_M = \inf\{t : |\mathbf{u}(t)|_{s,q} \geq M\}$. We obtain easily the statement by Proposition 3 in [38] applied to $\mathbb{H}^s_q$-solution $\boldsymbol{\xi}$ of the equation

$$\partial_t \boldsymbol{\xi}(t) = \partial_i(a^{ij}(t)\partial_j \boldsymbol{\xi}(t)) - \Psi^\varepsilon(u^k\,(t \wedge \tau_M))\partial_k \boldsymbol{\xi}(t) + \mathbf{D}(\boldsymbol{\xi}(t), t)$$

$$- \mathcal{G}(\partial_i(a^{ij}(t)\partial_j \boldsymbol{\xi}(t)) - \Psi^\varepsilon(u^k\,(t \wedge \tau_M))\partial_k \boldsymbol{\xi}^l(t) + \mathbf{D}(\boldsymbol{\xi}(t), t))$$

$$[\sigma^k(t)\partial_k \boldsymbol{\xi}(t) + \mathbf{G}(\boldsymbol{\xi}(t), t) - \mathcal{G}(\sigma^k(t)\partial_k \boldsymbol{\xi}(t) + \mathbf{G}(\boldsymbol{\xi}(t), t))] \cdot \dot{W},$$

$$\boldsymbol{\xi}(0) = \mathbf{u}_0,\ \operatorname{div} \xi = 0.$$

According to Proposition 2 in [38],

$$\mathbf{E}\sup_{r\leq\tau\wedge t}|\mathbf{u}(r)|^p_{s,q}\leq C\mathbf{E}[|\mathbf{u}_0|^p_{s,q}+\int_0^{t\wedge\tau}(|\mathbf{D}(\mathbf{0},r)|^p_{s-1,q}+$$

$$+|\Psi^\varepsilon(u^k(r))\mathbf{u}(r)|^p_{s,q}+|\,|\mathbf{G}(\mathbf{0},r)|\,|^p_{s,q})\,dr],$$

Also,

$$|\Psi^\varepsilon(u^k(r))\mathbf{u}(r)|^p_{s,q}\leq CM|\mathbf{u}(r)|^p_{s,q},$$

and the statement follows by Gronwall lemma. ▮

Now we point out a simple case when $\zeta=\infty$ $\mathbf{P}$-a.s.

Proposition 3. *Assume* ***A****,* ***A1***$(0,2)$*-**A3***$(0,2)$ *are satisfied and* $\mathbf{E}|\mathbf{u}_0|_2^2<\infty$. *Let* $(\mathbf{u}(t),\zeta)$ *be a maximal* $\mathbb{H}^0_2=\mathbb{L}_2$*-solution to (3.15).*

Then the stopping time $\zeta=\infty$ $\mathbf{P}$*-a.s. Moreover, for each* $T>0$, *there is a constant* C, *so that for all stopping times* $\tau\leq T$,

$$\mathbf{E}[\sup_{r\leq\tau}|\mathbf{u}(r)|_2^2+\int_0^\tau|\nabla\mathbf{u}(r)|_2^2\,dr]\leq C\mathbf{E}[|\mathbf{u}_0|_2^2+\int_0^\tau|\mathbf{D}(\mathbf{0},r)|^2_{-1,2}+||\mathbf{G}(\mathbf{0},r)||_2^2\,dr].$$

Proof. Let $M>1,\tau_M=\inf\{t:|\mathbf{u}(t)|_{s,2}\geq M),\mathbf{u}_M(t)=\mathbf{u}(t\wedge\tau_M)\}$. By Itô formula (see [38]) we have

$$|\mathbf{u}_M(t)|_2^2=|\mathbf{u}(0)|_2^2+2\int_0^{t\wedge\tau_M}\langle\mathbf{u}(r),\mathbf{D}(\mathbf{u}(r),r)\rangle_1\,ds$$

$$-2\int_0^{t\wedge\tau_M}\int a^{ij}(r)\partial_i u^l(r))\partial_j u^l(r)\,dx\,dr$$

$$+2\int_0^{t\wedge\tau_M}(\int u^l(r)\tilde{b}^l(r)\,dx)\cdot dW_r+\int_0^{t\wedge\tau_M}\int\sum_i|b^i(r)|_Y^2\,dx\,dr$$

where $\tilde{b}^k(r)=\sigma^i(r)\partial_i u^k(r)+Q^k(\mathbf{u},r)$, $b^k(r)=\sigma^i(r)\partial_i u^k(r)+Q^k(\mathbf{u},r)-\mathcal{G}(\sigma^i(r)\partial_i u^k(r)+Q^k(\mathbf{u},r))$. Therefore, for each T, there is a constant C, independent of M, so that for all stopping times $\tau\leq T$,

$$\mathbf{E}\sup_{r\leq\tau}|\mathbf{u}_M(r)|_2^2+\int_0^\tau|\nabla\mathbf{u}_M(r)|_2^2\,dr\leq C\mathbf{E}[|\mathbf{u}_0|_2^2+\int_0^\tau(|\mathbf{u}_M(r)|_2^2$$

$$+|\mathbf{D}(\mathbf{0},r)|^2_{-1,2}+||\mathbf{G}(\mathbf{0},r)||_2^2\,dr],$$

and the statement follows. ▮

3.4. **Approximating sequence.** Given a scalar function v on $\mathbf{R}^d$, we define

$$\Psi_n(v)(x)=\Psi^{1/n}(v)(x)=\int v(x-y)\psi_{1/n}(y)\,dy,$$

where $\psi_\varepsilon(x)=\varepsilon^{-d}\psi(x/\varepsilon)$. Similarly, for a vector function $\mathbf{v}$:

$$\Psi_n(\mathbf{v})(x)=\Psi^{1/n}(\mathbf{v})(x).$$

We construct a sequence of approximations to (3.7) by solving for $\mathbf{u}=(u^l)_{1\leq l\leq d}=\mathbf{u}_n=\left(u_n^l\right)_{1\leq l\leq d}$ the equation

$$\left\{\begin{array}{l}\partial_t\mathbf{u}\left(t\right)=\mathcal{S}[\partial_i\left(a^{ij}\left(t\right)\partial_j\mathbf{u}\left(t\right)\right)-\Psi_n(u^i(t))\partial_i\mathbf{u}(t)\\[2ex] +b^i(t)\partial_i\mathbf{u}(t)+\mathbf{F}\left(\mathbf{u}(t),t\right)]+\mathcal{S}[\sigma^i(t)\partial_i\mathbf{u}\left(t\right)+\mathbf{G}\left(\mathbf{u}(t),t\right)]\dot{W}_t,\\[2ex] \mathbf{u}\left(0\right)=\mathbf{u}_{0,n},\end{array}\right. \tag{3.23}$$

where $\mathbf{u}_{0,n}=\mathbf{u}_0*\psi_{1/n},\Psi_n(v)(x)=\Psi^{1/n}(v)(x)=\int v(x-y)\psi_{1/n}(y)\,dy$. Alternatively, we may write it as

$$\begin{aligned}&\partial_t\mathbf{u}\left(t\right)=\partial_i\left(a^{ij}\left(t\right)\partial_j\mathbf{u}\left(t\right)\right)-\Psi_n(u^i\left(t\right))\partial_i\mathbf{u}(t)-\boldsymbol{\nabla}P\left(t\right)+\\ &b^i(t)\partial_i\mathbf{u}(t)+\mathbf{F}\left(\mathbf{u}\left(t\right),t\right)+[\sigma^i(t)\partial_i\mathbf{u}\left(t\right)+\mathbf{G}\left(\mathbf{u}\left(t\right),t\right)-\boldsymbol{\nabla}\tilde{P}\left(t\right)]\dot{W}_t,\\ &\mathbf{u}\left(0\right)=\mathbf{u}_{0,n},\operatorname{div}\mathbf{u}=0.\end{aligned} \tag{3.24}$$

Proposition 4. *a) Let* ***B1****,* ***B2(p)*** *be satisfied* ($p\geq 2$)*,* $\mathbf{E}(|\mathbf{u}_0|_{1,p}^p)<\infty$*. Then for each* $n>1$ *there is a unique maximal* $\mathbb{H}_p^1$*-solution* $(\mathbf{u},\boldsymbol{\zeta})=(\mathbf{u}_n,\zeta_n)$ *of (3.24).*

b) Let ***B1****,* ***B2(p)*** ($p>2$)*, and* ***B2***(2)*,***B2***$(2,p)$ *be satisfied, and* $\mathbf{E}(|\mathbf{u}_0|_{1,p}^p+|\mathbf{u}_0|_{1,2}^p)<\infty$*. Then for each* $n>1$ *the unique maximal* $\mathbb{H}_p^1$*-solution* $(\mathbf{u},\boldsymbol{\zeta})=(\mathbf{u}_n,\zeta_n)$ *is also a unique maximal* $\mathbb{H}_2^1$*-solution of (3.24). Moreover,* $\zeta=\zeta_n=\infty$ $\mathbf{P}$*-a.s.*

Proof. Fix n. For each $p\geq 2$, the conditions **B1, B2**(p) imply the assumptions **A, A1(**$0,p$)-**A3**($0,p$) and **A1(**$1,p$)-**A2**($1,p$) with

$$\mathbf{D}(\mathbf{v},t)=b^i\partial_i\mathbf{v}+\mathbf{F}(\mathbf{v},t),$$

We apply Propositions 1 and 2 to (3.24) in order to obtain part a) of the statement.

Part b) follows by Corollary 1 and Propositions 1-3. ∎

Applying curl operator to both sides of (3.24), we obtain obviously the following statement.

Remark 3. *Under the assumptions of Proposition 4, for each stopping time* S *such that* $[0,S]\subseteq[0,\zeta)$ η*=curl* $\mathbf{u}$ *(definition and properties of curl and crossproduct for* $d>3$ *are given in Appendix, subsection 5.5) satisfies in* $[[0,S]]$ *the equation*

$$\begin{aligned}&\partial_t\boldsymbol{\eta}\left(t\right)=\partial_i\left(a^{ij}\left(t\right)\partial_j\boldsymbol{\eta}\left(t\right)\right)-\Psi_n(u^i(t))\partial_i\boldsymbol{\eta}(t)\\ &+\mathbf{r}_n(\mathbf{u}(t))+b^i(t)\partial_i\boldsymbol{\eta}(t)+curl\{\mathbf{F}\left(\mathbf{u}(t),t\right)\}+\mathbf{r}(\mathbf{u}(t),t)\\ &+[\sigma^i(t)\partial_i\boldsymbol{\eta}\left(t\right)+\tilde{\mathbf{r}}(\mathbf{u}(t),t)+curl\{\mathbf{G}\left(\mathbf{u}(t),t\right)\}]\dot{W}_t,\\ &\boldsymbol{\eta}\left(0\right)=curl\,\mathbf{u}_{0,n},\end{aligned} \tag{3.25}$$

where

$$\mathbf{r}(\mathbf{v},t)=\partial_i(\boldsymbol{\nabla}a^{ij}(t)\times\partial_j\mathbf{v})+(\boldsymbol{\nabla}b^i(t))\times\partial_i\mathbf{v},$$

$$\mathbf{r}_n(\mathbf{v})=-\boldsymbol{\nabla}\Psi_n(v^i)\times\partial_i\mathbf{v},\ \tilde{\mathbf{r}}(\mathbf{v},t)=\boldsymbol{\nabla}\sigma^i(t)\times\partial_i\mathbf{v},\mathbf{v}\in\mathbb{H}_p^1.$$

The equation is linear in $\boldsymbol{\eta}$. *In fact, each component of (3.25) has a unique* L_p-*solution in* $[[0,S]]$ *of a corresponding linear equation (It is also* L_2-*solution in* $[[0,S]]$ *in the case b)).*

For $\mathbf{v} \in \mathbb{H}_p^1$ we set

$$\mathbf{L}_n(\mathbf{v}) = \mathcal{G}[\Psi_n(v^i)\partial_i \mathbf{v}],$$

$$\tilde{\mathbf{G}}(\mathbf{v}, r) = (\tilde{G}^l(\mathbf{v}, r))_{1 \le l \le d} = \mathcal{S}(\mathbf{G}(\mathbf{v}, r)) - \mathcal{G}(\sigma^i(r)\partial_i \mathbf{v}),$$

$$\tilde{\mathbf{F}}(\mathbf{v}, r) = \mathcal{S}(\mathbf{F}(\mathbf{v}, r)) + \mathcal{S}(b^i(r)\partial_i \mathbf{v}) - \partial_i \mathcal{G}(a^{ij}(r)\partial_j \mathbf{v}).$$

Also, we define

$$\mathbf{H}(\mathbf{v}, t) = \operatorname{curl}\{\mathbf{F}\,(\mathbf{v},t)\} + \mathbf{r}(\mathbf{v}, t), \mathbf{B}(\mathbf{v}, t) = \tilde{\mathbf{r}}(\mathbf{v}, t) + \operatorname{curl}\{\mathbf{G}\,(\mathbf{v}, t)\},$$

where

$$\mathbf{r}(\mathbf{v}, t) = \partial_i(\boldsymbol{\nabla} a^{ij}(t) \times \partial_j \mathbf{v}) + (\boldsymbol{\nabla} b^i(t)) \times \partial_i \mathbf{v}, \tilde{\mathbf{r}}(\mathbf{v}, t) = \boldsymbol{\nabla}\sigma^i(t) \times \partial_i \mathbf{v}.$$

Then we can rewrite (3.24) as

$$\begin{aligned} &\partial_t \mathbf{u}\,(t) = \partial_i \left(a^{ij}\,(t)\,\partial_j \mathbf{u}\,(t)\right) - \Psi_n(u^i\,(t))\partial_i \mathbf{u}(t) \\ &+\tilde{\mathbf{F}}\,(\mathbf{u}\,(t)\,, t) + \mathbf{L}_n(\mathbf{u}(t) + [\sigma^i(t)\partial_i \mathbf{u}\,(t) + \tilde{\mathbf{G}}\,(\mathbf{u}\,(t)\,, t)]\,\dot{W}_t, \\ &\mathbf{u}\,(0) = \mathbf{u}_{0,n}, \operatorname{div} \mathbf{u} = 0. \end{aligned} \tag{3.26}$$

Similarly, the equation (3.25) can be written as

$$\begin{aligned} &\partial_t \boldsymbol{\eta}\,(t) = \partial_i \left(a^{ij}\,(t)\,\partial_j \boldsymbol{\eta}\,(t)\right) - \Psi_n(u^i(t))\partial_i \boldsymbol{\eta}(t) \\ &+\mathbf{r}_n(\mathbf{u}(t)) + b^i(t)\partial_i \boldsymbol{\eta}(t) + \mathbf{H}\,(\mathbf{u}(t),t) + [\sigma^i(t)\partial_i \boldsymbol{\eta}\,(t) + \mathbf{B}\,(\mathbf{u}(t), t)\}]\,\dot{W}_t, \\ &\boldsymbol{\eta}\,(0) = \operatorname{curl} \mathbf{u}_{0,n}, \end{aligned} \tag{3.27}$$

where

$$\mathbf{r}_n(\mathbf{v}) = -\boldsymbol{\nabla}\Psi_n(v^i) \times \partial_i \mathbf{v}, \mathbf{v} \in \mathbb{H}_p^1.$$

For the estimate of L_p-norm of $\mathbf{u}$, we will need some simple estimates of $\tilde{\mathbf{F}}, \tilde{\mathbf{G}}, \mathbf{H}, \mathbf{B}$.

Lemma 2. *Assume* ***B2****(p) holds. Then*
a) there is a constant C *so that for all* $\mathbf{v} \in \mathbb{H}_p^1, t$,

$$|\tilde{\mathbf{F}}(\mathbf{v}, t)|_{-1,p} \le C(|\mathbf{F}(\mathbf{0}, t)|_p + |\mathbf{v}|_p),$$

$$||\tilde{\mathbf{G}}(\mathbf{v}, t)||_p \le C(||G(\mathbf{0}, t)||_p + |\mathbf{v}|_p),$$

$$|\mathbf{H}(\mathbf{v}, t)|_{-1,p} \le C(|\mathbf{F}(\mathbf{0}, t)|_p + |\mathbf{v}|_p + |\nabla \mathbf{v}|_p),$$

$$||\mathbf{B}(\mathbf{v}, t)||_p \le C(||\mathbf{G}\,(\mathbf{0}, t)\,||_{1,p} + |\mathbf{v}|_p + |\nabla \mathbf{v}|_p);$$

b) there is a constant C so that for all $\mathbf{v}, \bar{\mathbf{v}} \in \mathbb{H}^1_p, t \geq 0$

$$|\tilde{\mathbf{F}}(\mathbf{v},t) - \tilde{\mathbf{F}}(\bar{\mathbf{v}},t)|_{-1,p} \leq C|\mathbf{v}|_p,$$

$$||\tilde{\mathbf{G}}(\mathbf{v},t) - \tilde{\mathbf{G}}(\bar{\mathbf{v}},t)||_p \leq C|\mathbf{v}|_p),$$

$$|\mathbf{H}(\mathbf{v},t) - \mathbf{H}(\bar{\mathbf{v}},t)|_{-1,p} \leq C(|\mathbf{v} - \bar{\mathbf{v}}|_p + |\nabla\mathbf{v} - \boldsymbol{\nabla}\bar{\mathbf{v}}|_p),$$

$$||\mathbf{B}(\mathbf{v},t) - \mathbf{B}(\bar{\mathbf{v}},t)||_p \leq C(|\mathbf{v} - \bar{\mathbf{v}}|_p + |\nabla\mathbf{v} - \boldsymbol{\nabla}\bar{\mathbf{v}}|_p);$$

Proof. By our assumption and Lemma 1 there is a constant C so that

$$|\mathcal{S}(\mathbf{F}(\mathbf{v},t))|_p \leq |\mathbf{F}(\mathbf{0},t)|_p + C|\mathbf{v}|_p),$$

$$||\mathcal{S}(\mathbf{G}(\mathbf{v},t))||_p \leq ||\mathbf{G}(\mathbf{0},t)||_p + C|\mathbf{v}|_p)$$

By Corollary 6 and Lemma 21 (see Appendix), there is a constant C so that

$$|\partial_i \mathcal{G}(a^{ij}(r)\partial_j \mathbf{v})|_{-1,p} \leq C|\mathbf{v}|_p, \ |\mathcal{G}(\sigma^i(r)\partial_i \mathbf{v})|_p \leq C|\mathbf{v}|_p.$$

Also,

$$|\mathcal{S}(b^i(r)\partial_i \mathbf{v})|_{-1,p} = |\partial_i \mathcal{S}(b^i(r)\mathbf{v}) - \mathcal{S}(\partial_i b^i(r)\mathbf{v})|_{-1,p} \leq C|\mathbf{v}|_p,$$

and the statement, obviously, follows. ∎

The following standard estimate will be needed later as well.

Lemma 3. *Let* $p \geq 2$.

a) There is a constant C such that for all $\mathbf{v} \in H^1_p(\mathbf{R}^d, \mathbf{R}^m)$,

$$|\bar{\mathbf{v}}|_{1,p'} \leq C[\left(\int |\mathbf{v}|^{p-2}|\nabla \mathbf{v}|^2 \, dx\right)^{1/2} |\mathbf{v}|_p^{(p-2)/2} + |\mathbf{v}|_p^{p-1}]$$

where $\bar{\mathbf{v}} = |\mathbf{v}|^{p-2}\mathbf{v}, (p')^{-1} + p^{-1} = 1$.

For each $\varepsilon > 0$ there is a constant C_ε such that for all $\mathbf{v} \in H^1_p(\mathbf{R}^d, \mathbf{R}^m)$,

$$|\bar{\mathbf{v}}|_{1,p'}|\mathbf{v}|_p \leq \varepsilon \left(\int |\mathbf{v}|^{p-2}|\nabla \mathbf{v}|^2 \, dx\right) + C_\varepsilon |\mathbf{v}|_p^p.$$

b) For each $\varepsilon > 0$ there is a constant C_ε such that for all $\mathbf{v} \in H^1_p(\mathbf{R}^d, \mathbf{R}^m), \mathbf{h} \in L_p(\mathbf{R}^d, \mathbf{R}^m)$

$$\int |\mathbf{v}|^{p-2}|\nabla \mathbf{v}||\mathbf{h}| \, dx \leq \boldsymbol{\varepsilon} \left(\int |\mathbf{v}|^{p-2}|\nabla \mathbf{v}|^2 \, dx\right) + C_\varepsilon(|\mathbf{v}|_p^p + |\mathbf{h}|_p^p).$$

c) If for all t, x, y

$$|\boldsymbol{\sigma}(t,x) - \boldsymbol{\sigma}(t,y)|_Y \leq K|x - y|,$$

then there is a constant C such that for all $\mathbf{v} \in H^1_p(\mathbf{R}^d, \mathbf{R}^m), \mathbf{h} \in L_p(\mathbf{R}^d, \mathbf{R}^m)$

$$|\int (|\mathbf{v}|^{p-2}\mathbf{v}, \sigma^k \partial_k \mathbf{v} + \mathbf{h}) \, dx| \leq C(|\mathbf{v}|_p^p + |\mathbf{v}|_p^{p-1}|\mathbf{h}|_p).$$

Proof. We have

$$|\bar{\mathbf{v}}|_{1,p'} \le C(|\bar{\mathbf{v}}|_{p'} + \sum_k |\partial_k \bar{\mathbf{v}}|_{p'}),$$

and, obviously, $|\bar{\mathbf{v}}|_{p'} = |\mathbf{v}|_p^{p-1}$. By Hölder inequality,

$$\begin{aligned}
|\partial_k \bar{\mathbf{v}}|_{p'} &\le C|\,|\mathbf{v}|^{p-2}|\nabla \mathbf{v}|\,|_{p'} = C\left(\int |\mathbf{v}|^{p'(p-2)}|\nabla \mathbf{v}|^{p'}\right)^{1/p'} \\
&= C\left(\int |\mathbf{v}|^{p'(p-2)/2}(|\nabla \mathbf{v}|^{p'}|\mathbf{v}|^{p'(p-2)/2})\right)^{1/p'} \\
&\le C\left(\int (|\nabla \mathbf{v}|^{p'}|\mathbf{v}|^{p'(p-2)/2})^{2/p'}\right)^{1/2}\left(\int (|\mathbf{v}|^{p'(p-2)/2})^{2/(2-p')}\right)^{(2-p')/2} \\
&= C(\int |\mathbf{v}|^{p-2}|\nabla \mathbf{v}|^2\,dx)^{1/2}|\mathbf{v}|_p^{(p-2)/2}.
\end{aligned}$$

Therefore

$$|\bar{\mathbf{v}}|_{1,p'}|\mathbf{v}|_p \le C(\int |\mathbf{v}|^{p-2}|\nabla \mathbf{v}|^2\,dx)^{1/2}|\mathbf{v}|_p^{p/2} + |\mathbf{v}|_p^p),$$

and the part a) follows.

For each $\varepsilon > 0$ there is a constant C_ε such that

$$\int |\mathbf{v}|^{p-2}|\nabla \mathbf{v}||\mathbf{h}|\,dx \le \boldsymbol{\varepsilon}\left(\int |\mathbf{v}|^{p-2}|\nabla \mathbf{v}|^2\,dx\right) + C_\varepsilon \int |\mathbf{v}|^{p-2}|\mathbf{h}|^2\,dx$$

and part b) follows by Hölder inequality.

Inegrating by parts, we obtain easily c). ∎

For the estimates of L_p-norms, we will need the following important quantity. For $\mathbf{v} \in H_p^1(\mathbf{R}^d, \mathbf{R}^m)$, we define

$$\begin{aligned}
N_p(\mathbf{v},t) = &-\int \{[|\mathbf{v}|^{p-2} v^l \partial_i(a^{ij}(t)\partial_j v^l) + 2^{-1}[(p-2)|\mathbf{v}|^{p-4} v^i(s) v^j(s) \\
&+ |\mathbf{v}|^{p-2}\delta_{ij}]\sigma^k(t)\cdot\sigma^m(t)\partial_k v^i \partial_m v^j]\}\,dx \\
&= \int |\mathbf{v}|^{p-2}\partial_i v^l A^{ij}(t)\partial_j v^l\,dx \\
&+ (p-2)\int |\mathbf{v}|^{p-4}\, v^m \partial_i v^m A^{ij}(t) v^l \partial_j v^l\,dx,
\end{aligned} \tag{3.28}$$

where

$$A^{ij}(t) = a^{ij}(t) - \frac{1}{2}\sigma^i(t)\cdot\sigma^j(t).$$

Notice,

$$(p-2)\int |\mathbf{v}|^{p-4}\, v^m \partial_i v^m A^{ij} v^l \partial_j v^l\,dx = [4(p-2)/p^2] a^{ij}\partial_i(|\mathbf{v}|^{p/2})\partial_j(|\mathbf{v}|^{p/2}).$$

3.5. **Estimates of approximations.**

3.5.1. *Estimate of* $\mathbb{L}_p$*-norm of* $\mathbf{u}$. For estimating $\mathbb{L}_p$-norms of the approximations, we need some auxiliary statements. We start with some interpolation inequalities.

Lemma 4. *(see* [12]*) a) Given* $\mathbf{v} \in \mathbb{H}^1_p, p > d > 2$,

$$\int |\mathbf{v}|^{p+2}\,dx \le C|\mathbf{v}|_p^{p+2-d} H(\mathbf{v})^{d/p},$$

where $H(\mathbf{v}) = |\boldsymbol{\nabla}(|\mathbf{v}|^{p/2})|_2^2$;
b) Given $\mathbf{v} \in \mathbb{H}^1_p, p > d = 2$,

$$(\int |\mathbf{v}|^{2p}\,dx)^{2/p} \le 2^{2/p}|\mathbf{v}|_p^2 H(\mathbf{v})^{2/p}.$$

Proof. a) is proved in [12]: one applies the inequality

$$|\phi|_{2(p+2)/p} \le c|\phi|_2^{1-d/(p+2)}|\boldsymbol{\nabla}\phi|_2^{d/(p+2)}$$

for the scalar function $\phi = |\mathbf{v}|^{p/2}$.

In the case b) we apply the inequality

$$\int \phi^4\,dx \le 2\int \phi^2\,dx \int |\boldsymbol{\nabla}\phi|^2\,dx$$

for the scalar function $\phi = |\mathbf{v}|^{p/2}$. We have

$$\int |\mathbf{v}|^{2p}\,dx \le 2\int |\mathbf{v}|^p\,dx \int |\boldsymbol{\nabla}(|\mathbf{v}|^{p/2})|^2\,dx = 2|\mathbf{v}|_p^p H(\mathbf{v}).$$

∎

Notice $H(\mathbf{v}) \le C\int |\mathbf{v}|^{p-2}|\boldsymbol{\nabla}\mathbf{v}|^2\,dx \le C|\mathbf{v}|_{1,p}^p$. We have also the following obvious statement.

Corollary 4. *a) Let* $p > d > 2$*. For each* ε *there is a constant* C_ε *such that for all* $\mathbf{v} \in \mathbb{H}^1_p$,

$$\int |\mathbf{v}|^{p+2}\,dx \le \varepsilon H(\mathbf{v}) + C_\varepsilon(|\mathbf{v}|_p^p)^{1+\mu},$$

where $\mu = 2/(p-d)$.
b) Let $p > d = 2$*. For each* ε *there is a constant* C_ε *such that for all* $\mathbf{v} \in \mathbb{H}^1_p$,

$$|\mathbf{v}|_p^{p-2}(\int |\mathbf{v}|^{2p}\,dx)^{2/p} \le \varepsilon H(\mathbf{v}) + C_\varepsilon(|\mathbf{v}|_p^p)^{1+\mu},$$

where $\mu = 2/(p-d)$.

Lemma 5. *For each* ε *there is a constant* C_ε *independent of* n *such that for all* $\mathbf{v} \in \mathbb{H}^1_p$

$$|\int |\mathbf{v}|^{p-2}(\mathbf{v}, \mathbf{L}_n(\mathbf{v}))\,dx| \le \varepsilon\int |\mathbf{v}|^{p-2}|\boldsymbol{\nabla}\mathbf{v}|^2\,dx + C_\varepsilon(|\mathbf{v}|_p^p)^{1+\mu}, \tag{3.29}$$

where $\mu = 2/(p-d)$.

Proof. Denote D^{il} the l-th component of $\mathbf{D}^i = \mathcal{G}(\Psi_n(v^i)\mathbf{v})$. Since

$$\mathcal{G}(\Psi_n(v^i)\partial_i\mathbf{v}) = \partial_i\mathcal{G}(\Psi_n(v^i)\mathbf{v}),$$

integrating by parts, we get that for each ε there is a constant C_ε independent of n such that

$$|\int |\mathbf{v}|^{p-2}(\mathbf{v},\mathbf{L}_n(\mathbf{v}))\,dx\,| = |\int \partial_i(|\mathbf{v}|^{p-2}v^l)D^{il}\,dx| \le C\int |\mathbf{v}|^{p-2}|\nabla\mathbf{u}||\mathbf{D}^i|\,dx$$

$$\le \varepsilon\int |\mathbf{v}|^{p-2}|\nabla\mathbf{v}|^2\,dx + C_\varepsilon\int |\mathbf{v}|^{p-2}|\mathbf{D}^i|^2\,dx.$$

We need to estimate the term $B = \int|\mathbf{v}|^{p-2}|\mathbf{D}^i|^2\,dx \ = \int|\mathbf{v}|^{p-2}|\mathbf{D}^i|^2$. By Hölder inequality and Lemma 1,

$$B \le (\int |\mathbf{v}|^{p+2})^{\frac{p-2}{p+2}}(\int |\mathbf{D}^i|^{\frac{p+2}{2}})^{\frac{4}{p+2}} \le \int |\mathbf{v}|^{p+2})^{\frac{p-2}{p+2}}(\int\{|\Psi_n(v^i)||\mathbf{v}|\}^{\frac{p+2}{2}})^{\frac{4}{p+2}}$$

(3.30)

$$\le (\int |\mathbf{v}|^{p+2})^{\frac{p-2}{p+2}}(\int |\Psi_\kappa(v^i)|^{p+2})^{\frac{2}{p+2}}(\int |\mathbf{v}|^{p+2})^{\frac{2}{p+2}} \le \int |\mathbf{v}|^{p+2}.$$

On the other hand,

$$B \le (\int |\mathbf{v}|^p)^{\frac{p-2}{p}}(\int |\mathbf{D}^i|^p)^{\frac{2}{p}} \le C|\mathbf{v}|_p^{p-2}(\int\{|\Psi_\kappa(v^i)||\mathbf{v}|\}^p)^{\frac{2}{p}}$$

(3.31)

$$\le C|\mathbf{v}|_p^{p-2}(\int |\Psi_n(v^i)|^{2p})^{\frac{1}{p}}(\int |\mathbf{v}|^{2p})^{\frac{1}{p}} \le C|\mathbf{v}|_p^{p-2}(\int |\mathbf{v}|^{2p})^{\frac{2}{p}}.$$

By Corollary 4 (using (3.30) for $d > 2$, and (3.31) for $d = 2$), for each ε there is a constant C_ε such that

$$B \le \varepsilon H(\mathbf{v}) + C_\varepsilon(|\mathbf{v}|_p^p)^{1+\mu},$$

where $\mu = 2/(p-d)$ and

$$H(\mathbf{v}) = |\,\nabla(|\mathbf{v}|^{p/2})|_2^2 \le C\int |\mathbf{v}|^{p-2}|\boldsymbol{\nabla}\mathbf{v}|^2\,dx \le C(|\mathbf{v}|_p^p + |\nabla\mathbf{v}|_p^p.$$

∎

Using the Itô formula we estimate the $\mathbb{L}_p$-norm of the solution.

Proposition 5. *a) Let* ***B1****,* ***B2(p)*** *be satisfied,* $p > d$, $\mathbf{E}|\mathbf{u}_0|_{1,p}^p < \infty$. *Then for some* $\mathbb{F}$*-adapted functions* $a(s), b(s)$ *(in which* $a(s)$ *is real valued and* $b(s)$ *is* Y*-valued)* $\mathbf{P}$*-a.s. in* $[0,\zeta_n)$

$$|\mathbf{u}(t)|_p^p = |\mathbf{u}_{0,n}|_p^p + \int_0^t a(r)\ ds + \int_0^t \gamma(r)\cdot \dot{W}_s ds. \tag{3.32}$$

Moreover, there is a constant C *independent of* n *such that*

$$a(r) \le C[|\mathbf{u}(r)|_p^p + (|\mathbf{u}(r)|_p^p)^{1+\mu} + |\mathbf{G}(\mathbf{0},r)|_p^p + |\mathbf{F}(\mathbf{0},r)|_p^p],$$

(3.33)

$$|\gamma(r)|_Y \le C[|\mathbf{u}(r)|_p^p + |\mathbf{u}(r)|_p^{p-1}|\mathbf{G}(\mathbf{0},r)|_p),$$

where $\mu = 2/(p-d)$.

b) a) Let ***B1, B2(p)*** *and* ***B2(**2), be satisfied,* $p > d$*, and*

$$\mathbf{E}(|\mathbf{u}_0|^p_{1,p} + |\mathbf{u}_0|^p_{1,2}) < \infty,$$

$$\int_0^t (||\mathbf{G}(\mathbf{0}, r)||^p_{1,2} + |\mathbf{F}(\mathbf{0}, r)|^p_{1,2})\, dr < \infty$$

$\mathbf{P}$*-a.s. for all* t*. Then (3.32), (3.33) hold and for some* $\mathbb{F}$*-adapted functions* $\tilde{a}(s), \tilde{\gamma}(s)$ *and all* t*,*

$$|\mathbf{u}(t)|^p_2 = |\mathbf{u}_{0,n}|^p_2 + \int_0^t \tilde{a}(r)\ dr + \int_0^t \tilde{\gamma}(r) \cdot dW_r.$$

$\mathbf{P}$*-a.s. Moreover, there is a constant* C *independent of* n *such that*

$$\tilde{a}(r) \le C[|\mathbf{u}(r)|^p_2 + |\mathbf{G}(\mathbf{0}, r)|^p_2 + |\mathbf{F}(\mathbf{0}, r)|^p_2],$$

$$|\tilde{\gamma}(r)|_Y \le C[|\mathbf{u}(r)|^p_2 + |\mathbf{u}(r)|^{p-1}_2 |\mathbf{G}(\mathbf{0}, r)|_2)$$

Proof. According to Proposition 4, there is a solution $\mathbf{u} = \mathbf{u}_n$ to (3.24) such that $\mathbf{P}$-a.s. for all T

$$\sup_{t \le T} |\mathbf{u}(t)|^p_{1,p} + \int_0^T |\partial^2 \mathbf{u}(t)|^p_p\, dt < \infty.$$

Denoting $\mathbf{c}(r) = (c^i(r))_{1 \le i \le d} = \sigma^k \partial_k \mathbf{u}(r) + \tilde{\mathbf{G}}(\mathbf{u}(r), r)$, and applying Itô formula to $\mathbf{u}$ satisfying (3.26) (see [38]), we find that

$$|\mathbf{u}(t)|^p_p = |\mathbf{u}_0|^p_p - p \int_0^t N_p(\mathbf{u}(r), r)\, dr + p \int_0^t \int |\mathbf{u}(r)|^{p-2} (\mathbf{u}(r), \mathbf{L}_n(\mathbf{u}(r)))\, dx\, dr$$

$$+ p \int_0^t (\int |\mathbf{u}(r)|^{p-2} (\mathbf{u}(r), \tilde{\mathbf{F}}(\mathbf{u}(r), r))\, dx + \int_0^t p(\int |\mathbf{u}(r)|^{p-2} u^l(r) c^l(r)\, dx \dot{W}\, ds$$

$$+ \frac{p}{2} \int_0^t (\int [(p-2)|\mathbf{u}(r)|^{p-4} u^i(r) u^j(r) + |\mathbf{u}(r)|^{p-2} \delta_{ij}] \bar{b}^{ij}(r)\, dx)\, ds,$$

where

$$\bar{b}^{ij}(r) = \sigma^k(r) \partial_k u^i(r) \cdot d^j(r) + \sigma^k(r) \partial_k u^j(r) \cdot d^i(r) + d^i(r) \cdot d^j(r),$$

and $d^i(r) = \tilde{G}^i(\mathbf{u}(r), r)$. By Lemmas 2 and 3, for each $\varepsilon > 0$ there is a constant C_ε such that

$$|\int |\mathbf{u}(r)|^{p-2} (\mathbf{u}(r), \tilde{\mathbf{F}}(\mathbf{u}(r), r))\, dx| \le \varepsilon \int |\mathbf{u}(r)|^{p-2} |\nabla \mathbf{u}(r)|^2\, dx + C_\varepsilon (|\mathbf{u}(r)|^p_p + |\mathbf{F}(\mathbf{0}, r)|^p_p),$$

$$|\int [(p-2)|\mathbf{u}(r)|^{p-4} u^i(r) u^j(r) + |\mathbf{u}(r)|^{p-2} \delta_{ij}] \bar{b}^{ij}(r)\, dx|$$

(3.34)

$$\le \varepsilon \int |\mathbf{u}(r)|^{p-2} |\nabla \mathbf{u}(r)|^2\, dx + C_\varepsilon (|\mathbf{u}(r)|^p_p + |\mathbf{G}(\mathbf{0}, r)|^p_p).$$

By Lemma 3

$$|\int |\mathbf{u}(r)|^{p-2} u^l(r) c^l(r)\, dx| \le C(|\mathbf{u}(r)|^p_p + |\mathbf{u}(r)|^{p-1}_p ||\mathbf{G}(\mathbf{0}, r)|_p). \tag{3.35}$$

So, (3.33) follows by Lemma 5.
In the case b), applying Itô formula (see [38]) we obtain

$$|\mathbf{u}(t)|_2^p = |\mathbf{u}(0)|_2^p - p\int_0^t |\mathbf{u}(r)|_2^{p-2} N_2(\mathbf{u}(r), r)\, dr +$$

$$p\int_0^t |\mathbf{u}(r)|_2^{p-2} \int (\mathbf{u}(r), \tilde{\mathbf{F}}(\mathbf{u}(r), r))\, dx\, dr + p\int_0^t |\mathbf{u}(r)|_2^{p-2} (\int u^l(r) c^l(r)\, dx)\, dW_r +$$

$$p/2 \int_0^t |\mathbf{u}(r)|_2^{p-2} (\int \bar{b}^{ii}(r)\, dx)\, dr + \tfrac{p}{2}(p-2)\int_0^t |\mathbf{u}(r)|_2^{p-4} |\int u^l(r) c^l(r)\, dx|_Y^2\, dr.$$

Since (**??**)-(3.35) holds for $p = 2$ as well, the assertion of part b) follows. ▮

Remark 4. *There is a constant $C = C(K, d, p)$ independent of δ such that*

$$a(s) \le C[|\mathbf{u}(s)|_p^p + |\nabla \mathbf{u}(r)|_p^p + (|\mathbf{u}(s)|_p^p)^{1+\mu} + |\mathbf{G}(\mathbf{0}, s)|_p^p + |\mathbf{F}(\mathbf{0}, s)|_p^p]$$

$$|\gamma(s)|_Y \le C[|\mathbf{u}(s)|_p^p + |\mathbf{u}(s)|_p^{p-1} |\mathbf{G}(\mathbf{0}, s)|_p).$$

3.5.2. ***Estimate of $\mathbb{L}_p$-norm of $\boldsymbol{\nabla}\mathbf{u}$***. Since by Biot-Savaret law (see Proposition 8 in Appendix), for each $p > 1$

$$|\boldsymbol{\nabla}\mathbf{u}|_p \le C|\boldsymbol{\eta}|_p, (\boldsymbol{\eta} = \operatorname{curl}\mathbf{u}),$$

we need to estimate $|\boldsymbol{\eta}|_p$. According to Remark 3, $\boldsymbol{\eta}$ satisfies the linear equation (3.25) or (3.27).

Proposition 6. *a) Let **B1**, **B2**(p) be satisfied ($p > d$), $\mathbf{E}(|\mathbf{u}_0|_{1,p}^p) < \infty$, and $\mathbf{u}$ be the solution of (3.26). Then for some $\mathbb{F}$-adapted functions $h(t), \kappa(t)$ ($h(t)$ is real valued and $\kappa(t)$ is Y-valued) $\mathbf{P}$-a.s. in $[0, \zeta_n)$*

$$|\boldsymbol{\eta}(t)|_p^p = |curl\,\mathbf{u}_{0,n}|_p^p + \int_0^t h(r)\ ds + \int_0^t \kappa(r) \cdot dW_s.$$

Moreover, there is a constant C independent of n such that

$$h(r) \le C[|\boldsymbol{\eta}(r)|_p^p + (|\boldsymbol{\eta}(r)|_p^p)^{1+\mu} + |\mathbf{u}(r)|_p^p + |\mathbf{F}(\mathbf{0}, r)|_{1,p}^p + ||\mathbf{G}(\mathbf{0}, r)||_{1,p}^p],$$

$$|\kappa(r)|_Y \le C[|\boldsymbol{\eta}(r)|_p^p + |\mathbf{u}(r)|_p^p + |\boldsymbol{\eta}(r)|_p^{p-1} |\mathbf{G}(\mathbf{0}, r)|_{1,p}),$$

where $\mu = 2/(p-d)$.

*b) Let **B1**, **B2**(p) and **B2**(2) be satisfied ($p > d = 2$), and*

$$\mathbf{E}(|\mathbf{u}_0|_{1,p}^p + |\mathbf{u}_0|_{1,2}^p) < \infty,$$

$$\int_0^t (||\mathbf{G}(\mathbf{0}, r)||_{1,2}^p + |\mathbf{F}(\mathbf{0}, r)|_{1,2}^p)\, dr < \infty$$

$\mathbf{P}$-a.s. for all t. Let $\mathbf{u}$ be the solution of (3.26). Then for some $\mathbb{F}$-adapted functions $\tilde{a}(s), \tilde{b}(s)$ ($\tilde{a}(s)$ is real valued and $\tilde{b}(s)$ is Y-valued) $\mathbf{P}$-a.s. for all t

$$|\boldsymbol{\eta}(t)|_2^p = |curl\,\mathbf{u}_{0,n}|_2^p + \int_0^t \tilde{a}(r)\ ds + \int_0^t \tilde{b}(r) \cdot dW_s.$$

Moreover, there is a constant C independent of n such that

$$\tilde{a}(r) \leq C[|\boldsymbol{\eta}(r)|_2^p + (|\boldsymbol{\eta}(r)|_2^p)^{1+2/p} + |\mathbf{u}(r)|_2^p + |\mathbf{F}(\mathbf{0},r)|_{1,2}^p + ||\mathbf{G}(\mathbf{0},r)||_{1,2}^p],$$

$$|\tilde{b}(r)|_Y \leq C[|\boldsymbol{\eta}(r)|_2^p + |\boldsymbol{\eta}(r)|_2^{p-1}(|\mathbf{u}(r)|_2 + ||\mathbf{G}(\mathbf{0},r)||_{1,2})].$$

Proof. By applying Itô formula to function $\boldsymbol{\eta}\,(t)$, which verifies (3.27), we find that

$$|\boldsymbol{\eta}(t)|_p^p = |\boldsymbol{\eta}(0)|_p^p - p\int_0^t N_p(\boldsymbol{\eta}(r),r)\,ds - p\int_0^t \int |\boldsymbol{\eta}(r)|^{p-2}(\boldsymbol{\eta}(r),\mathbf{r}_n(\mathbf{u}(r))\,dx\,dr$$

$$+\,p\int_0^t \left\langle |\boldsymbol{\eta}(r)|^{p-2}\boldsymbol{\eta}(r),\mathbf{H}(r)\right\rangle_{1,p} dr + \int_0^t p(\int |\boldsymbol{\eta}(r)|^{p-2}\eta^l(r)c^l(r)\,dx\dot{W}_r dr$$

$$+\,\frac{p}{2}\int_0^t (\int [(p-2)|\boldsymbol{\eta}(r)|^{p-4}\eta^i(r)\eta^j(r) + |\boldsymbol{\eta}(r)|^{p-2}\delta_{ij}]\bar{b}^{ij}(r)\,dx)\,dr,$$

where $\mathbf{c}(r) = (c^i(r))_i = \sigma^k(r)\partial_k\boldsymbol{\eta}(r) + \mathbf{B}(\mathbf{u}(r),r),$

$$\mathbf{H}(r) = (H^l(r))_l = b^i(r)\partial_i\boldsymbol{\eta}(r)) + \mathbf{H}(\mathbf{u}(r),r),$$

and

$$\bar{b}^{ij}(r) = \sigma^k\partial_k\eta^i d^j(r) + \sigma^k\partial_k\eta^j d^i(r) + d^i(r)d^j(r),$$

and $\mathbf{d}(r)= \mathbf{B}(\mathbf{u}(r),r)$.

According to Lemmas 2 and 3, for every $\varepsilon > 0$, there is a constant C_ε so that

$$|\int [(p-2)|\boldsymbol{\eta}(r)|^{p-4}\eta^i(r)\eta^j(r) + |\boldsymbol{\eta}(r)|^{p-2}\delta_{ij}]\bar{b}^{ij}(r)\,dx|$$

(3.36)

$$\leq \varepsilon\int |\boldsymbol{\eta}|^{p-2}|\nabla\boldsymbol{\eta}|^2\,dx + C_\varepsilon(|\mathbf{G}(\mathbf{0},r)|_{1,p}^p + |\mathbf{u}(r)|_{1,p}^p),$$

and

(3.37)

$$|\left\langle |\boldsymbol{\eta}(r)|^{p-2}\boldsymbol{\eta}(r),\mathbf{H}(r)\right\rangle_{1,p}| \leq \varepsilon\int |\boldsymbol{\eta}|^{p-2}|\nabla\boldsymbol{\eta}|^2\,dx + C_\varepsilon(|\mathbf{F}(\mathbf{0},r)|_{1,p}^p + |\mathbf{u}(r)|_{1,p}^p).$$

Also,

$$(3.38)\quad |\int |\boldsymbol{\eta}(r)|^{p-2}\eta^l(r)c^l(r)\,dx|_Y \leq C(|\boldsymbol{\eta}(r)|_p^p + |\mathbf{u}(r)|_{1,p}^p + |\mathbf{G}(\mathbf{0},r)|_{1,p}|\boldsymbol{\eta}(r)|_p^{p-1})$$

It remains to estimate the term

$$A = \int |\boldsymbol{\eta}(r)|^{p-2}(\boldsymbol{\eta}(r),\mathbf{r}_n(\mathbf{u}(r))\,dx$$

$$= \int |\boldsymbol{\eta}(r)|^{p-2}(\boldsymbol{\eta}(r),\boldsymbol{\nabla}(\Psi_n(u^i(r))) \times \partial_i\mathbf{u}(r))\,dx\,.$$

We have

$$|A| \leq C(\int |\boldsymbol{\eta}|^p + \int |\boldsymbol{\eta}|^{p-2}|\boldsymbol{\nabla}\Psi_\kappa(\mathbf{u})|^2|\boldsymbol{\nabla}\mathbf{u}|^2),$$

and by Hölder inequality

$$\int |\boldsymbol{\eta}|^{p-2}|\boldsymbol{\nabla}\Psi_\kappa(\mathbf{u})|^2|\boldsymbol{\nabla}\mathbf{u}|^2 \leq |\boldsymbol{\eta}|_{p+2}^{p-2}|\boldsymbol{\nabla}\Psi_n(\mathbf{u})|_{p+2}^2|\boldsymbol{\nabla}\mathbf{u}|_{p+2}^2 \leq \int |\boldsymbol{\eta}|^{p+2},$$

or

$$\int |\boldsymbol{\eta}|^{p-2}|\boldsymbol{\nabla}\Psi_\kappa(\mathbf{u})|^2|\boldsymbol{\nabla}\mathbf{u}|^2 \leq C|\boldsymbol{\eta}|_p^{p-2}(\int |\boldsymbol{\eta}|^{2p})^{2/p}.$$

So, by Corollary 4, for each ε there is a constant C_ε independent of n such that

$$|A| \leq \varepsilon \int |\boldsymbol{\eta}|^{p-2}|\boldsymbol{\nabla}\boldsymbol{\eta}|^2\,dx + C_\varepsilon(|\boldsymbol{\eta}|_p^p)^{1+\mu}, \tag{3.39}$$

where $\mu = 2/(p-d)$. The part a) of the statement obviously follows by summarizing all the estimates.

In the case b), we apply Itô formula to $|\boldsymbol{\eta}(t)|_2^p$:

$$|\boldsymbol{\eta}(t)|_2^p = |\boldsymbol{\eta}(0)|_2^p - p\int_0^t |\boldsymbol{\eta}(r)|_2^{p-2}N_2(\boldsymbol{\eta}(r),r)\,dr - p\int_0^t |\boldsymbol{\eta}(r)|_2^{p-2}\int(\boldsymbol{\eta}(r),\mathbf{r}_n(\mathbf{u}(r)))\,dx\,dr$$

$$+p\int_0^t |\boldsymbol{\eta}(r)|_2^{p-2}\int(\boldsymbol{\eta}(r),\mathbf{H}(r))\,dx\,dr + \int_0^t p|\boldsymbol{\eta}(r)|_2^{p-2}(\int \eta^l(r)c^l(r)\,dx)\,dW_r$$

$$+\tfrac{p}{2}\int_0^t |\boldsymbol{\eta}(r)|_2^{p-2}(\int \bar{b}^{ii}(r)\,dx)\,dr + \tfrac{p}{2}(p-2)\int_0^t |\boldsymbol{\eta}(r)|_2^{p-4}|\int \eta^l(r)c^l(r)\,dx|_Y^2\,dr.$$

Since (3.36)-(3.38) hold for $p=2$ as well, it remains to estimate $A = \int(\boldsymbol{\eta}(r),\boldsymbol{\nabla}(\Psi_n(\mathbf{u})^i)\times \partial_i\mathbf{u}(r))\,dx$:

$$|A| \leq C|\boldsymbol{\eta}(r)|_2|\boldsymbol{\eta}(r)|_4^2 \leq C|\boldsymbol{\eta}(r)|_2^2|\boldsymbol{\nabla}\boldsymbol{\eta}(r)|_2, \text{ if } d=2.$$

So,

$$|\boldsymbol{\eta}(r)|_2^{p-2}|A| \leq \varepsilon|\boldsymbol{\eta}(r)|_2^{p-2}|\boldsymbol{\nabla}\boldsymbol{\eta}(r)|_2^2 + C_\varepsilon(|\boldsymbol{\eta}(r)|_2^p)^{1+2/p}.$$

Now, the part b) follows. ∎

Remark 5. *a) Consider a scalar process* $y_t = |\mathbf{u}(t)|_p^p + |\boldsymbol{\eta}(t)|_p^p$. *Then, according to the part a) of Proposition 5 and 6, for some adapted functions* $h(t)$ *and* $\kappa(t)$ *(*κ *is* Y*-valued) in* $[0,\zeta_n)$

$$y_t = y_0 + \int_0^t h_r\,dr + \int_0^t \kappa_r\cdot\dot{W}_r dr, \tag{3.40}$$

and there is a constant $C = C(\delta,K,d,p)$ *such that*

$$h_r \leq C(y_r + y_r^{1+\mu} + z_r), \tag{3.41}$$

$$|\kappa_r|_Y \leq C(y_r + y_r^{1-1/p}\tilde{z}_r^{1/p}),$$

where $z_r = |\mathbf{F}(\mathbf{0},r)|_{1,p}^p + ||\mathbf{G}(\mathbf{0},r)||_{1,p}^p$, $\tilde{z}_s = ||\mathbf{G}(\mathbf{0},r)||_{1,p}^p$.

b) Consider a scalar process $\tilde{y}_t = |\mathbf{u}(t)|_2^p + |\boldsymbol{\eta}(t)|_2^p$. *Then, according to the part b) of Proposition 5 and 6, for some adapted functions* $\tilde{h}(t)$ *and* $\tilde{\kappa}(t)$ *(*$\tilde{\kappa}$ *is* Y*-valued)*

$$\tilde{y}_t = \tilde{y}_0 + \int_0^t \tilde{h}_r\,dr + \int_0^t \tilde{\kappa}_r\cdot\dot{W}_r dr,$$

and there is a constant $C = C(\delta, K, d, p)$ *such that*

$$\tilde{h}_r \le C(\tilde{y}_r + \tilde{y}_r^{1+2/p} + \tilde{z}_r),$$

$$|\tilde{\kappa}_r|_Y \le C(\tilde{y}_r + \tilde{y}_r^{1-1/p}(\tilde{z}'_r)^{1/p}),$$

where $\tilde{z}_r = |\mathbf{F}(\mathbf{0}, r)|_{1,2}^p + ||\mathbf{G}(\mathbf{0}, r)||_{1,2}^p$, $\tilde{z}'_r = ||\mathbf{G}(\mathbf{0}, r)||_{1,2}^p$.

Introduce the following smooth scalar function

$$G(y) = \int_0^y (1 + x + x^{1+\mu})^{-1}\, dx.$$

Notice $G'(y) = (1 + y + y^{1+\mu})^{-1} > 0$, $G''(y) \le 0$.

Remark 6. *In the context of the previous Remark we obtain by Itô formula*

$$G(y_t) = G(y_0) + \int_0^t \bar{r}_s ds + \int_0^t \bar{b}_s \cdot \dot{W}_s\, ds,$$

where $r_s = G'(y_s) r_s + 2^{-1} G''(y_s) |\bar{b}_s|_Y^2 \le C(1 + z_s)$, $|\bar{b}_s|_Y = G'(y_s)|b_s|_Y \le C(1 + \tilde{z}_s^{1/p})$.

A similar observation holds for $\tilde{y}_r$.

3.5.3. *Convergence of approximations.* The following two auxiliary statements will be needed later.

Lemma 6. *a) Let* $\mathbf{v}, \mathbf{g}, \mathbf{f} \in \mathbb{H}_p^1$. *For each* $\varepsilon > 0$ *there is a constant* C_ε *such that*

$$|\int (\mathcal{S}[(\Psi_n(v^k) - \Psi_{n'}(\bar{v}^k))\partial_k \mathbf{g}], \mathbf{f}|\mathbf{f}|^{p-2})\, dx| \le \varepsilon \int |\nabla \mathbf{f}|^2 |\mathbf{f}|^{p-2}\, dx + C_\varepsilon(|\mathbf{f}|_p^p + |\mathbf{g}A|_p^p),$$

$$|\int (\mathcal{S}[\Psi_{n'}(\bar{v}^k))\partial_k \mathbf{g}], \mathbf{f}|\mathbf{f}|^{p-2})\, dx| \le \varepsilon \int |\nabla \mathbf{f}|^2 |\mathbf{f}|^{p-2}\, dx + C_\varepsilon(|\mathbf{f}|_p^p + |\mathbf{g}B|_p^p),$$

where $A = |\Psi_n(v^k) - \Psi_{n'}(\bar{v}^k)|, B = |\Psi_{n'}(\bar{v}^k)|$;

b) Let $\mathbf{v} \in \mathbb{H}_p^1, \mathbf{f} = (f^l), \mathbf{g} = (g^l) \in H_p^1(\mathbf{R}^d, \mathbf{R}^{d(d-1)/2})$. *For each* $\varepsilon > 0$ *there is a constant* C_ε *such that*

$$|\int (\Psi_n(v^k) - \Psi_{n'}(\bar{v}^k))\partial_k g^l f^l |\mathbf{f}|^{p-2}\, dx| \le \varepsilon \int |\nabla \mathbf{f}|^2 |\mathbf{f}|^{p-2}\, dx + C_\varepsilon(|A\mathbf{g}|_p^p + |\mathbf{f}|_p^p),$$

where $\mathbf{g} = (g^l), \mathbf{f} = (f^l), A = |\Psi_n(v^k) - \Psi_{n'}(\bar{v}^k)|$.

Proof. a) Indeed, we have

$$|\int (\mathcal{S}[(\Psi_n(v^k) - \Psi_{n'}(\bar{v}^k))\partial_k \mathbf{g}], \mathbf{f}|\mathbf{f}|^{p-2})\, dx|$$

$$= |\int (\mathcal{S}[(\Psi_n(v^k) - \Psi_{n'}(\bar{v}^k))\mathbf{g}], \partial_k(f^l |\mathbf{f}|^{p-2}))\, dx|$$

$$\le \varepsilon \int |\nabla \mathbf{f}|^2 |\mathbf{f}|^{p-2}\, dx + C_\varepsilon(|\mathbf{f}|_p^p + |\mathbf{g}A|_p^p).$$

Similarly, the second estimate follows.

b) We have

$$|\int (\Psi_n(v^k)-\Psi_{n'}(\bar v^k))\partial_k g^l f^l |\mathbf{f}|^{p-2}\,dx| = |\int (\Psi_n(v^k)-\Psi_{n'}(\bar v^k)) g^l \partial_k (f^l|\mathbf{f}|^{p-2})\,dx|$$

$$\le \varepsilon \int |\nabla \mathbf{f}|^2 |\mathbf{f}|^{p-2}\,dx + C_\varepsilon \int A^2 |\mathbf{g}|^2 |\mathbf{f}|^{p-2}\,dx \le \varepsilon \int |\nabla \mathbf{f}|^2 |\mathbf{f}|^{p-2}\,dx + C_\varepsilon |A\mathbf{g}|_p^2 |\mathbf{f}|_p^{p-2}$$

$$\le \varepsilon \int |\nabla \mathbf{f}|^2 |\mathbf{f}|^{p-2}\,dx + C_\varepsilon (|A\mathbf{g}|_p^p + |\mathbf{f}|_p^p),$$

and the statement follows. ∎

Lemma 7. *a) There is a constant C so that for all $\mathbf{v}, \bar{\mathbf{v}} \in \mathbb{H}_p^1, n' \ge n > 1$,*

$$|\Psi_n(v^k) - \Psi_{n'}(\bar v^k)|_p \le C(|\mathbf{v}-\bar{\mathbf{v}}|_{1,p} + n^{-1}(|\nabla \mathbf{v}|_p + |\nabla \bar{\mathbf{v}}|_p)].$$

b)Let $p > d$. Then there is a constant C so that for all $\mathbf{v}, \bar{\mathbf{v}} \in \mathbb{H}_p^1, n' \ge n > 1$,

$$|\Psi_n(v^k) - \Psi_{n'}(\bar v^k)|_\infty \le C[|\mathbf{v}-\bar{\mathbf{v}}|_{1,p} + n^{-\nu}|\mathbf{v}|_{1,p}],$$

where $\nu = 1 - d/p$.

Proof. By Sobolev's embedding theorem there is a constant C so that for all $\mathbf{v} \in \mathbb{H}_p^1$

$$\sup_x |\mathbf{v}(\,x)| + \sup_{x,y} |\mathbf{v}(\,x) - \mathbf{v}(\,y)||x-y|^\nu \le C|\mathbf{v}|_{1,p},$$

where $\nu = 1 - d/p$. Therefore,

$$\sup_x |\Psi_n(v^k) - \Psi_{n'}(\bar v^k)| \le \sup_x |\Psi_n(v^k) - \Psi_n(\bar v^k)| + \sup_x |\Psi_n(\bar v^k) - \bar v^k|$$

$$+ \sup_x |\bar v^k - \Psi_{n'}(\bar v^k)| \le C[\sup_x |\mathbf{v}(x) - \bar{\mathbf{v}}(x)| + ((1/n)^\nu + (1/n')^\nu)|\bar{\mathbf{v}}|_{1,p}],$$

and the statement follows. ∎

We will need the following equalities and estimates later.

Lemma 8. *Let $\mathbf{v}, \mathbf{d} \in \mathbb{H}_p^1, \eta = (\eta^{jl})_{j<l} \in H_p^1(\mathbf{R}^d, \mathbf{R}^{d(d-1)/2}), \bar\eta = \eta|\eta|^{p-2}, p > d$. Then*

$$\int \left(\nabla d^k \times \partial_k \mathbf{v}, \bar\eta\right)\,dx = \int \left(d^k \partial_k \mathbf{v} \times \boldsymbol{\nabla}, \bar\eta\right)\,dx + \int (\boldsymbol{\nabla} \times \mathbf{v}, d^k \partial_k \bar\eta)\,dx.$$

Also, for each ε there is a constant C_ε such that for all $\mathbf{v}, \mathbf{d} \in \mathbb{H}_p^1, \eta = (\eta^{jl})_{j<l} \in H_p^1(\mathbf{R}^d,\ \mathbf{R}^{d(d-1)/2})$

$$\int |\left(\nabla d^k \times \partial_k \mathbf{v}, \bar\eta\right)\,dx| \le \varepsilon \int |\eta|^{p-2}|\nabla \eta|^2\,dx + C_\varepsilon(|\eta|_p^p + |\nabla \mathbf{v}|_p^p |\mathbf{d}|_\infty^p).$$

Proof. It is enough to prove the statement for $\mathbf{v}, \mathbf{d} \in \mathbb{C}_0^\infty, \eta^{jl} \in C_0^\infty$. Integrating by parts, we have

$$\int \left(\nabla d^k \times \partial_k \mathbf{v}, \bar{\eta}\right) dx = \int \varepsilon_{jk}(\partial_j d^k \partial_k v^l - \partial_l d^k \partial_k v^j)\bar{\eta}^{jl}\, dx$$

$$= -\int \varepsilon_{jk} d^k(\partial_k \partial_j v^l - \partial_k \partial_l v^j)\bar{\eta}^{jl}\, dx - \int \varepsilon_{jk} d^k(\partial_k v^l \partial_j \bar{\eta}^{jl} - \partial_k v^j \partial_l \,\bar{\eta}^{jl})\, dx$$

$$= \int \varepsilon_{jk}(d^k \partial_j v^l - d^k \partial_l v^j)\partial_k \bar{\eta}^{jl}\, dx + \int \varepsilon_{jk}(d^k \partial_k v^j \partial_l \,\bar{\eta}^{jl} - d^k \partial_k v^l \partial_j \bar{\eta}^{jl})\, dx,$$

where $\varepsilon_{jk} = (-1)^{j+k-1}$. Therefore, for each ε there is a constant C_ε such that

$$|\int \left(\nabla d^k \times \partial_k \mathbf{v}, \bar{\eta}\right) dx| \leq \varepsilon \int |\eta|^{p-2} |\nabla \eta|^2\, dx + C_\varepsilon \int |\eta|^{p-2} |\mathbf{d}|^2 |\nabla \mathbf{v}|^2\, dx,$$

and the statement follows by Hölder inequality. ▮

Remark 7. *If $d = 2$, then for all $\mathbf{v} \in \mathbb{H}_p^1$*

$$\nabla v^k \times \partial_k \mathbf{v} = 0.$$

Let $\mathbf{u} = \mathbf{u}_n = (u_n^l) = (u^l)$ be a maximal $\mathbb{H}_p^1$-solution to (3.24). Let $\boldsymbol{\eta} = \boldsymbol{\eta}_n = (\eta_n^l) = \text{curl}\ \mathbf{u}_n$. Fix a large number $M > 0$ and $T > 0$. Given a positive integer n, let $\mathcal{T}_n = \mathcal{T}_n^{M,T}$ be the set of all stopping times $\tau \leq T \wedge \zeta_n$ such that $\mathbf{P}$-a.s.

$$\sup_{s \leq \tau} (|\mathbf{u}_n(s)|_p + |\boldsymbol{\eta}_n(s)|_p) \leq M.$$

In the case $d = 2$ we also introduce the set $\tilde{\mathcal{T}}_n = \tilde{\mathcal{T}}_n^{M,T}$ of all stopping times $\tau \leq T$ such that

$$\sup_{s \leq \tau} (|\mathbf{u}_n(s)|_p + |\boldsymbol{\eta}_n(s)|_p + |\mathbf{u}_n(s)|_2 + |\boldsymbol{\eta}_n(s)|_2) \leq M.$$

Let $\mathcal{T}_{n,n'} = \mathcal{T}_n \cap \mathcal{T}_{n'}, \tilde{\mathcal{T}}_{n,n'} = \tilde{\mathcal{T}}_n \cap \tilde{\mathcal{T}}_n$.

Lemma 9. *a) Let* ***B1****,* ***B2****(p),* ***B3****(p) be satisfied ($p > d$), $\mathbf{E}(|\mathbf{u}_0|_{1,p}^p) < \infty$. Let $\mathbf{u} = \mathbf{u}_n = (u_n^l) = (u^l)$ be $\mathbb{H}_p^{‘1}$-solutions to (3.24). Then*

$$\lim_n \sup\{\mathbf{E} \sup_{s \leq \tau} |\mathbf{u}_{n'} - \mathbf{u}_n|_{1,p}^p : n' \geq n, \tau \in \mathcal{T}_{n',n}\} = 0$$

where $\mathcal{T}_{n',n} = \mathcal{T}_n \cap \mathcal{T}_{n'}$.

b) Let ***B1****,* ***B2****(p),* ***B3****(p),* ***B2****(2),* ***B3****(2) be satisfied ($p > d = 2$), and*

$$\mathbf{E}(|\mathbf{u}_0|_{1,p}^p + |\mathbf{u}_0|_{1,2}^p) < \infty,$$

$$\int_0^t (||\mathbf{G}(\mathbf{0}, r)||_{1,2}^p + |\mathbf{F}(\mathbf{0}, r)|_{1,2}^p)\, dr < \infty$$

$\mathbf{P}$*-a.s. for all t. Let $\mathbf{u} = \mathbf{u}_n = (u_n^l) = (u^l)$ be $\mathbb{H}_p^1 \cap \mathbb{H}_2^1$-solution to (3.24). Then*

$$\lim_{n \to \infty} \sup\{\mathbf{E} \sup_{s \leq \tau} (|\mathbf{u}_{n'} - \mathbf{u}_n|_{1,p}^p + |\mathbf{u}_{n'} - \mathbf{u}_n|_{1,2}^p) : n' \geq n, \tau \in \tilde{\mathcal{T}}_{n',n}\} = 0,$$

where $\tilde{\mathcal{T}}_{n',n} = \tilde{\mathcal{T}}_n \cap \tilde{\mathcal{T}}_{n'}$.

Proof. Let $\tau \in \mathcal{T}_{n',n}, n' \geq n$. Consider

$$\mathbf{w}(t) = \mathbf{u}_{\mathbf{n}'}(t \wedge \tau) - \mathbf{u}_n(t \wedge \tau),\ \boldsymbol{\xi}(t) = \boldsymbol{\eta}_{n'}(t \wedge \tau) - \boldsymbol{\eta}_n(t \wedge \tau).$$

Denote $\bar{\mathbf{u}} = \mathbf{u}_{n'}$, $\mathbf{u} = \mathbf{u}_n$.

Applying Itô formula (see [38]) we have

(3.42)

$$|\mathbf{w}(t)|_p^p = |\mathbf{w}(0)|_p^p - p\int_0^{t\wedge\tau} N_p(\mathbf{w}(r), r)\,dr + p\int_0^{t\wedge\tau}\int |\mathbf{w}(r)|^{p-2}(\mathbf{w}(r), \mathbf{a}(r)\,dx\,dr$$

$$+p\int_0^{t\wedge\tau}\int |\mathbf{w}(r)|^{p-2} w^l(r) c^l(r)\,dW_r$$
$$+\frac{p}{2}\int_0^{t\wedge\tau}(\int[(p-2)|\mathbf{w}(r)|^{p-4}w^i(r)w^j(r) + |\mathbf{w}(r)|^{p-2}\delta_{ij}]\bar{c}^{ij}(r)\,dx)\,dr$$
$$-p\int_0^{t\wedge\tau}(\int |\mathbf{w}(r)|^{p-2}(\mathbf{w}(r), \mathcal{S}[\partial_k\mathbf{w}(r)\Psi_{n'}(\bar{u}^k) + (\Psi_{n'}(\bar{u}^k) - \Psi_n(u^k))\partial_k\mathbf{u}]\,dx)\,dr$$

where $\mathbf{c}(r) = (c^i(r))_i = \sigma^k\partial_k\mathbf{w}(r) + \tilde{\mathbf{G}}(\bar{\mathbf{u}}, r) - \tilde{\mathbf{G}}(\mathbf{u}, r)$,

$$\mathbf{a}(r) = (a^l(r))_l = \tilde{F}^l(\mathbf{v}, r) - \tilde{F}^l(\mathbf{u}, r);$$

and

$$\bar{c}^{ij}(r) = \sigma^k(r)\partial_k w^i(r)\cdot d^j(r) + \sigma^k(r)\partial_k w^j(r)\cdot d^i(r) + d^i(r)\cdot d^j(r)$$

where $d^i(r) = \tilde{G}^i(\bar{\mathbf{u}}(r), r) - \tilde{G}^i(\mathbf{u}(r), r)$.

Also, by Itô formula (see [38]),

$$|\boldsymbol{\xi}(t)|_p^p = |\boldsymbol{\xi}(0)|_p^p - p\int_0^{t\wedge\tau} N_p(\boldsymbol{\xi}(r), r)\,dr + p\int_0^{t\wedge\tau}(\int |\boldsymbol{\xi}(r)|^{p-2}\xi^l(r)H^l(r)\,dx\,dr$$

$$+p\int_0^{t\wedge\tau}\int |\boldsymbol{\xi}(r)|^{p-2}(\boldsymbol{\xi}(r), \mathbf{r}_{n'}(\bar{\mathbf{u}}(r)) - \mathbf{r}_n(\mathbf{u}(r))\,dx\,dr$$

$$+p\int_0^{t\wedge\tau}\int |\boldsymbol{\xi}(r)|^{p-2}\xi^l(r)\kappa^l(r)\,dx\,dW_r$$

$$+\frac{p}{2}\int_0^{t\wedge\tau}(\int[(p-2)|\boldsymbol{\xi}(r)|^{p-4}\xi^i(r)\xi^j(r) + |\boldsymbol{\xi}(r)|^{p-2}\delta_{ij}]\bar{\kappa}^{ij}(r)\,dx)\,dr$$

$$-p\int_0^{t\wedge\tau}\int |\boldsymbol{\xi}(s)|^{p-2}\xi^l(s)[\partial_k\xi^l(s)\Psi_{n'}(\bar{u}^k) + (\Psi_{n'}(\bar{u}^k) - \Psi_n(u^k))\partial_k\eta_n^l\,dx\,ds$$

where $\boldsymbol{\kappa}(r) = (\kappa^i(r))_i = \sigma^k\partial_k\boldsymbol{\xi}(r) + \mathbf{B}(\bar{\mathbf{u}}, r) - \mathbf{B}(\mathbf{u}, r))$,

$$\mathbf{H}(r) = (H^l(r))_l = b^i\partial_i\boldsymbol{\xi}(r) + \mathbf{H}(\bar{\mathbf{u}}(r), r) - \mathbf{H}(\mathbf{u}(r), r));$$

and

$$\bar{\kappa}^{ij}(r) = \sigma^k(r)\partial_k\xi^i(r)\cdot D^j(r) + \sigma^k(r)\partial_k\xi^j(r)\cdot D^i(r) + D^i(r)\cdot D^j(r)$$

where $D^i(r) = B^i(\bar{\mathbf{u}}(r), r) - B^i(\mathbf{u}(r), r)$.

By Lemmas 6 a) and 7 b) and Sobolev embedding theorem

$$\bar L_1 = |\int |\mathbf{w}(r)|^{p-2}(\mathbf{w}(r), \mathcal{S}[\partial_k \mathbf{w}(r)\Psi_{n'}(\bar u^k) + (\Psi_{n'}(\bar u^k) - \Psi_n(u^k))\partial_k \mathbf{u}]\, dx|$$

$$\le \varepsilon \int |\nabla \mathbf{w}(r)|^2 |\mathbf{w}(r)|^{p-2}\, dx + C_\varepsilon(|\mathbf{w}(r)|_p^p + |\mathbf{w}(r)A|_p^p + |\mathbf{w}(r)B|_p^p) \tag{3.43}$$

$$\le \varepsilon \int |\nabla \mathbf{w}(r)|^2 |\mathbf{w}(r)|^{p-2}\, dx + C_\varepsilon(M)(|\mathbf{w}(r)|_p^p + |\boldsymbol{\xi}(r)|_p^p + n^{-\nu p})$$

where $A = |\Psi_{n'}(\bar u^k) - \Psi_n(u^k)|, B = |\Psi_{n'}(\bar u^k)|, \nu = 1 - d/p$.

Similarly, by Lemmas 6 b) and 7 b),

$$\bar L_2 = |\int |\boldsymbol{\xi}(s)|^{p-2}\xi^l(s)(\Psi_{n'}(\bar u^k) - \Psi_n(u^k))\partial_k \eta_n^l\, dx|$$

$$\le \varepsilon \int |\nabla \xi(r)|^2 |\boldsymbol{\xi}(r)|^{p-2}\, dx + C_\varepsilon(|A\boldsymbol{\eta}_n(r)|_p^p + |\boldsymbol{\xi}(r)|_p^p) \tag{3.44}$$

$$\le \varepsilon \int |\nabla \xi(r)|^2 |\boldsymbol{\xi}(r)|^{p-2}\, dx + C_\varepsilon(M)(|\mathbf{w}(r)|_p^p + |\boldsymbol{\xi}(r)|_p^p + n^{-p\nu}).$$

Obviously,

$$\bar L_3 = |\int (|\boldsymbol{\xi}(r)|^{p-2}\boldsymbol{\xi}(r), \boldsymbol{\nabla}(\Psi_{n'}(\bar u^i)) \times \partial_i \bar{\mathbf{u}}(r) - \boldsymbol{\nabla}(\Psi_n(u^i)) \times \partial_i \mathbf{u}(r))\, dx|$$

$$\le |\int (|\boldsymbol{\xi}(r)|^{p-2}\boldsymbol{\xi}(r), (\boldsymbol{\nabla}(\Psi_{n'}(\bar u^i)) - \boldsymbol{\nabla}(\Psi_n(u^i)) \times \partial_i \bar{\mathbf{u}}(r))\, dx|$$

$$+ |\int (|\boldsymbol{\xi}(r)|^{p-2}\boldsymbol{\xi}(r), \boldsymbol{\nabla}(\Psi_{n'}(\bar u^i)) \times \partial_i \mathbf{w}(r))\, dx| = \bar L_{31} + \bar L_{32}.$$

By Lemmas 8 and 7 (part b)),

$$\bar L_{31} \le \varepsilon \int |\boldsymbol{\xi}(r)|^{p-2}|\nabla \boldsymbol{\xi}(r)|^2\, dx + C_\varepsilon(|\boldsymbol{\xi}(r)|_p^p + |\nabla \bar{\mathbf{u}}(r)|_p^p |A|_\infty^p)$$

$$\le \varepsilon \int |\boldsymbol{\xi}(r)|^{p-2}|\nabla \boldsymbol{\xi}(r)|^2\, dx + C_\varepsilon(M)(|\boldsymbol{\xi}(r)|_p^p + |\mathbf{w}(r)|_p^p + n^{=p\nu}),$$

(3.45)

$$\bar L_{32} \le \varepsilon \int |\boldsymbol{\xi}(r)|^{p-2}|\nabla \boldsymbol{\xi}(r)|^2\, dx + C_\varepsilon(|\boldsymbol{\xi}(r)|_p^p + |\nabla \mathbf{w}(r)|_p^p |B|_\infty^p)$$

$$\le \varepsilon \int |\boldsymbol{\xi}(r)|^{p-2}|\nabla \boldsymbol{\xi}(r)|^2\, dx + C_\varepsilon(M)(|\boldsymbol{\xi}(r)|_p^p).$$

Let $Z_t = |\mathbf{w}(t)|_p^p + |\boldsymbol{\xi}(t)|_p^p$. Using (3.43)-(3.45) and estimating the remaining terms by Lemmas 2 and 3, we obtain that for some adapted functions $a(t)$, $b(t)$

($b(t)$ is Y-valued)

$$dZ_t = a(t)\,dt + b(t)\cdot dW_t, \tag{3.46}$$

and there is a constant C such that on $[0,\tau]$ for all $\tau \in \mathcal{T}_{n',n}$, $n' \geq n$

$$a(t) \leq C(Z_t + (1/n^{\nu p})),\ \ |b(t)|_Y \leq CZ_t. \tag{3.47}$$

(We use **B3(**p) to estimate the term

$$\int [(p-2)|\boldsymbol{\xi}(r)|^{p-4}\xi^i(r)\xi^j(r) + |\boldsymbol{\xi}(r)|^{p-2}\delta_{ij}]\bar{\kappa}^{ij}(r)\,dx.)$$

Now the part a) of the statement follows by Lemma 18.

In the case b) we have ($\tau \in \tilde{\mathcal{T}}_{n',n}$)

$$|\mathbf{w}(t)|_2^p = |\mathbf{w}(0)|_2^p - p\textstyle\int_0^{t\wedge\tau} |\mathbf{w}(\mathbf{s})|_2^{p-2} N_2(\mathbf{w}(r),r)\,dr$$

$$+p\textstyle\int_0^{t\wedge\tau} |\mathbf{w}(r)|_2^{p-2}\int(\mathbf{w}(r),\mathbf{a}(r))\,dx\,dr + p\int_0^{t\wedge\tau} |\mathbf{w}(r)|_2^{P-2}(\int w^l(r)c^l(r)\,dx)\,dW_r$$

$$+p/2\left(\textstyle\int_0^{t\wedge\tau} |\mathbf{w}(r)|_2^{p-2}(\int \bar{c}^{ii}(r)\,dx)\,dr + (p-2)\int_0^{t\wedge\tau} |\mathbf{w}(r)|_2^{p-4}|\int w^l(r)c^l(r)\,dx|_Y^2\,dr\right)$$

$$-p\textstyle\int_0^{t\wedge\tau} |\mathbf{w}(r)|_2^{p-2}(\int w^l(r)[\partial_k w^l(r)\Psi_{n'}(\bar{u}^k(r)) + (\Psi_{n'}(\bar{u}^k) - \Psi_\kappa(u^k))\partial_k u^l]\,dx)\,dr.$$

Similarly,

$$|\boldsymbol{\xi}(t)|_2^p = |\boldsymbol{\xi}(0)|_2^p - p\int_0^{t\wedge\tau} |\boldsymbol{\xi}(r)|_2^{p-2} N_2(\boldsymbol{\xi}(r),r)\,ds$$

$$-p\int_0^{t\wedge\tau} |\boldsymbol{\xi}(r)|_2^{p-2}\int(\boldsymbol{\xi}(r), \boldsymbol{\nabla}(\Psi_n(\bar{u}^i(r))) \times \partial_i\mathbf{w}(r)$$

$$+ (\boldsymbol{\nabla}(\Psi_{n'}(u^i(r)) - \boldsymbol{\nabla}(\Psi_n(u^i(r))) \times \partial_i\mathbf{u}(r))\,dx\,dr$$

$$p\int_0^{t\wedge\tau} |\boldsymbol{\xi}(r)|_2^{p-2}(\int(\boldsymbol{\xi}(r),\mathbf{H}(r))\,dx\,dr + \int_0^{t\wedge\tau} p|\boldsymbol{\xi}(r)|_2^{p-2}(\int \xi^l(r)\kappa^l(r)\,dx)\,dW_r$$

$$+\frac{p}{2}\int_0^{t\wedge\tau} |\boldsymbol{\xi}(r)|_2^{p-2}(\int \bar{\kappa}^{ii}(r)\,dx)\,dr + \frac{p}{2}(p-2)\int_0^{t\wedge\tau} |\boldsymbol{\xi}(r)|_2^{p-4}|\int \boldsymbol{\xi}^l(r)\kappa^l(r)\,dx|_Y^2\,dr.$$

Let $n' \geq n$. According to Lemmas 6 (part a)) and 7 (part a)) and Sobolev embedding theorem, for each $\varepsilon > 0$ there is a constant C_ε such that $\mathbf{P}$-a.s. on $[0, \tau[$

$$H = |\int w^l(r)(\Psi_{n'}(\bar{u}^k(r)) - \Psi_\kappa(u^k(r)))\partial_k u^l(r)]\,dx|$$

$$\leq \varepsilon|\nabla \mathbf{w}(r)|_2^2 + C_\varepsilon|\mathbf{u}(r)|_\infty^2|\Psi_{n'}(\bar{u}^k(r)) - \Psi_n(u^k(r)|_2^2$$

$$\leq \varepsilon|\nabla \mathbf{w}(r)|_2^2 + C_\varepsilon M^2(|\mathbf{w}(r)|_{1,2}^2 + n^{-2}(|\nabla \bar{\mathbf{u}}(r)|_2^2 + |\nabla \bar{\mathbf{u}}(r)|_2^2)$$

$$\leq \varepsilon|\nabla \mathbf{w}(r)|_2^2 + C_\varepsilon M^2(|\mathbf{w}(r)|_{1,2}^2 + n^{-2}),$$

and

$$|\mathbf{w}(r)|_2^{p-2}H \leq \varepsilon|\nabla \mathbf{w}(r)|_2^2|\mathbf{w}(r)|_2^{p-2} + C_\varepsilon(M)(|\mathbf{w}(r)|_2^p + |\boldsymbol{\xi}(r)|_2^p + n^{-p}). \tag{3.48}$$

By Lemma 8 and Sobolev imbedding theorem,

$$L_1 = |\int(\boldsymbol{\xi}(r), \boldsymbol{\nabla}(\Psi_n(\bar{u}^i(r))) \times \partial_i \mathbf{w}(r)\,dx|$$

$$\leq \varepsilon|\nabla \boldsymbol{\xi}(r)|_2^2 + C_\varepsilon(|\boldsymbol{\xi}(r)|_2^2 + |\Psi_n(\bar{u}^i(r))|_\infty^2|\nabla \mathbf{w}(r)|_2^2) \tag{3.49}$$

$$\leq \varepsilon|\nabla \boldsymbol{\xi}(r)|_2^2 + C_\varepsilon(1 + M^2)|\boldsymbol{\xi}(r)|_2^2.$$

By Lemma 8 and Lemma 7 b),

$$L_2 = |\int(\boldsymbol{\xi}(r), (\boldsymbol{\nabla}(\Psi_{n'}(\bar{u}^i(r)) - \boldsymbol{\nabla}(\Psi_n(u^i(r))) \times \partial_i \mathbf{u}(r))\,dx|$$

$$\begin{aligned} &\leq \varepsilon|\nabla \boldsymbol{\xi}(r)|_2^2 + C_\varepsilon(|\boldsymbol{\xi}(r)|_2^2 + |\Psi_n(\bar{u}^i(r)) - \Psi_{n'}(\bar{u}^i(r))|_\infty^2|\nabla \mathbf{u}(r)|_2^2) \\ &\leq \varepsilon|\nabla \boldsymbol{\xi}(r)|_2^2 + C_\varepsilon[|\boldsymbol{\xi}(r)|_2^2 + M^2(|\mathbf{w}(r)|_{1,p}^2 + n^{-2\nu}M^2)] \end{aligned} \tag{3.50}$$

$$\leq \varepsilon|\nabla \boldsymbol{\xi}(r)|_2^2 + C_\varepsilon(M)(|\boldsymbol{\xi}(r)|_2^2 + |\mathbf{w}(r)|_{1,p}^2 + n^{-2\nu})$$

where $\nu = 1 - 2/p$. So,

$$\begin{aligned} &|\boldsymbol{\xi}(r)|_2^{p-2}L_2 \leq \varepsilon|\nabla \boldsymbol{\xi}(r)|_2^2|\boldsymbol{\xi}(r)|_2^{p-2} + C_\varepsilon(M)(|\boldsymbol{\xi}(r)|_2^p + |\mathbf{w}(r)|_{1,p}^p + n^{-2\nu}), \\ &|\boldsymbol{\xi}(r)|_2^{p-2}L_1 \leq \varepsilon|\nabla \boldsymbol{\xi}(r)|_2^2|\boldsymbol{\xi}(r)|_2^{p-2} + C_\varepsilon(M)|\boldsymbol{\xi}(r)|_2^p \end{aligned} \tag{3.51}$$

Let $K_t = |\mathbf{w}(t)|_2^p + |\boldsymbol{\xi}(t)|_2^p$. Using (3.48)-(3.51) and estimating the remaining terms by Lemmas 2 and 3, we obtain that for some adapted functions $\bar{a}(t)$, $\bar{b}(t)$ ($\bar{b}(t)$ is Y-valued)

$$dK_t = \bar{a}(t)\,dt + \bar{b}(t) \cdot dW_t, \tag{3.52}$$

and there is a constant C such that on $[0,\tau]$ for all $\tau\in\mathcal{T}_{n',n}$, $n'\geq n$

$$\bar a(t)\leq C(K_t+|\mathbf{w}(t)|_{1,p}^p+(1/n^p)),\ \ |\bar b(t)|_Y\leq CK_t. \tag{3.53}$$

(We use **B3**(2) to estimate $\int\bar b^{ii}(r)\,dx$).

Combining (3.46), (3.52), (3.47), (3.53), we find that for some adapted functions $\tilde a(t),\tilde b(t)$

$$R_t=|\mathbf{w}(t)|_2^p+|\boldsymbol{\xi}(t)|_2^p+|\mathbf{w}(t)|_p^p+|\boldsymbol{\xi}(t)|_p^p$$

$$=R(0)+\int_0^t\tilde a(r)\,dr+\int_0^t\tilde b(r)\ \cdot dW_r,$$

and there is a constant C, such that on any $[0,\tau],\tau\in\tilde{\mathcal{T}}_{n',n}$

$$\tilde a(t)\leq C[R(t)+(1/n^p)+(1/n^{p\nu})],\ \ |\bar b(t)|_Y\leq CR(t).$$

Now the part b) of the statement follows by Lemma 18 (see Appendix). ∎

3.6. **Local existence and uniqueness.**

3.6.1. *Uniqueness.*

Proposition 7. *Let τ be a bounded stopping time, $p>d$. Let **B1**, **B2**(p) be satisfied. Assume $\mathbf{u}(t)$ and $\bar{\mathbf{u}}(t)$ are $\mathbb{L}_p$-solutions of (3.7) in $[[0,\tau]]$ and also $\mathbb{H}_p^1$-valued and continuous.*

Then $\mathbf{P}$-a.s. $\mathbf{u}(t\wedge\tau)=\bar{\mathbf{u}}(t\wedge\tau)$ for all t.

Proof. For $\mathbf{v}\in\mathbb{H}_p^1$ we set

$$\tilde{\mathbf{G}}(\mathbf{v},r)=(\tilde G^l(\mathbf{v},r))_{1\leq l\leq d}=\mathcal{S}(\mathbf{G}(\mathbf{v},r))-\mathcal{G}(\sigma^i(r)\partial_i\mathbf{v}),$$

$$\tilde{\mathbf{F}}(\mathbf{v},r)=\mathcal{S}(\mathbf{F}(\mathbf{v},r)+b^i(r)\partial_i\mathbf{v})-\partial_i\mathcal{G}(a^{ij}(r)\partial_j\mathbf{w}).$$

Then for all $\mathbf{v},\bar{\mathbf{v}}\in\mathbb{H}_p^1$,

$$\begin{aligned}|\tilde{\mathbf{G}}(\mathbf{v},r)-\tilde{\mathbf{G}}(\bar{\mathbf{v}},r)|_p&\leq C|\mathbf{v}-\bar{\mathbf{v}}|_p,\\ |\tilde{\mathbf{F}}(\mathbf{v},r)-\tilde{\mathbf{F}}(\bar{\mathbf{v}},r)|_{-1,p}&\leq C|\mathbf{v}-\bar{\mathbf{v}}|_p.\end{aligned}\tag{3.54}$$

Let

$$N_p(\mathbf{v},r)=-\int\{[|\mathbf{v}|^{p-2}v^l\partial_i(a^{ij}(r)\partial_jv^l)+2^{-1}[(p-2)|\mathbf{v}|^{p-4}v^iv^j$$

$$+|\mathbf{v}|^{p-2}\delta_{ij}]\sigma^k(r)\cdot\sigma^m(r)\partial_kv^i\partial_mv^j]\}\,dx.$$

Obviously (see (3.28)),

$$N_p(\mathbf{v},r)\geq\delta\int|\mathbf{v}|^{p-2}|\nabla\mathbf{v}|^2\,dx.$$

Let $\mathbf{w}(t) = \mathbf{u}(t \wedge \tau) - \bar{\mathbf{u}}(t \wedge \tau)$. By Itô formula (see [38]),

$$\begin{aligned}
|\mathbf{w}(t)|_p^p = {} & -p \int_0^{t\wedge\tau} N_p(\mathbf{w}(r), r)\, dr + p \int_0^{t\wedge\tau} \int |\mathbf{w}(r)|^{p-2} w^l(r) a^l(r)\, dx\, dr \\
& + p \int_0^{t\wedge\tau} \int |\mathbf{w}(r)|^{p-2} w^l(r) c^l(s)\, dW_s \\
& + \frac{p}{2} \int_0^{t\wedge\tau} \Big(\int [(p-2)|\mathbf{w}(r)|^{p-4} w^i(r) w^j(r) + |\mathbf{w}(r)|^{p-2}\delta_{ij}] \bar{c}^{ij}(r)\, dx\Big)\, dr \\
& - p \int_0^{t\wedge\tau} \Big(\int |\mathbf{w}(r)|^{p-2} (\mathbf{w}(r), \mathcal{S}[\partial_k \mathbf{w}(r) \bar{u}^k(r) + w^k(r) \partial_k \mathbf{u}(r)]\, dx\Big)\, dr
\end{aligned}$$

where $\mathbf{c}(r) = (c^i(r))_{1 \le i \le d} = \sigma^k \partial_k \mathbf{w}(r) + \tilde{\mathbf{G}}(\bar{\mathbf{u}}, r) - \tilde{\mathbf{G}}(\mathbf{u}, r)$,

$$\mathbf{a}(r) = (a^l(r))_l = \tilde{F}^l(\bar{\mathbf{u}}, r) - \tilde{F}^l(\mathbf{u}, r),$$

and

$$\bar{c}^{ij}(r) = \sigma^k(r) \partial_k w^i(r) \cdot d^j(r) + \sigma^k(r) \partial_k w^j(r) \cdot d^i(r) + d^i(r) \cdot d^j(r)$$

where $d^i(r) = \tilde{G}^i(\bar{\mathbf{u}}(r), r) - \tilde{G}^i(\mathbf{u}(r), r)$.

By (3.54), for each $\varepsilon > 0$ there is a constant C_ε such that

$$|\int |\mathbf{w}(r)|^{p-2} w^l(r) a^l(r)\, dx| \le \varepsilon \int |\mathbf{w}(r)|^{p-2} |\nabla \mathbf{w}(r)|^2\, dx + C_\varepsilon |\mathbf{w}(r)|_p^p. \tag{3.55}$$

Integrating by parts, using Sobolev embedding theorem ($p > d$) and Hölder inequality, we obtain that for each $\varepsilon > 0$ there is a constant C_ε such that

$$\begin{aligned}
& |\int |\mathbf{w}(r)|^{p-2} (\mathbf{w}(r), \mathcal{S}[\partial_k \mathbf{w}(r) \bar{u}^k(r) + w^k(r) \partial_k \mathbf{u}(r)]\, dx| \\
& = |\int |\mathbf{w}(r)|^{p-2} (\mathbf{w}(r), \partial_k \mathcal{S}[\mathbf{w}(r) \bar{u}^k(r) + w^k(r) \mathbf{u}(r)] \\
& \le \varepsilon \int |\mathbf{w}(r)|^{p-2} |\nabla \mathbf{w}(r)|^2\, dx + C_\varepsilon \int |\mathbf{w}(r)|^{p-2} |\mathcal{S}[\mathbf{w}(r) \bar{u}^k(r) + w^k(r) \mathbf{u}|^2\, dx \\
& \le \varepsilon \int |\mathbf{w}(r)|^{p-2} |\nabla \mathbf{w}(r)|^2\, dx + C_\varepsilon |\mathbf{w}(r)|_p^p (|\bar{\mathbf{u}}(r)|_{1,p} + |\mathbf{u}(r)|_{1,p}).
\end{aligned} \tag{3.56}$$

By (3.54), for each $\varepsilon > 0$ there is a constant C_ε such that

$$\begin{aligned}
& |\int [(p-2)|\mathbf{w}(r)|^{p-4} w^i(r) w^j(r) + |\mathbf{w}(r)|^{p-2} \delta_{ij}] \bar{c}^{ij}(r)\, dx| \\
& \le \varepsilon \int |\mathbf{w}(r)|^{p-2} |\nabla \mathbf{w}(r)|^2\, dx + C_\varepsilon |\mathbf{w}(r)|_p^p.
\end{aligned} \tag{3.57}$$

Integrating by parts and by (3.54) we have

$$|\int |\mathbf{w}(r)|^{p-2}w^l(r)c^l(r)\,dx| \leq C|\mathbf{w}(r)|_p^p. \tag{3.58}$$

Let $M>1$ and $\tau_M = \inf(t : |\bar{\mathbf{u}}(t)|_{1,p} + |\mathbf{u}(t)|_{1,p} \geq M) \wedge \tau$. Since (3.55)-(3.58) hold, the assumptions of Lemma 18 (see Appendix) are satisfied with

$$Z_t = |\mathbf{w}(t)|_p^p, c_t = \int |\mathbf{w}(t)|^{p-2}|\nabla \mathbf{w}(t)|^2\,dx, f_t = g_t = 0, Z_0 = 0.$$

Therefore, $\mathbf{P}$-a.s. $\mathbf{w}(t\wedge\tau_M) = \mathbf{0}$ for all t. Since M is arbitrary, the pathwise uniqueness follows. ∎

3.6.2. *Existence.* Now we extract a converging subsequence.

Lemma 10. *a) Let* ***B1****,* ***B2****(p),* ***B3****(p) be satisfied ($p>d$),* $\mathbf{E}(|\mathbf{u}_0|_{1,p}^p) < \infty$. *Then there is a bounded stopping time τ such that $\mathbf{P}(\tau > 0) = 1$ and a unique $\mathbb{L}_p$-solution $\mathbf{u}(t)$ of (3.7) in $[[0,\tau]]$ which is also $\mathbb{H}_p^1$-valued continuous process such that*

$$\mathbf{E}\sup_{t\leq\tau}|\mathbf{u}(t)|_{1,p}^p < \infty.$$

b) Let ***B1****,* ***B2****(p),* ***B2****(2),* ***B3****(p),* ***B3****(2) be satisfied ($p>d=2$), and*

$$\mathbf{E}(|\mathbf{u}_0|_{1,p}^p + |\mathbf{u}_0|_{1,2}^p) < \infty,$$

$$\int_0^t (||\mathbf{G}(\mathbf{0},r)||_{1,2}^p + |\mathbf{F}(\mathbf{0},r)|_{1,2}^p)\,dr < \infty$$

$\mathbf{P}$*-a.s. for all t. Then there is a stopping time τ such that $\mathbf{P}(\tau>0)=1$ and a unique $\mathbb{L}_p\cap\mathbb{L}_2$-solution $\mathbf{u}(t)$ of (3.7) in $[[0,\tau]]$, which is also $\mathbb{H}_p^1\cap\mathbb{H}_2^1$-valued continuous process such that*

$$\mathbf{E}\sup_{t\leq\tau}(|\mathbf{u}(t)|_{1,p}^p + |\mathbf{u}(t)|_{1,2}^p) < \infty.$$

Proof. We apply Lemma 19 (see Appendix) to extract a converging subsequence. We choose the Banach space

$$B = \begin{cases} \mathbb{H}_p^1, & \text{in the case a),} \\ \mathbb{H}_p^1\cap\mathbb{H}_2^1, & \text{in the case b).} \end{cases}$$

In $\mathbb{H}_p^1\cap\mathbb{H}_2^1$ we use the norm:

$$|\mathbf{v}|_B = (|\mathbf{v}|_p^p + |\mathrm{curl}\,\mathbf{v}|_p^p + |\mathbf{v}|_2^p + |\mathrm{curl}\,\mathbf{v}|_2^p)^{1/p}.$$

In $\mathbb{H}_p^1$ the norm $|\mathbf{v}|_B = (|\mathbf{v}|_p^p + |\mathrm{curl}\,\mathbf{v}|_p^p)^{1/p}$ is used.

Fix arbitrary $T_0>0, M_0>1$. Since Lemma 9 holds, according to Lemma 19, it is enough to prove that

$$\lim_{T\to 0}\sup_{n,\tau\in\mathcal{T}_n^{M_0,T_0}}\mathbf{P}(\sup_{s\leq\tau\wedge T}|\mathbf{u}_n(s)|_B > |\mathbf{u}_n(0)|_B + M_0 - 1) = 0. \tag{3.59}$$

where $\mathcal{T}_n^{M_0,T_0}$ is the set of all stopping times $\tau\leq T_0$ such that $\sup_{s\leq\tau}|\mathbf{u}_n(s)|_B \leq M_0 + |\mathbf{u}_n(0)|_B$. Let $T<T_0$,

$$S_n = \inf(t : |\mathbf{u}_n(t)|_B > |\mathbf{u}_n(0)|_B + M_0 - 1).$$

Let

$$K_t = \begin{cases} \int_0^t (||\mathbf{G}(\mathbf{0},r)||_{1,p}^p + |\mathbf{F}(\mathbf{0},r)|_{1,p}^p)\, dr, & \text{in the case a)}, \\ \int_0^t (||\mathbf{G}(\mathbf{0},r)||_{1,p}^p + |\mathbf{F}(\mathbf{0},r)|_{1,p}^p + ||\mathbf{G}(\mathbf{0},r)||_{1,2}^p + |\mathbf{F}(\mathbf{0},r)|_{1,2}^p)\, dr, & \text{in the case b)}. \end{cases}$$

Define $\tau^M = \inf(t : K_t \geq M) \wedge T_0$. By Propositions 5 and 6, for $\tau \in \mathcal{T}_n^{M_0,T_0}$, we have for each M

$$\mathbf{P}\left(\sup_{t\leq\tau\wedge T} |\mathbf{u}_n(t)|_B > |\mathbf{u}_n(0)|_B + M_0 - 1\right)$$

$$\leq \mathbf{P}(|\mathbf{u}_n(S_n \wedge T)|_B > |\mathbf{u}_n(0)|_B + M_0 - 1)$$

$$\leq \mathbf{P}(|\mathbf{u}_n(S_n \wedge T)|_B^p > |\mathbf{u}_n(0)|_B^p + (M_0 - 1)^p)$$

$$\leq \mathbf{P}(|\mathbf{u}_n(S_n \wedge \tau^M \wedge T)|_B^p > |\mathbf{u}_n(0)|_B^p + (M_0 - 1)^p)$$

$$+ \mathbf{P}\left(\tau^M < T_0\right) \leq C(M_0)[T + \mathbf{E}K_{\tau^M\wedge T}] + \mathbf{P}\left(\tau^M < T_0\right).$$

Therefore, for each M

$$\limsup_{T\to 0} \sup_{n,\tau\in\mathcal{T}_n^{M_0,T_0}} \mathbf{P}\left(\sup_{s\leq\tau\wedge T} |\mathbf{u}_n(s)|_B > |\mathbf{u}_n(0)|_B + M_0 - 1\right) \leq \mathbf{P}\left(\tau^M < T_0\right),$$

and (3.59) follows. By Lemma 19, there is a stopping time τ such that $\mathbf{P}(\tau > 0) = 1$, a B-valued stochastic process $\mathbf{u}$ on the interval $[0,\tau]$ and a subsequence $\mathbf{u}_{n_k}$ converging uniformly on $[0,\tau]$ to $\mathbf{u}$. Obviously, $\mathbf{u}(t)$ is an $\mathbb{L}_p$-solution (respectively, $\mathbb{L}_p \cap \mathbb{L}_2$-solution) of (3.7) in $[[0,\tau]]$ which is also $\mathbb{H}_p^1$ (respectively, $\mathbb{H}_p^1 \cap \mathbb{H}_2^1$) valued and continuous. Uniqueness follows by Proposition 7. ∎

The following almost obvious statement is a straightforward generalization of Lemma 10 .

Lemma 11. *a) Let* ***B1****,* ***B2****(p),* ***B3****(p) be satisfied ($p > d$) and* $\mathbf{E}(|\mathbf{u}_0|_{1,p}^p) < \infty$. *Assume that* $\mathbf{u}(t)$ *is a* $\mathbb{H}_p^1$*-valued continuous* $\mathbb{L}_p$*-solution of (3.7) on* $[0,S]$*, where* S *is a finite stopping time and*

$$\mathbf{E}\sup_{t\leq S} |\mathbf{u}(t)|_{1,p}^p < \infty.$$

Then there exist a finite stopping time τ *and a* $\mathbb{H}_p^1$*-valued continuous* $\mathbb{L}_p$*-solution* $\mathbf{v}(t)$ *to (3.7) in* $[[0,\tau]]$ *such that* $\mathbf{P}(\tau > S) = 1$ *and* $\mathbf{v}$ *coincides with* $\mathbf{u}$ *on* $[0,S]$ *and*

$$\mathbf{E}\sup_{t\leq\tau} |\mathbf{v}(t)|_{1,p}^p < \infty.$$

b) Let ***B1****,* ***B2****(p),* ***B3****(p),* ***B2****(2),* ***B3****(2) be satisfied, and*

$$\mathbf{E}(|\mathbf{u}_0|_{1,p}^p + |\mathbf{u}_0|_{1,2}^p) < \infty,$$

$$\int_0^t (||\mathbf{G}(\mathbf{0},r)||_{1,2}^p + |\mathbf{F}(\mathbf{0},r)|_{1,2}^p)\, dr < \infty$$

P-*a.s. for all* t*. Assume that* $\mathbf{u}(t)$ *is a* $\mathbb{H}^1_p \cap \mathbb{H}^1_2$*-valued continuous* $\mathbb{L}_p \cap \mathbb{L}_2$*-solution of (3.7) in* $[[0, S]]$*, where* S *is a finite stopping time and*

$$\mathbf{E} \sup_{t \le S} (|\mathbf{u}(t)|^p_{1,p} + |\mathbf{u}(t)|^p_{1,2}) < \infty.$$

Then there exist a finite stopping time τ *and a* $\mathbb{H}^1_p \cap \mathbb{H}^1_2$*-valued continuous* $\mathbb{L}_p \cap \mathbb{L}_2$*-solution* $\mathbf{v}(t)$ *of (3.7) in* $[[0, \tau]]$ *such that* $\mathbf{P}(\tau > S) = 1$*, and* $\mathbf{v}$ *coincides with* $\mathbf{u}$ *on* $[0, S]$ *and*

$$\mathbf{E} \sup_{t \le \tau} (|\mathbf{v}(t)|^p_{1,p} + |\mathbf{v}(t)|^p_{1,2}) < \infty.$$

Now we can prove the main result.

3.6.3. *Proof of Theorem 1.* We follow here the proof of Theorem 14.21 in [22]. Consider the set $\mathcal{S}$ of all finite stopping times S such that a $\mathbb{H}^1_p$ (respectively $\mathbb{H}^1_p \cap \mathbb{H}^1_2$)-valued continuous $\mathbb{L}_p$ (respectively $\mathbb{L}_p \cap \mathbb{L}_2$)-solution $\mathbf{u}(t)$ of (3.7) exists in $[[0, S]]$ and

$$\mathbf{E} \sup_{t \le S} |\mathbf{u}(t)|^p_{1,p} < \infty \text{ (respectively } \mathbf{E} \sup_{t \le S} (|\mathbf{u}(t)|^p_{1,p} + |\mathbf{u}(t)|^p_{1,2}) < \infty).$$

By Lemma 10, $\mathcal{S}$ is not empty. It is closed with respect to the finite minimum and finite maximum operations. Let ζ be the essential upper bound of the set $\mathcal{S}$. So, there is a sequence $T_n \in \mathcal{S}$ increasing to ζ. Let $\mathbf{U}_n$ be a corresponding sequence of solutions on $[0, T_n]$. Since Proposition 7 holds, the sequence $\mathbf{U}_n$ defines a continuous process $\mathbf{u}$ on $\cup_n [0, T_n]$.

Let $y_t = |\mathbf{U}(t)|_{1,p}($ respectively $y_t = |\mathbf{u}(t)|_{1,p} + |\mathbf{u}(t)|_{1,2})$. Let $R_m = \zeta \wedge \inf(t : y_t \ge m)$. Then $T_q \wedge R_m \in \mathcal{S}$ and $\mathbf{u}(\,\cdot \wedge T_q \wedge R_m)$ is a solution in $[[0, T_q \wedge R_m]]$. Passing to a limit as $q \to \infty$, we obtain that $R_m \in \mathcal{S}$ and $\mathbf{u}(\cdot \wedge R_m)$ is a solution in $[[0, R_m]]$. If $\mathbf{P}(R_m = \zeta < \infty) > 0$, Lemma 11 would imply that there is a stopping time $S \in \mathcal{S}$ such that $S \ge R_m$ and $\mathbf{P}(R_m = \zeta < S) > 0$. This would contradict the definition of ζ. Thus $\mathbf{P}$-a.s. $R_m < \zeta$ on $\{\zeta < \infty\}$, and, obviously, $\limsup_{t \uparrow \zeta} y_t = \infty$ on $\{\zeta < \infty\}$. So, the sequence (R_m) "announces" ζ and ζ is a predictable stopping time. Obviously, $[0, S] \subseteq [0, \zeta)$ for all $S \in \mathcal{S}$. Let S be a stopping time such that $\mathbf{P}$-a.s. $S < \zeta$. Then $T_q \wedge S \in \mathcal{S}$ and $\mathbf{u}(\cdot \wedge T_q \wedge S)$ is a solution in $[[0, T_q \wedge S]]$. Passing to the limit as $q \to \infty$ we obtain that $\mathbf{u}(\cdot \wedge S)$ is a solution in $[[0, S]]$.

Let, in addition, $\mathbf{E}(|\mathbf{u}_0|^p_{2-2/p,p}) < \infty$. Let S be a stopping time such that $S < \zeta$ $\mathbf{P}$-a.s. Consider a linear equation in $[[0, S]]$ for $\mathbf{v}(t)$

(3.60)
$$\begin{cases} \partial_t \mathbf{v}(t) = \mathcal{S}[\partial_i \left(a^{ij}(t) \partial_j \mathbf{v}(t)\right) - u^k(t) \partial_k \mathbf{u}(t) + b^i(t) \partial_i \mathbf{u}(t) + \mathbf{F}(\mathbf{u}(t), t)] \\ \\ +\mathcal{S}[\sigma^i(t) \partial_i \mathbf{v}(t) + \mathbf{G}(\mathbf{u}(t), t)] \dot{W}_t, \\ \\ \mathbf{v}(0) = \mathbf{u}_0. \end{cases}$$

By Theorem 3.3 in [37], in the case a) there is a unique $\mathbb{H}^1_p$-solution of (3.60) which is also a unique $\mathbb{L}_p$-solution. So, $\mathbf{u}(t) = \mathbf{v}(t)$ on $[0, S]$ and $\mathbf{u}(t)$ is an $\mathbb{H}^1_p$-solution to (3.7) in $[[0, S]]$. In the case b) we do the same using Corollary 3.7 in [37] and obtain that $\mathbf{u}(t)$ is a $\mathbb{H}^1_p \cap \mathbb{H}^1_2$-solution in $[[0, S]]$.

It remains to prove that, in the case a), $\lim_{t \uparrow \zeta} |\mathbf{u}(t)|_{1,p} = \infty$ on $\{\zeta < \infty\}$, if $\mathbf{E}(|\mathbf{u}_0|^p_{2-2/p,p}) < \infty$.

Fix an arbitrary $m > 1$. Let

$$\tau_{m+1} = \inf(t : y_t \geq m+1).$$

Define a sequence of stopping times

$$S_1 = \inf(t > \tau_{m+1} : y_t \leq m) \wedge \zeta, \;\; S_{2n} = \inf(t > S_{2n-1} : y_t \geq m+1) \wedge \zeta,$$

$$S_{2n+1} = \inf(t > S_{2n} : y_t \leq m) \wedge \zeta.$$

Let $S = \lim_n S_n$. Applying Itô formula (see proof of Proposition 5) we find that for some adapted $a(r), b(r)$

$$\begin{aligned} y(t \wedge S) &= |\mathbf{u}(t \wedge S)|_p^p + |\mathrm{curl}\, \mathbf{u}(t \wedge S)|_p^p \\ &= y(0) + \int_0^{t \wedge S} a(r)\, dr + \int_0^{t \wedge S} b(r) \cdot dW_r, \end{aligned} \tag{3.61}$$

and

$$a_r \leq C(y_r + y_r^{1+\mu} + z_r), \;\; |b_r|_Y \leq C(y_r + y_r^{1-1/p} \tilde{z}_r^{1/p}) \tag{3.62}$$

where $z_r = |\mathbf{F}(\mathbf{0}, r)|_{1,p}^p + ||\mathbf{G}(\mathbf{0}, r)||_{1,p}^p$, $\tilde{z}_r = ||\mathbf{G}(\mathbf{0}, r)||_{1,p}^p, \mu > 0$.

We will prove that $\mathbf{P}(S = \zeta < \infty) = 0$. Since m is arbitrary, this will imply that $\mathbf{P}$-a.s. $\lim_{t \uparrow \zeta} |\mathbf{u}(t)|_{1,p} = \infty$ on $\{\zeta < \infty\}$.

For $M > 1$ we set

$$\tau^M = \inf(t : \int_0^t (|\mathbf{F}(\mathbf{0}, r)|_{1,p}^p + ||\mathbf{G}(\mathbf{0}, r)||_{1,p}^p)\, dr \geq M).$$

It is enough to prove that for all q, M,

$$\mathbf{P}(S = \zeta < q \wedge \tau^M) = 0.$$

If $\mathbf{P}(S = \zeta < q \wedge \tau^M) > 0$ for some M, q, then by (3.61), (3.62),

$$\infty = \mathbf{E} \sum_{k \geq 1} [y(S_{2k}) - y(S(2k-1)] \leq C(m, M) < \infty.$$

The statement follows.

3.7. **Stochastic Navier-Stokes Equation in $2D$.**

Lemma 12. *Let **B1**, **B2**(p), **B3**(p), **B2**(2), **B3**(2) be satisfied,* $p > d = 2$, *and*

$$\mathbf{E}(|\mathbf{u}_0|_{2-2/p,p}^p + |\mathbf{u}_0|_{1,2}^p) < \infty,$$

$$\int_0^t (||\mathbf{G}(\mathbf{0}, r)||_{1,2}^p + |\mathbf{F}(\mathbf{0}, r)|_{1,2}^p)\, dr < \infty$$

$\mathbf{P}$-*a.s. for all* t.

Then there is a maximal unique $\mathbb{H}_p^1 \cap \mathbb{H}_2^1$*-solution* $\mathbf{u}(t)$ *of (3.7) and for some* $\mathbb{F}$*-adapted functions* $a_l(t), b_l(t)$ *(*$a_l(t)$ *is real valued and* $b_l(t)$ *is* Y*-valued),* $l = p, 2$,

$\mathbf{P}$-*a.s. on* $[0, S] \subseteq [0, \zeta)$

$$|\boldsymbol{\eta}(t)|_l^p = |\boldsymbol{\eta}(0)|_l^p + \int_0^t a_l(r)\ ds + \int_0^t b_l(r) \cdot dW_s,$$

$$|\mathbf{u}(t)|_2^p = |\mathbf{u}(0)|_2^p + \int_0^t a(r)\ ds + \int_0^t b(r) \cdot dW_s,$$

$l = p, 2$, $\boldsymbol{\eta}(t) =$*curl*$\mathbf{u}(t)$. *Moreover, there is a constant* C *independent of* S *such that*

$$|a_l(r)| \leq C[|\boldsymbol{\eta}(r)|_2^p + |\mathbf{u}(r)|_l^p + ||\mathbf{G}(\mathbf{0}, r)||_{1,l}^p + |\mathbf{F}(\mathbf{0}, r)|_{1,l}^p],$$

$$|b_l(r)|_Y \leq C[|\boldsymbol{\eta}(r)|_l^p + |\mathbf{u}(r)|_l^p + ||\boldsymbol{\eta}(r)|_l^{p-1}||\mathbf{G}(\mathbf{0}, r)|_{1,l}),$$

$$|a(r)| \leq C[|\mathbf{u}(r)|_2^p + ||\mathbf{G}(\mathbf{0}, r)||_{1,2}^p + |\mathbf{F}(\mathbf{0}, r)|_{1,2}^p],$$

$$|b(r)|_Y \leq C[|\mathbf{u}(r)|_2^p + |\mathbf{u}(r)|_2^{p-1}||\mathbf{G}(\mathbf{0}, r)||_2),$$

$l = p, 2.$

Proof. The existence of a unique maximal $\mathbb{H}_p^1 \cap \mathbb{H}_2^1$-solution $(\mathbf{u}(t), \zeta)$ is guaranteed by Theorem 1. Since in 2D $\nabla v^i \times \partial_i \mathbf{v} = \mathbf{0}$, the following "regular growth" equation holds for $\boldsymbol{\eta}(t)$ in any $[[0, S]] \subseteq [0, \zeta)$ (cf (3.27):

$$\partial_t \boldsymbol{\eta}\,(t) = \partial_i \left(a^{ij}\,(t)\,\partial_j \boldsymbol{\eta}\,(t)\right) - u^i\,(t)\,\partial_i \boldsymbol{\eta}(t) + b^i(t)\partial_i \boldsymbol{\eta}(t) + \mathbf{H}\,(\mathbf{u}(t), t)$$

$$+ [\sigma^i(t)\partial_i \boldsymbol{\eta}\,(t) + \mathbf{B}\,(\mathbf{u}(t), t))]\,dW_t, \quad \boldsymbol{\eta}\,(0) = \operatorname{curl} \mathbf{u}_0.$$

By Itô formula (see [38]),

$$|\mathbf{u}(t \wedge S)|_2^p = |\mathbf{u}(0)|_2^p - p \textstyle\int_0^{t\wedge S} |\mathbf{u}(r)|_2^{p-2} N_2(\mathbf{u}(r))\,ds +$$

$$p \textstyle\int_0^{t\wedge S} |\mathbf{u}(r)|_2^{p-2} \int (\mathbf{u}(r), \bar{\mathbf{a}}(r))\,dx\,dr + p \int_0^{t\wedge S} |\mathbf{u}(r)|_2^{p-2} (\int u^l(r) c^l(r)\,dx)\,dW_r +$$

$$p/2 \textstyle\int_0^{t\wedge S} |\mathbf{u}(r)|_2^{p-2} (\int \bar{b}^{ii}(r)\,dx)\,dr + \frac{p}{2}(p-2) \int_0^{t\wedge S} |\mathbf{u}(r)|_2^{q-4} |\int u^l(r) c^l(r)\,dx|_Y^2\,dr$$

where

$$\bar{\mathbf{a}}(r) = b^i(r)\partial_i \mathbf{u}(r) + \mathbf{F}(\mathbf{u}(r), r),$$

$$\bar{b}^{ij}(r) = \sigma^k(r)\partial_k u^i(r) \cdot d^j(r) + \sigma^k(r)\partial_k u^j(r) \cdot d^i(r) + d^i(r) \cdot d^j(r),$$

and $\mathbf{d}(r) = (d^i(r)) = \tilde{\mathbf{G}}(\mathbf{u}(r), r))$, $\mathbf{c}(r) = (c^l(r)) = \sigma^k(r)\partial_k \mathbf{u}(r) + \mathbf{d}(r)$.

Similarly, for each stopping time S so that $[0,S]\subseteq[0,\zeta)$

$$|\boldsymbol{\eta}(t\wedge S)|_2^p=|\boldsymbol{\eta}(0)|_2^p-p\int_0^{t\wedge S}|\boldsymbol{\eta}(r)|_2^{p-2}N_2(\boldsymbol{\eta}(r),r)\,dr+p\int_0^{t\wedge S}|\boldsymbol{\eta}(r)|_2^{p-2}\int(\boldsymbol{\eta}(r),\mathbf{H}(r)\,dx\,dr$$

$$+\int_0^{t\wedge S}p|\boldsymbol{\eta}(r)|_2^{p-2}\int\eta(r)\gamma(r)\,dx)\,dW_r+\frac{p}{2}\int_0^{t\wedge S}|\boldsymbol{\eta}(r)|_2^{p-2}(\int\bar{\gamma}(r)\,dx)\,dr$$

$$+\frac{p}{2}(p-2)\int_0^{t\wedge S}|\boldsymbol{\eta}(r)|_2^{p-4}|\int\eta(r)\gamma(r)\,dx|_Y^2\,dr,$$

where

$$\mathbf{H}(r)=b^i(r)\partial_i\boldsymbol{\eta}(r)+\mathbf{H}(\mathbf{u}(r),r)),\ \bar{\gamma}(r)=2\sigma^k(r)\partial_k\eta\cdot\bar{d}(r)+|\bar{d}(r)|_Y^2,$$

and $\bar{d}(r)=\boldsymbol{\nabla}\sigma^i\times\partial_i\mathbf{u}+\operatorname{curl}(\mathbf{G}(\mathbf{u},t))$, $\gamma(r)=\sigma_k(r)\partial_k\boldsymbol{\eta}(r)+\bar{d}(r)$. By Lemmas 2 and 3,

$$|\boldsymbol{\eta}(t\wedge S)|_2^p=|\boldsymbol{\eta}(0)|_2^p+\int_0^{t\wedge S}a_2(r)\ ds+\int_0^{t\wedge S}b_2(r)\cdot dW_s,$$

$$|\mathbf{u}(t\wedge S)|_2^p=|\mathbf{u}(0)|_2^p+\int_0^{t\wedge S}a(r)\ ds+\int_0^{t\wedge S}b(r)\cdot dW_s,$$

and there is a constant C independent of S such that

$$a_2(r)\leq C[|\boldsymbol{\eta}(r)|_2^p+|\mathbf{u}(r)|_2^p+||\mathbf{G}(\mathbf{0},r)||_{1,2}^p+|\mathbf{F}(\mathbf{0},r)|_{1,2}^p],$$

$$|b_2(r)|_Y\leq C[|\boldsymbol{\eta}(r)|_2^p+|\mathbf{u}(r)|_2^p+||\boldsymbol{\eta}(r)|_2^{p-1}||\mathbf{G}(\mathbf{0},r)|_{1,2}),$$

$$a(r)\leq C[|\mathbf{u}(r)|_2^p+||\mathbf{G}(\mathbf{0},r)||_{1,2}^p+|\mathbf{F}(\mathbf{0},r)|_{1,2}^p],$$

$$|b(r)|_Y\leq C[|\mathbf{u}(r)|_2^p+|\mathbf{u}(r)|_2^{p-1}||\mathbf{G}(\mathbf{0},r)||_2^p).$$

Also by Itô formula

$$|\boldsymbol{\eta}(t)|_p^p=|\boldsymbol{\eta}(0)|_p^p-p\int_0^t N_p(\boldsymbol{\eta}(r),r)\,ds+p\int_0^t\left\langle|\boldsymbol{\eta}(r)|^{p-2}\boldsymbol{\eta}(r),\bar{\mathbf{a}}(r)\right\rangle_{1,p}dr$$

$$+\int_0^t p(\int|\boldsymbol{\eta}(r)|^{p-2}\eta(r)c(r)\,dxd\dot{W}_r+p\int_0^t(\int(p-2)|\boldsymbol{\eta}(r)|^{p-2}\bar{b}(r)\,dx)\,dr$$

where

$$\bar{\mathbf{a}}(r)=b^i\partial_i\boldsymbol{\eta}+\operatorname{curl}(\mathbf{F}(\mathbf{u},r))+\partial_i(\boldsymbol{\nabla}a^{ij}\times\partial_j\mathbf{u})+(\boldsymbol{\nabla}b^i)\times\partial_i\mathbf{u}$$

$$=b^i\partial_i\boldsymbol{\eta}+\operatorname{curl}(\mathbf{F}(\mathbf{u},r))+\mathbf{r}(\mathbf{u}(r),r),$$

$$\bar{b}(r)=2\sigma^k\partial_k\eta\cdot d(r)+|\bar{d}(r)|_Y^2,$$

and $\bar{\mathbf{d}} = \boldsymbol{\nabla}\sigma^i \times \partial_i \mathbf{u} + \operatorname{curl}(\mathbf{G}(\mathbf{u}, t))$, $c(r) = \sigma^k(r)\partial_k \eta(r) + \bar{\mathbf{d}}(r)$. Using Lemmas 2 and 3, we obtain

$$|\boldsymbol{\eta}(t)|_p^p = |\boldsymbol{\eta}(0)|_p^p + \int_0^t a_p(r)\ ds + \int_0^t b_p(r) \cdot dW_s.$$

and there is a constant C independent of S such that

$$|a_p(r)| \le C[|\boldsymbol{\eta}(r)|_p^p + |\mathbf{u}(r)|_p^p + ||\mathbf{G}(\mathbf{0}, r)||_{1,p}^p + |\mathbf{F}(\mathbf{0}, r)|_{1,p}^p],$$

$$|b_p(r)|_Y \le C[|\boldsymbol{\eta}(r)|_p^p + |\mathbf{u}(r)|_p^p + ||\boldsymbol{\eta}(r)|_p^{p-1} ||\mathbf{G}(\mathbf{0}, r)|_{1,p}).$$

So, the statement follows. ▮

Now we can complete the proof of Theorem 2.

3.7.1. *Proof of Theorem 2.* We have immediately the existence of a maximal solution by Theorem 1. It remains to prove that $\mathbf{P}(\zeta = \infty) = 1$ and the estimate. Let

$$y_t = |\mathbf{u}(t)|_2^p + |\operatorname{curl}\mathbf{u}(t)|_2^p + |\operatorname{curl}\mathbf{u}(t)|_p^p,$$

$R_m = \inf(t : y_t \ge m) \wedge \zeta$. Since in 2D $\mathbb{L}_p$ is continuously embedded into $\mathbb{H}_2^1$ ($\mathbb{L}_p \subseteq \mathbb{H}_2^1$), we obtain by Lemma 12 that for some adapted functions $h(t), \kappa(t)$,

$$y_{t\wedge R_m} = y_0 + \int_0^{t\wedge R_m} h(r)\, dr + \int_0^{t\wedge R_m} \kappa(r) \cdot dW_r,$$

and there is a constant C independent of m such that

$$h(r) \le C(y_r + z_r),\ \ \kappa(r) \le C(y_r + y_r^{1-1/p} \tilde{z}_r^{1/p})$$

where

$$z_r = ||\mathbf{G}(\mathbf{0}, r)||_{1,p}^p + |\mathbf{F}(\mathbf{0}, r)|_{1,p}^p + ||\mathbf{G}(\mathbf{0}, r)||_{1,2}^p + |\mathbf{F}(\mathbf{0}, r)|_{1,2}^p,$$

$$\tilde{z}_r = ||\mathbf{G}(\mathbf{0}, r)||_{1,p}^p + ||\mathbf{G}(\mathbf{0}, r)||_{1,2}^p.$$

By Lemma 18 (see Appendix), for each T there is a constant C (independent of m) so that for all stopping times $\tau \le T$

$$\begin{aligned} \mathbf{E}\sup_{t\le\tau} y_t \le C\mathbf{E}[y_0 + \int_0^\tau (||\mathbf{G}(\mathbf{0}, r)||_{1,p}^p + |\mathbf{F}(\mathbf{0}, r)|_{1,p}^p \\ + ||\mathbf{G}(\mathbf{0}, r)||_{1,2}^p + |\mathbf{F}(\mathbf{0}, r)|_{1,2}^p)\, dr]. \end{aligned} \tag{3.63}$$

Let

$$K_t = \int_0^t (||\mathbf{G}(\mathbf{0}, r)||_{1,p}^p + |\mathbf{F}(\mathbf{0}, r)|_{1,p}^p + ||\mathbf{G}(\mathbf{0}, r)||_{1,2}^p + |\mathbf{F}(\mathbf{0}, r)|_{1,2}^p)\, dr.$$

For $M > 1$ set $\tau^M = \inf(t : K_t \ge M)$. Since the sequence R_m "announces" the predictable stopping time ζ, for each $T > 0$ and $M > 1$ we have

$$\mathbf{P}(R_m < T) \le \mathbf{P}(y_{Rm\wedge T\wedge\tau^M} \ge m) + \mathbf{P}(\tau^M < T)$$

$$\le m^{-1}\mathbf{E} y_{R_m\wedge T\wedge\tau^M} + \mathbf{P}(\tau^M < T).$$

So, by (3.63), for each $M > 1$

$$\limsup_m \mathbf{P}(R_m < T) \leq \mathbf{P}(\tau^M < T).$$

Therefore $\lim_m \mathbf{P}(R_m < T) = 0$, and $\mathbf{P}(\zeta = \infty) = 1$. The statement follows.

4. Wiener Chaos and Moment Theory

4.1. **Preliminaries.** In this Section we continue the study of global solutions of stochastic Navier-Stokes equations. We will deal with the equation

$$\begin{aligned} &\partial_t \mathbf{u} = \partial_i \left(a^{ij}\partial_j \mathbf{u}\right) + b^i \partial_i \mathbf{u} - u^k \partial_k \mathbf{u} + \mathbf{h}^i \cdot \mathcal{G}^i \left(\sigma^{ik}\partial_i \mathbf{u} + \mathbf{g}\right) - \\ &\boldsymbol{\nabla} P + \mathbf{f} + [\sigma^{ik}\partial_i \mathbf{u} + \mathbf{g} - \boldsymbol{\nabla}\tilde{P}] \cdot \dot{W}_t,\ div\, \mathbf{u} = 0, \\ &\mathbf{u}\left(0, x\right) = \mathbf{u}_0\left(x\right) \end{aligned} \tag{4.1}$$

with the free forces $\mathbf{f} = \mathbf{f}\left(t, x\right)$ and $\mathbf{g} = \mathbf{g}\left(t, x\right)$ that do not include a solution as an independent variable. Since the existence of global solutions is proved only for $d = 2$, we restrict ourselves to this case.

Our goal now is to investigate how the SNS equation (4.1) propagates the chaos generated by the driving Brownian motion and randomness in the initial conditions. We are particularly interested in deriving formulas for the statistical moments of a solution to (4.1).

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space. Let $W\left(t\right)$ and ξ^0 be a cylindrical Brownian motion and a cylindrical Gaussian random variable in Y. We assume that $W\left(t\right)$ and ξ^0 are defined on $(\Omega, \mathcal{F}, \mathbf{P})$ and independent.

Let us fix a positive number $T < \infty$. Let $\mathcal{F}_T$ be a $\mathbf{P}$-completion of $\boldsymbol{\sigma}\{\xi^0, W\left(r\right) : r \leq t\}$ and $\mathcal{F}_t$ be the a $\boldsymbol{\sigma}$-algebra generated by $\cap_{t<s\leq T}\boldsymbol{\sigma}\{\xi^0, W\left(r\right), r \leq s\}$ and all the negligible sets of $\mathcal{F}$. The filtration of right continuous $\boldsymbol{\sigma}$-algebras $(\mathcal{F}_t)_{t\leq T}$ will be denoted by $\mathbb{F}$.

We will assume in the future that the initial value $\mathbf{u}_0\left(x\right)$ is random but its randomness is due solely to its dependence on ξ^0.

To begin with we shall introduce additional notation and recall some basic facts of the Wiener chaos theory (see e.g. [19],[20],[34], [36], etc).

Let us fix a positive number $T < \infty$. Let $\{m_k, k \geq 1\}$ be an orthonormal basis in $L_2\left(0, T\right)$ and $\{\ell_k, k \geq 1\}$ an orthonormal basis in Y. Write $\xi_i^k = \int_0^T m_i\left(s\right) dw^k\left(t\right)$ where $w^k\left(t\right) = \left(W\left(t\right), \ell_k\right)_Y$.

Without any loss of generality, we assume that $\mathcal{F}_0$ is generated by the sequence of independent standard (i.e. $\mathcal{N}\left(0,1\right)$) Gaussian random variables $\left\{\xi_i^0, i \geq 1\right\}$

Let $\alpha = \left\{\alpha_i^k,\ k \geq 0; i \geq 1\right\}$ be a multiindex, i.e. for every $\left(i, k\right)$, $\alpha_i^k \in \mathbb{N} = \{0, 1, 2, \ldots\}$. We shall consider only such α that $|\alpha| = \sum_{k,i} \alpha_i^k < \infty$, i.e., only a finite number of α_i^k is non-zero, and we denote by $\mathcal{J}$ the set of all such multiindices. Obviously, if $\alpha \in \mathcal{J}$, the number $\alpha! = \Pi_{k,i}\alpha_i^k!$ is well defined. We also write $\alpha! = \prod_{k,l}(\alpha_k^l!)$.

For $\alpha \in \mathcal{J}$, write

$$\zeta_\alpha := \prod_{i=1}^{\infty}\prod_{k=0}^{\infty} H_{\alpha_i^k}(\xi_i^k)$$

where H_n is the n^{th} Hermite polynomial $H_n(x) = (-1)^n \left(\frac{d^n}{dx^n} e^{-\frac{x^2}{2}}\right) e^{\frac{x^2}{2}}$.
The random variable ζ_α is often referred to as (unnormalized) α^{th} Wick polynomial.

Let $\mathcal{J}_0$ be a subset of $\mathcal{J}$ consisting of all multiindices of the form

$$\alpha = \left\{\alpha_i^k, k \geq 0,\ i \geq 1 : \alpha_i^k = 0 \text{ if } k \neq 0\right\}$$

We will often denote a multiindex from $\mathcal{J}_0$ by α^0. Obviously, for $\alpha^0 \in \mathcal{J}$,

$$\zeta_{\alpha^0} := \prod_{i=1}^{\infty} H_{\alpha_i^0}(\xi_i^0).$$

It is a standard fact that

$$E\zeta_\alpha\zeta_\beta = \left\{\begin{pmatrix} 0 \text{ if } \alpha \neq \beta \\ \alpha! \text{ if } \alpha = \beta \end{pmatrix}\right. . \tag{4.2}$$

The most important feature of the Wick polynomials ζ_α is that the set $\left\{\zeta_\alpha/\sqrt{\alpha!}, \alpha \in \mathcal{J}\right\}$ is an orthonormal basis in $L_2\left(\Omega, \mathcal{F}_T, \mathbf{P}\right)$ (see e.g. [6],[36]) . This result is often referred to as the Cameron-Martin theorem. The following lemma is an obvious extension of the Cameron-Martin theorem to the vector case.

Let $\mathcal{H}$ be a separable Hilbert space and $\{h_i, i \geq 1\}$ be an orthonormal basis in $\mathcal{H}$.

Lemma 13. *Let $\eta : \Omega \longrightarrow \mathcal{H}$ be an $\mathcal{F}$-measurable random variable so that $\mathbf{E}\left\|\eta\right\|_{\mathcal{H}}^2 < \infty$. Then, η admits the Wiener chaos expansion in $L^2(\Omega; \mathcal{H})$:*

$$\eta = \sum_{\boldsymbol{\alpha} \in \mathcal{J}} \frac{\hat{\eta}_\alpha}{\alpha!} \zeta_{\boldsymbol{\alpha}} \tag{4.3}$$

where $\hat{\eta}_\alpha = \mathbf{E}[\eta\zeta_{\boldsymbol{\alpha}}] := \sum_{i=1}^{\infty} \mathbf{E}[(\eta, h_i)\zeta_{\boldsymbol{\alpha}}]h_i$.

Moreover,

$$\mathbf{E}\left\|\eta\right\|_{\mathcal{H}}^2 = \sum_{\boldsymbol{\alpha} \in \mathcal{J}} \frac{\left|\hat{\eta}_\alpha\right|^2}{\alpha!} = \sum_{\boldsymbol{\alpha} \in \mathcal{J}} \sum_{i=1}^{\infty} \frac{1}{\alpha!} \mathbf{E}[(\eta, h_i)_{\mathcal{H}} \zeta_{\boldsymbol{\alpha}}]^2. \tag{4.4}$$

In the future, we will refer to the functions $\hat{\eta}_\alpha$ as unnormalized Hermite-Fourier coefficients, or simply, Hermite-Fourier coefficients, of η **.**

4.2. **Propagator.** Suppose that the assumptions of Theorem 2 hold. Then, equation (4.1) has a unique $\mathbb{F}-$adapted global solution in $\mathbb{H}_p^1\left(\mathbf{R}^2\right) \bigcap \mathbb{H}_2^1\left(\mathbf{R}^2\right)$ and

$$\mathbf{E}\sup_{s \leq T}\left(|\mathbf{u}(t)|_{1,p}^2 + |\mathbf{u}(t)|_{1,2}^p\right) < \infty.$$

By (4.3), a solution of (4.1) allows the Wiener Chaos expansion $\mathbf{u}\left(t, x\right) = \sum_{\boldsymbol{\alpha} \in \mathcal{J}} \frac{\hat{\mathbf{u}}_\alpha(t,x)}{\alpha!} \zeta_{\boldsymbol{\alpha}}$.

This equality holds for all t in $L_2\left(\Omega; \mathbb{H}_2^1\left(R^2\right)\right)$ as well as for all t, x in $L_2\left(\Omega; R^2\right)$. The latter is due to the well-known imbedding $H_{1,p} \subset C^{1-2/p}$.

Of course, the main problem of interest is how to characterize the Hermite-Fourier coefficients $\mathbf{u}_\alpha\left(t, x\right)$. It will be shown below that these coefficients verify a certain nonlinear system of equations. This system describes the pattern of deterministic propagation of randomness in (4.1).

In this Section we shall make the following additional assumptions:

(C1) *The initial value* u_0 *is a measurable* $\mathcal{F}_0$*- adapted function.*

(C2) *The coefficients* α^{ij} *and* b^i *are measurable functions on* $[0,T)\times\mathbf{R}^2$*;* f^l *are predictable functions on* $[0,T)\times\mathbf{R}^2\times\Omega; \sigma^l, h^{l,i}$ *are* $Y-$*valued measurable functions on* $[0,T)\times\mathbf{R}^2$*,* g^l *are* $Y-$*valued predictable functions on* $[0,T)\times\mathbf{R}^2\times\Omega$*, and for all* t,x $\sum_{i=0}^1\left(\left|\partial_x^k h^{l,i}(t,x)\right|_Y+\left|\partial_x^k\boldsymbol{\sigma}(t,x)\right|\right)\le C.$

(**C3**) *For* $p>2$ and $l=2,p$,

$$\int_0^T\mathbf{E}\left(|\mathbf{g}(t)|_{1,l}^p+\sum_i|\mathbf{f}(t)|_{1,l}^p\right)dt<\infty.$$

Note that, in contrast to the previous sections, we postulate that a^{ij} , b^i and $h^{l,i}$ are non-random.

Now we introduce some additional notation.

Write

$$\widehat{u^i\partial_i\mathbf{u}_\alpha}(t)=\sum_{p\in\mathcal{J}}\sum_{0\le\beta\le\alpha}\frac{1}{p!}\binom{\alpha}{\beta}\hat{u}^i_{p+\beta}(t)\,\partial_i\hat{\mathbf{u}}_{p+\alpha-\beta}(t), \tag{4.5}$$

$$\mathcal{M}(\hat{\mathbf{u}}_\alpha,t)=\sigma^j(t)\partial_j\hat{\mathbf{u}}_\alpha(t)+\hat{\mathbf{g}}_\alpha(t).$$

For $j\neq0$, we define multiindex $\alpha(i,j)\in\mathcal{J}$ by the formula

$$\alpha(i,j)_l^k=\begin{cases}\alpha_l^k \text{ if } (k,l)\neq(j,i) \text{ or } k=0\\ \left(\alpha_l^k-1\right)\wedge0 \text{ if } (k,l)=(j,i)\end{cases}, \tag{4.6}$$

i.e. the multiindex $\alpha(i,j)$ might differ from α only by its (i,j) entry which is equal to $\left(\alpha_j^i-1\right)\vee0$.

Set

$$D\mathcal{M}(\hat{\mathbf{u}}_\alpha,t)=\begin{cases}\sum_{k\neq0}\left(\hat{\mathbf{g}}_{\alpha(i,k)}(t)+\sigma^j(t)\,\partial_j\hat{\mathbf{u}}_{\alpha(i,k)}(t)\right)\alpha_i^k m_i(t) & \text{if } \alpha\notin\mathcal{J}_0\\ 0 & \text{otherwise.}\end{cases}$$

and

$$\mathcal{L}_0(\hat{\mathbf{u}}_\alpha,t)=\partial_i\left(a^{ij}(t)\,\partial_j\hat{\mathbf{u}}_\alpha(t)\right)+b^i(t)\partial_i\hat{\mathbf{u}}_\alpha(t)+\hat{\mathbf{f}}_\alpha(t)+\mathbf{h}^i(t)\mathcal{G}^i(\mathcal{M}(\hat{\mathbf{u}}_\alpha,t).$$

Theorem 3. *Assume that* $C1-C3$ *as well as the assumptions of Theorem 2 hold.*

Then the Fourier-Hermite coefficients $\hat{\mathbf{u}}_\alpha$ *of the global solution of SNS* (4.1)*are continuous* $\mathbb{H}_p^1\left(\mathbf{R}^2\right)\bigcap\mathbb{H}_2^1\left(\mathbf{R}^2\right)$*-valued functions on* $[0,T]$*. Moreover, the set of functions* $\{\hat{\mathbf{u}}_\alpha(t,x),\alpha\in\mathcal{J}\}$ *verifies the following system of equations*

$$\begin{cases}(\hat{\mathbf{u}}_\alpha(t),\boldsymbol{\varphi})_2=(\hat{\mathbf{u}}_\alpha(0),\boldsymbol{\varphi})_2+\int_0^t\{\langle\mathcal{L}_0(\hat{\mathbf{u}}_\alpha,s),\boldsymbol{\varphi}\rangle+\\ \\ (-\widehat{u^i\partial_i\mathbf{u}_\alpha}(s)+D\mathcal{M}(\hat{\mathbf{u}}_\alpha,s),\boldsymbol{\varphi})_2\}ds;\ div\hat{\mathbf{u}}_\alpha(t)=0\\ \\ \textit{for all } t\le T \textit{ and } \boldsymbol{\varphi}\in\left(C_0^\infty\left(\mathbf{R}^2\right)\right)^2 \textit{ so that } div\boldsymbol{\varphi}=0.\end{cases} \tag{4.7}$$

4.2.1. *Proof of Theorem 3.* To begin with, we remark that for $\alpha=0$, $\hat{\mathbf{u}}_\alpha(0)=\mathbf{E}\mathbf{u}_0$, and if α has at least one positive entry α_i^k with $k\neq0, \hat{\mathbf{u}}_\alpha(0)=0$.

By Theorem 2 we have that $\mathbf{u}(t)$, a solution of equation (4.1), is a continuous $\mathbb{H}_p^1\left(\mathbf{R}^2\right)\bigcap\mathbb{H}_2^1\left(\mathbf{R}^2\right)$-valued function and

$$\mathbf{E}\sup_{t\le T}(|\mathbf{u}(t)|_{1,p}^p+|\mathbf{u}(t)|_{1,2}^p)<\infty. \tag{4.8}$$

Owing to (4.8) ,we have by Hölder inequality that for $l = 2, p$

$$\sup_{t \leq T} \left(|\hat{\mathbf{u}}_\alpha(t)|_l + |(\partial_i \mathbf{u})_\alpha(t)|_l \right) < \infty \ . \tag{4.9}$$

By Fubini Theorem, for all $\boldsymbol{\varphi} \in \left(C_0^\infty \left(\mathbf{R}^2 \right) \right)^2$ and $t \leq T$,

$$\left((\partial_i \mathbf{u}(t))_\alpha , \boldsymbol{\varphi} \right)_2 = -\mathbf{E} \left[\left(\mathbf{u} \left(t \right) , \partial_i \boldsymbol{\varphi} \right)_2 \zeta_\alpha \right] = - \left(\hat{\mathbf{u}}_\alpha \left(t \right) , \partial_i \boldsymbol{\varphi} \right)_2 .$$

Thus,

$$\partial_i \hat{\mathbf{u}}_\alpha(t) = (\widehat{\partial_i \mathbf{u}\,(t)})_\alpha . \tag{4.10}$$

Now, by (4.9) and (4.10)we have that for all t, $\hat{\mathbf{u}}_\alpha \left(t \right) \in \mathbb{H}_p^1 \left(\mathbf{R}^2 \right) \bigcap \mathbb{H}_2^1 \left(\mathbf{R}^2 \right)$. Since for integer n and $q \geq 1$, the norm $\left| \cdot \right|_{n,q}$ is equivalent to the norm of the Sobolev space $W^{n,q}$,by (4.10) and Hölder inequality, we have that for $l = 2, p$

$$|\hat{\mathbf{u}}_\alpha(t) - \hat{\mathbf{u}}_\alpha(s)|_{1,l} \leq$$

$$C \left(|\mathbf{E} \left(\mathbf{u}(t) - \mathbf{u}(s) \right) \zeta_\alpha|_l + \textstyle\sum_i |\mathbf{E} \left(\partial_i \mathbf{u}(t) - \partial_i \mathbf{u}(s) \right) \zeta_\alpha|_l \right) \leq$$

$$C' \mathbf{E} |\mathbf{u}(t) - \mathbf{u}(s)|_{1,l} .$$

Thus, by the Dominated Convergence Theorem we have that the Fourier-Hermite coefficients $\hat{\mathbf{u}}_\alpha \left(t \right)$ are continuous in $\mathbb{H}_p^1 \left(\mathbf{R}^2 \right) \bigcap \mathbb{H}_2^1 \left(\mathbf{R}^2 \right)$.

Owing to (4.10), we also have that for every $\alpha \in \mathcal{J}$, $div \hat{\mathbf{u}}_\alpha(t) = 0$.

We continue with two simple but useful lemmas.

Lemma 14. *Let η and ψ be $\mathcal{F}$-measurable $\mathcal{H}$-valued random variables, and* $\mathbf{E} \left[\|\eta\|_{\mathcal{H}}^2 + \|\psi\|_{\mathcal{H}}^2 \right] < \infty$.

Then,

$$\begin{aligned} (\psi, \eta)_{\mathcal{H}} &= \textstyle\sum_{\boldsymbol{\gamma}, \boldsymbol{\beta} \in \mathcal{J}} \left(\hat{\psi}_\gamma, \hat{\eta}_\beta \right)_{\mathcal{H}} \sum_{p \leq \gamma \wedge \beta} \left((\gamma - p)! \, (\beta - p)! p! \right)^{-1} \zeta_{\gamma + \beta - 2p} \\ &= \textstyle\sum_{\alpha, p \in \mathcal{J}} \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} \frac{1}{\alpha! p!} \left(\hat{\psi}_{p + \alpha - \beta}, \hat{\eta}_{p + \beta} \right)_{\mathcal{H}} \zeta_\alpha . \end{aligned} \tag{4.11}$$

Proof. It is a standard fact (see e.g. [36]) that

$$\zeta_\gamma \zeta_\beta = \sum_{p \leq \gamma \wedge \beta} \binom{\gamma}{p} \binom{\beta}{p} p! \zeta_{\gamma + \beta - 2p} . \tag{4.12}$$

By Lemma 13 and (4.12) , we have

$$\begin{gathered} (\psi, \eta)_{\mathcal{H}} = \textstyle\sum_{\boldsymbol{\gamma}, \boldsymbol{\beta} \in \mathcal{J}} \frac{1}{\gamma! \beta!} \left(\hat{\psi}_\gamma, \hat{\eta}_\beta \right)_{\mathcal{H}} \zeta_\gamma \zeta_\beta = \\ \textstyle\sum_{\boldsymbol{\gamma}', \boldsymbol{\beta}' \in \mathcal{J}} \left(\hat{\psi}_{\gamma'}, \hat{\eta}_{\beta'} \right)_{\mathcal{H}} \sum_{p \leq \gamma' \wedge \beta'} \left((\gamma' - p)! \, (\beta' - p)! p! \right)^{-1} \zeta_{\gamma' + \beta' - 2p} \end{gathered} \tag{4.13}$$

By making the change of variables $\alpha = \gamma' + \beta' - 2p, \beta = \beta' - p$ in (4.13) and observing that $\gamma' - p = \alpha - \beta$, we arrive at

$$(\psi, \eta)_{\mathcal{H}} = \sum_{\alpha, p \in \mathcal{J}} \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} \frac{1}{\alpha! p!} \left(\hat{\psi}_{p + \alpha - \beta}, \hat{\eta}_{p + \beta} \right)_{\mathcal{H}} \zeta_\alpha .$$

∎

Lemma 15. *(see [43]) The process $\zeta_\alpha(t) = \mathbf{E}[\zeta_\alpha|\mathcal{F}_t]$ verifies the following equation:*

$$d\zeta_\alpha(t) = m_i(t)\,\alpha_i^k \zeta_{\alpha(i,k)}(t) dw^k(t). \tag{4.14}$$

Remark 8. *Write $D\zeta_\alpha(t) = m_i(t)\,\alpha_i^k\zeta_{\alpha(i,k)}(t)\ell_k$ where as before $(\ell_k)_{k\geq 1}$ is an orthonormal basis in Y. It is readily checked (cf. [45]) that $D\zeta_\alpha(t)$ is the Malliavin derivative of $\zeta_\alpha(t)$. Now we can rewrite equation (4.14) in the following more compact and maybe more insightful form:*

$$d\zeta_\alpha(t) = D\zeta_\alpha(t)\cdot dW(t).$$

Note that since , $(\zeta_\alpha(t), \mathcal{F}_t)$ is a uniformly integrable martingale, we can sharpen (4.3) as follows:

Corollary 5. *If $\mathbf{v} \in L^2(\Omega, \mathcal{F}_s, \mathbf{P}; \mathbb{L}_2)$ for some $s \in [0,T]$, then $\hat{\mathbf{v}}_\alpha = \mathbf{E}[\mathbf{v}\zeta_\alpha(s)]$, and*

$$\mathbf{v} = \sum_{\alpha\in\mathcal{J}} \frac{\hat{\mathbf{v}}_\alpha}{\alpha!}\zeta_\alpha(s)$$

in $L^2(\Omega, \mathcal{F}_s, \mathbf{P}; \mathbb{L}_2)$.

Write, $\mathcal{M}^k(\mathbf{u},t) = \sigma^{jk}(t)\partial_j\mathbf{u}(t) + \mathbf{g}^k(t)$ and $\mathcal{M}^k(\hat{\mathbf{u}}_\alpha, t) = \sigma^{jk}(t)\partial_j\hat{\mathbf{u}}_\alpha(t) + \hat{\mathbf{g}}_\alpha^k(t)$ where $\sigma^{jk} = \left(\sigma^i, \ell_k\right)_Y$, $\mathbf{g}^k = (\mathbf{g},\ell_k)_Y$, and $(\ell_k, k \geq 1)$ is an orthonormal basis in Y. By Itô formula, Lemma 15, and (4.1), we have

$$\begin{cases} d\left((\mathbf{u}(t), \boldsymbol{\varphi})_2\zeta_\alpha(t)\right) = [\zeta_\alpha(t)\langle\mathcal{L}(\mathbf{u},t), \boldsymbol{\varphi}\rangle + I_{\{\alpha\notin\mathcal{J}\}}\sum_{k\neq 0} m_i(t)\,\alpha_i^k\zeta_{\alpha(i,k)}(t)\left(\mathcal{M}^k(\mathbf{u},t), \boldsymbol{\varphi}\right)_2]dt \\ \\ + \left[\zeta_\alpha(t)\left(\mathcal{M}^k(\mathbf{u},t), \boldsymbol{\varphi}\right)_2 + I_{\{\alpha\neq\alpha^0\}}(\mathbf{u}(t), \boldsymbol{\varphi})_2 m_i(t)\,\alpha_i^k\zeta_{\alpha(i,k)}(t)\right] dw^k(t) \end{cases}$$

where

$$\langle\mathcal{L}(\mathbf{u}), \boldsymbol{\varphi}\rangle := [-\left(a^{ij}\partial_j\mathbf{u}, \partial_i\boldsymbol{\varphi}\right)_2 + (b^i\partial_i\mathbf{u} - u^k\partial_k\mathbf{u} + \mathbf{f} + \left(\mathbf{h}^i\cdot\mathcal{G}^i(\mathcal{M}(\mathbf{u})), \boldsymbol{\varphi}\right)_2].$$

Taking the expectations of both sides of the equation and using Corollary 5, we arrive at

(4.15)

$$\partial_t\hat{\mathbf{u}}_\alpha(t) = \mathbf{E}[\zeta_\alpha(t)\langle\mathcal{L}(\mathbf{u},t), \boldsymbol{\varphi}\rangle] + I_{\{\alpha\notin\mathcal{J}_0\}}\sum_{k\neq 0} m_i(t)\,\alpha_i^k\mathbf{E}\left[\zeta_{\alpha(i,k)}(t)\left(\mathcal{M}^k(\mathbf{u},t), \boldsymbol{\varphi}\right)_2\right].$$

Now we shall express $\mathbf{E}[\zeta_\alpha(t)\langle\mathcal{L}(\mathbf{u},t), \boldsymbol{\varphi}\rangle]$ and $\mathbf{E}\left[\zeta_{\alpha(i,k)}(t)\left(\mathcal{M}^k(\mathbf{u},t), \boldsymbol{\varphi}\right)_2\right]$ in terms of Hermite-Fourier coefficients of $\mathbf{u}, \mathbf{f}$,and $\mathbf{g}$.

Write $\left\langle\tilde{\mathcal{L}}_0(\mathbf{u},t), \boldsymbol{\varphi}\right\rangle = -\left(a^{ij}(t)\,\partial_j\mathbf{u}(t), \partial_i\boldsymbol{\varphi}\right)_2 + (b^i(t)\partial_i\mathbf{u}(t) + \mathbf{f}(t), \boldsymbol{\varphi})_2$. Obviously, $\langle\mathcal{L}_0(\mathbf{u},t), \boldsymbol{\varphi}\rangle = \left\langle\tilde{\mathcal{L}}_0(\mathbf{u},t), \boldsymbol{\varphi}\right\rangle + R(t, \boldsymbol{\varphi})$ where

$$R(t, \boldsymbol{\varphi}) = \left(\mathbf{h}^i(t)\cdot\mathcal{G}(\sigma^j(t)\partial_j\mathbf{u}(t) + \mathbf{g}(t)), \boldsymbol{\varphi}\right)_2. \tag{4.16}$$

It is easily seen that

$$\mathbf{E}\left[\zeta_\alpha(t)\left(\tilde{\mathcal{L}}_0(\mathbf{u},t), \boldsymbol{\varphi}\right)_2\right] = \left(\tilde{\mathcal{L}}_0(\hat{\mathbf{u}}_\alpha,t), \boldsymbol{\varphi}\right)_2, \tag{4.17}$$

and for $\alpha \notin \mathcal{J}_0$,

$$\sum_{k\neq 0} m_i\left(t\right)\mathbf{E}\left[\zeta_{\alpha(i,k)}(t)\alpha_i^k\left(\mathcal{M}^k\left(\mathbf{u},t\right),\boldsymbol{\varphi}\right)_2\right]=\sum_{k\neq 0} m_i\left(t\right)\alpha_i^k\left(\mathcal{M}^k\left(\hat{\mathbf{u}}_{\alpha(i,k)},t\right),\boldsymbol{\varphi}\right)_2. \tag{4.18}$$

Let us consider now the term $\left(u^i\left(t\right)\partial_i\mathbf{u}\left(t\right),\boldsymbol{\varphi}\right)_2$. By Schwartz inequality,

$$\mathbf{E}\int_0^t\int_{\mathbf{R}^2}\left|\zeta_\alpha u^i\left(s,x\right)\partial_i u^j\left(s,x\right)\boldsymbol{\varphi}^j\left(x\right)\right|dsdx\leq$$

$$\left(\int_0^t\int_{\mathbf{R}^2}\mathbf{E}\left|\zeta_\alpha u^i\left(s,x\right)\right|^2dsdx\right)^{1/2}\left(\int_0^t\int_{\mathbf{R}^2}\mathbf{E}\left|\partial_i u^j\left(s,x\right)\boldsymbol{\varphi}^j\left(x\right)\right|^2dsdx\right)^{1/2}<\infty.$$

Thus, by Fubini Theorem and Lemma 14, we have that

$$\mathbf{E}\left[\zeta_\alpha\left(t\right)\int_0^t\left(u^i\left(s\right)\partial_i\mathbf{u}\left(s\right),\boldsymbol{\varphi}\right)_2ds\right]=$$

$$\sum_{\alpha,p\in\mathcal{J}}\sum_{0\leq\beta\leq\alpha}\tfrac{1}{p!}\tbinom{\alpha}{\beta}\left(\hat{u}^i_{p+\beta}\left(t\right),\partial_i\hat{\mathbf{u}}_{p+\alpha-\beta}\left(t\right),\boldsymbol{\varphi}\right)_2. \tag{4.19}$$

It remains to evaluate

$$R\left(t\right)=\mathbf{E}\left[\boldsymbol{\zeta}_\alpha\left(t\right)\left(\mathbf{h}^i(t)\cdot\mathcal{G}(\sigma^j(t)\partial_j\mathbf{u}\left(t\right)+\mathbf{g}\left(t\right)),\boldsymbol{\varphi}\right)_{\mathbb{L}_2}\right].$$

To this end, we need the following simple result.

Lemma 16. *If* $\mathbf{v}\in L^2(\Omega,\mathcal{F}_s,\mathbf{P};\mathbb{L}_2)$, *then for all* $\alpha\in\mathcal{J}$, $\widehat{(\mathcal{G}\mathbf{v})}_\alpha=\mathcal{G}\hat{\mathbf{v}}_\alpha$ *and* $\widehat{(\mathcal{S}\mathbf{v})}_\alpha=\mathcal{S}\hat{\mathbf{v}}_\alpha$.

Proof. Since $\mathcal{S}\left(\mathbf{v}\right)=\mathbf{v}-\mathcal{G}\left(\mathbf{v}\right)$, it is sufficient to prove only the first equality. By Stein's Theorem, $\mathcal{G}\left(\mathbf{v}\right)$ is $\mathcal{F}_s-$measurable and $\left|\mathcal{G}\mathbf{v}\right|_{\mathbb{L}_2}\leq C\left|\mathbf{v}\right|_{\mathbb{L}_2}$, which yields that $\mathcal{G}\mathbf{v}\in L^2(\Omega,\mathcal{F}_s,\mathbf{P};\mathbb{L}_2)$. Thus by Fubini Theorem we have

$$\widehat{(\mathcal{G}\mathbf{v})}_\alpha=\mathbf{E}\left[\zeta_\alpha\nabla\int\Gamma_{x_i}(x-y)v^i(y)\,dy\right]=\nabla\int\Gamma_{x_i}(x-y)v^i_\alpha(y)\,dy=\mathcal{G}\hat{\mathbf{v}}_\alpha.$$

∎

It follows immediately from the lemma that

$$R\left(t,\boldsymbol{\varphi}\right)=\left(\mathbf{h}^i(t)\cdot\mathcal{G}(\sigma^j(t)\partial_j\hat{\mathbf{u}}_\alpha\left(t\right)+\mathbf{g}_\alpha\left(t\right)),\boldsymbol{\varphi}\right)_2.$$

This completes the proof of Theorem 3.

Now we shall derive another convenient representation for the term $\widehat{u^i\partial_i\mathbf{u}_\alpha}$.

For $\alpha,\beta\in\mathcal{J}$, define $\left|\alpha-\beta\right|=\left(\left|a_1-\beta_1\right|,\left|a_2-\beta_2\right|,...\right)$.

Definition 3. *We say that a triple of multiindices* (α,β,γ) *is complete, written* $(\alpha,\beta,\gamma)\in\mathcal{C}$, *if all the entries of the multiindex* $\alpha+\beta+\gamma$ *are even numbers and* $|\alpha-\beta|\leq\gamma\leq\alpha+\beta$.

Obviously, if (α,β,γ) is complete, then we have also that $|\alpha-\gamma|\leq\beta\leq\alpha+\gamma$, $|\gamma-\beta|\leq\alpha\leq\gamma+\beta$, and $\alpha+\beta-\gamma$, $\alpha-\beta+\gamma$, and $\beta+\gamma-\alpha$ are even multiindices.

It is readily checked that the following criterion holds:

Lemma 17. *A triple* (α,β,γ) *is complete if and only if* $\alpha+\beta+\gamma=2p$ *for some* $p\in\mathcal{J}$ *and* $p\leq\alpha\wedge\beta$.

For $(\alpha, \beta, \gamma) \in \mathcal{C}$, we define

$$\Phi(\alpha, \beta, \gamma) = \left(\left(\frac{\alpha - \beta + \gamma}{2} \right)! \left(\frac{\beta - \alpha + \gamma}{2} \right)! \left(\frac{\alpha + \beta - \gamma}{2} \right)! \right)^{-1}.$$

Obviously $\Phi(\alpha, \beta, \gamma)$ is invariant with respect to permutations of the arguments.

For $\alpha \in \mathcal{J}$, write $U^{\alpha} = \{\gamma, \beta \in \mathcal{J} : (\alpha, \beta, \gamma) \in \mathcal{C}\}$.

By Lemma 14,

$$\mathbf{E} u^i \partial_i u^j \zeta_\alpha = \sum_{\boldsymbol{\gamma}, \boldsymbol{\beta} \in \mathcal{J}} \hat{u}^i_\gamma \partial_i \hat{u}^j_\beta \sum_{p \le \gamma \wedge \beta} \left((\gamma - p)!\, (\beta - p)! p!\right)^{-1} \alpha! I_{(\alpha = \gamma + \beta - 2p)}. \tag{4.20}$$

Since $\gamma + \beta - \alpha = 2p$, then $\gamma + \beta + \alpha$ is also an even multiindex. Also, the inequality $p \le \gamma \wedge \beta$ implies $|\gamma - \beta| \le \alpha \le \gamma + \beta$. Thus (γ, β, α) is complete. Now, it follows from (4.20) that

$$\widehat{(u^i \partial_i \mathbf{u})}_\alpha = \sum_{\boldsymbol{\gamma}, \boldsymbol{\beta} \in U^\alpha} \alpha! \hat{u}^i_\gamma \partial_i \hat{\mathbf{u}}_\beta \Phi(\alpha, \beta, \gamma). \tag{4.21}$$

The propagators for advection type equations were studied in [33], [42], [45] (see also the references therein). Applications of Wiener chaos expansions to fluid mechanics have been sporadically discussed in the literature since 1960s. For example, the inertial range spectrum of low order Wiener Chaos truncations of a (random) Burgers equation were discussed in [10], [9], [48], [46]. There also exists a body of engineering literature on numerical aspects of Wiener Chaos approximations (see e.g. [32] , [23] and the references therein).

4.3. **Moments.** Making use of the Wiener chaos expansion [6] for a solution of the SNS (4.1), one can immediately compute the first two moments of the solution via the Hermite-Fourier coefficients given by equation (4.7) for the propagator. Indeed, let us assume that the assumptions of Theorem 3 hold. It was shown in the previous section that the Hermite-Fourier coefficients $\hat{\mathbf{u}}_\alpha(t)$ are $\mathbb{H}^1_p(\mathbf{R}^2) \bigcap \mathbb{H}^1_2(\mathbf{R}^2)$ −valued uniformly continuous functions of t. Owing to the imbedding $H_{1,p}(\mathbf{R}^2) \subset C^{1-2/p}(\mathbf{R}^2)$, we can interpret the Hermite-Fourier coefficients $\hat{\mathbf{u}}_\alpha(t, x)$ as continuous real functions on $\mathbf{R}^2 \times [0, T]$.

Since $\mathbf{E}\zeta_\alpha = 0$ for $\alpha \neq 0$ and $\mathbf{E}\zeta_{\mathbf{0}} = 1$ where $\mathbf{0}$ is the multiindex $\alpha \in \mathcal{J}$ such that $|\alpha| = 0$, we have that for all t, x,

$$\mathbf{E}\mathbf{u}(t, \mathbf{x}) = \hat{\mathbf{u}}_{\mathbf{0}}(t, \mathbf{x}).$$

By [6] and Parceval's identity, one has that for all $\mathbf{x}, \mathbf{y} \in \mathbf{R}^d$ and $t, s \in [0, T]$,

$$\mathbf{E}(\mathbf{u}(t, \mathbf{x}), \mathbf{u}(s, \mathbf{y})) = \sum_{\alpha \in \mathcal{J}} \frac{1}{\alpha!} (\hat{\mathbf{u}}_\alpha(t, \mathbf{x}), \hat{\mathbf{u}}_\alpha(s, \mathbf{y})).$$

Similarly, given the solution of the equation (4.7), the higher order moments of the solution to SNS equation (4.1) can be obtained by computing the moments of the Wick polynomials ζ_α.

Below we will derive some convenient formulas for these moments.

Let us consider the triple product $\zeta_\alpha \zeta_\beta \zeta_\gamma$. By (4.12)

$$\zeta_\alpha \zeta_\beta \zeta_\gamma = \sum_{p \le \alpha \wedge \beta} \binom{\alpha}{p} \binom{\beta}{p} p! \zeta_{\alpha + \beta - 2p} \zeta_\gamma. \tag{4.22}$$

It is readily checked that if f is a function on $\mathcal{J}$, then for $\alpha, \beta \in \mathcal{J}$,

$$\sum_{p\le\alpha\wedge\beta} f\left(\alpha+\beta-2p\right)\binom{\alpha}{p}\binom{\beta}{p}p! = \alpha!\beta!\sum_{r\in U^{\alpha,\beta}} f\left(r\right)\Phi\left(\alpha,\beta,r\right). \tag{4.23}$$

Therefore, from (4.22) , (4.23) , and (4.2) follows

$$\begin{gathered}\mathbf{E}\zeta_\alpha\zeta_\beta\zeta_\gamma = \alpha!\beta!\textstyle\sum_{r\in U^{\alpha,\beta}}\Phi\left(\alpha,\beta,r\right)\mathbf{E}\zeta_r\zeta_\gamma = \\ \alpha!\beta!\gamma!\textstyle\sum_{r\in U^{\alpha,\beta}}\Phi\left(\alpha,\beta,r\right)I_{(r=\gamma)} = \alpha!\beta!\gamma!\Phi\left(\alpha,\beta,\gamma\right)I_{\{(\alpha,\beta,\gamma)\in\mathcal{C}\}}.\end{gathered} \tag{4.24}$$

By induction, it is easy to verify that

$$\begin{gathered}\mathbf{E}\Pi_{i=1}^{m+1}\zeta_{\alpha^i} = \Pi_{i=0}^{m-3}r^i!\alpha^{m-i}!\textstyle\sum_{r^{i+1}\in U(\alpha^{m-i},r^i)}\Phi\left(\alpha^{m-i},r^i,r^{i+1}\right)\times \\ r^{m-2}!\alpha^2!\alpha^1!\Phi\left(\alpha^2,r^{m-2},\alpha^1\right)I_{\{(\alpha^2,r^{m-2},\alpha^1)\in\mathcal{C}\}}\end{gathered} \tag{4.25}$$

where $r^0=\alpha^{m+1}$(cf. [34] Thm. 5.3).

For example,

$$\mathbf{E}\zeta_\alpha\zeta_\beta\zeta_\gamma\zeta_\kappa = \alpha!\beta!\gamma!\kappa!\textstyle\sum_{r\in U(\alpha,\beta)}\Phi\left(\alpha,\beta,r\right)r!\Phi\left(r,\gamma,\kappa\right)I_{\{(r,\gamma,\kappa)\in\mathcal{C}\}}.$$

Formula (4.25) allows to compute pseudo-moments of orders higher then 2. Let v be an $\mathcal{F}_T-$measurable random variable and $\mathbf{E}v^3<\infty$.

Obviously, the set $\{\zeta_\alpha,\alpha\in J\}$ is total in all $L_p(\Omega)$. Given $v\in L_p(\Omega)$, there is a sequence of finite linear combinations $v^m=\sum_\alpha c_\alpha^m\xi_\alpha$ so that $\mathbf{E}|v-v^m|^p\to0$ as $m\to\infty$. If $p=3$, then, of course,

$$\mathbf{E}(v^m)^3 = \sum_{(\alpha,\beta,\gamma)\in\mathcal{C}}\hat{v}_\alpha^m\hat{v}_\beta^m\hat{v}_\gamma^m\Phi\left(\alpha,\beta,\gamma\right)\to\mathbf{E}v^3.$$

It is readily checked that $\hat{v}_\alpha^m\to\hat{v}_\alpha$ for all α. Therefore, we may define the third pseudo-moment $<v^3>$ by the formula

$$<v^3>=\sum_{(\alpha,\beta,\gamma)\in\mathcal{C}}\hat{v}_\alpha\hat{v}_\beta\hat{v}_\gamma\Phi\left(\alpha,\beta,\gamma\right).$$

Formula

$$<v^4>=\sum_{\alpha,\beta,\gamma,\kappa}\hat{v}_\alpha\hat{v}_\beta\hat{v}_\gamma\hat{v}_\kappa\sum_{r\in U(\alpha,\beta)}\Phi\left(\alpha,\beta,r\right)r!\Phi\left(r,\gamma,\kappa\right)I_{\{(r,\gamma,\kappa)\in\mathcal{C}\}} \tag{4.26}$$

as well as similar formulas for higher pseudo-moments could be proved by similar arguments.

Of course, the pseudo-moments $<v^p>$ coincide with the respective moments for $p=1,2$. However, if $p>2$ the relation between the moments and the related pseudo-moments is an open problem.

5. APPENDIX

5.1. **Non-negative semimartingales.** We will need also some estimates of non-negative semimartingales.

Lemma 18. *Let Z_t be a non-negative semimartingale such that*

$$Z_t = Z_0 + \int_0^t a_s\,ds + \int_0^t b_s \cdot dW_s.$$

Assume that there are non-negative measurable functions c_s, f_s, g_s, and a number $\delta \geq 0$ such that so that for any ε,

$$a_s \leq (-\delta + \varepsilon\delta)c_s + C_\varepsilon(Z_s + f_s),\ |b_s|_Y \leq \delta\varepsilon(c_s Z_s)^{1/2} + C_\varepsilon(Z_s + Z_s^{1-1/p} g_s^{1/p}),$$

where C_ε is a constant depending on ε.

Then for every $T > 0$, there is a constant $C = C(T)$ such that for all stopping times $\tau \leq T$

$$\mathbf{E}[\sup_{s\leq\tau}|Z_s| + (\delta/2)\int_0^\tau c_s\,ds] \leq C\mathbf{E}[Z_0 + \int_0^\tau (f_s + g_s)\,ds].$$

Proof. Let $s < t \leq T, t - s \leq 1$ and $\tilde\tau$ be a stopping time such that $\sup_{s\leq\tilde\tau} Z_s$ is bounded and

$$\mathbf{E}\int_0^{\tilde\tau}(f_s + g_s)\,ds < \infty.$$

Fix an arbitrary stopping time τ. Let $\bar\tau = \tilde\tau \wedge \tau$. Then by Burkhoelder's inequality

$$\mathbf{E}[\sup_{s\leq r\leq t} Z_{r\wedge\bar\tau} + \delta\int_{s\wedge\bar\tau}^{t\wedge\bar\tau} c_r\,dr] \leq \mathbf{E}Z_{s\wedge\bar\tau} + \mathbf{E}[\delta\varepsilon\int_{s\wedge\bar\tau}^{t\wedge\bar\tau} c_r\,dr + C_{\varepsilon,\delta}\sup_{s\leq r\leq t} Z_{r\wedge\bar\tau}(t-s) +$$

$$C_{\varepsilon,\delta}\int_{s\wedge\bar\tau}^{t\wedge\bar\tau} f_r\,dr + N(\int_{s\wedge\bar\tau}^{t\wedge\bar\tau}(\varepsilon^2\delta^2 Z_r c_r + C_{\varepsilon,\delta}^2 Z_r^2 + C_{\varepsilon,\delta}^2 Z_r^{2(1-1/p)} g_r^{2/p})\,dr)^{1/2}]$$

Obviously, for every $\varepsilon > 0$, there is a constant C_ε independent of T such that

$$(\int_{s\wedge\bar\tau}^{t\wedge\bar\tau} Z_r^{2(1-1/p)} g_s^{2/p}\,ds)^{1/2} \leq \sup_{s\leq r\leq t} Z_{r\wedge\bar\tau}^{1-1/p}(\int_{s\wedge\bar\tau}^{t\wedge\bar\tau} g_s^{2/p}\,ds)^{1/2}$$

$$\leq \varepsilon\sup_{s\leq r\leq t} Z_{r\wedge\bar\tau} + C_\varepsilon(\int_{s\wedge\bar\tau}^{t\wedge\bar\tau} g_s^{2/p}\,ds)^{p/2}.$$

Hence,

$$\mathbf{E}[\sup_{s\leq r\leq t} Z_{r\wedge\bar\tau} + \delta\int_{s\wedge\bar\tau}^{t\wedge\bar\tau} c_r\,dr] \leq \mathbf{E}Z_{s\wedge\bar\tau} + \mathbf{E}\{\delta\varepsilon\int_{s\wedge\bar\tau}^{t\wedge\bar\tau} c_r\,dr +$$

$$(N+1)\,C_{\varepsilon,\delta}\sup_{s\leq r\leq t} Z_{r\wedge\bar\tau}(t-s)^{1/2} + \tilde C_{\varepsilon,\delta}[\int_{s\wedge\bar\tau}^{t\wedge\bar\tau} f_r\,dr + (\int_{s\wedge\bar\tau}^{t\wedge\bar\tau} g_r^{2/p}\,dr)^{p/2}] +$$

$$\varepsilon\sup_{s\leq r\leq t} Z_{r\wedge\bar\tau} + 2^{-1}N\varepsilon\delta\sup_{s\leq r\leq t} Z_{r\wedge\bar\tau} + 2^{-1}N\varepsilon\delta\int_{s\wedge\bar\tau}^{t\wedge\bar\tau} c_r\,dr\}.$$

Let us take ε so that

$$\left(1 - \varepsilon\left(1 + 2^{-1}N\delta\right)\right) / \left(1 - \varepsilon\left(1 + 2^{-1}N\delta\right)\right) = 1/4.$$

Then, there is a constant $C = C(T)$ such that

$$\mathbf{E}\sup_{s\le r\le t} Z_{r\wedge\bar\tau} + (1/4)\delta\int_s^t c_r\,dr \le C\mathbf{E}[Z_{s\wedge\bar\tau} + \sup_{s\le r\le t} Z_{r\wedge\bar\tau}(t-s)^{1/2})$$

$$+\int_{s\wedge\bar\tau}^{t\wedge\bar\tau} f_r\,dr + (\int_{s\wedge\bar\tau}^{t\wedge\bar\tau} g_r^{2/p}\,dr)^{p/2}].$$

By choosing $\kappa = t - s$ small enough, we obtain

$$\mathbf{E}\sup_{s\le r\le t} Z_{r\wedge\bar\tau} + (1/2)\delta\int_{s\wedge\bar\tau}^{t\wedge\bar\tau} c_r\,dr \le C\mathbf{E}[Z_{s\wedge\bar\tau} + \int_{s\wedge\bar\tau}^{t\wedge\bar\tau}(f_s + g_s)\,ds]. \tag{5.1}$$

To prove a similar estimate for $s = 0$ and an arbitrary $t \le T$, we apply (5.1) estimate successively on the intervals, $[0,\kappa]$, $[\kappa, 2\kappa]$, ... Now, the statement follows. ∎

5.2. **Convergence lemma.** Let B be a Banach space with a norm $|\cdot|_B$. Let X_n be a sequence of B-valued continuous processes defined on $[0,\zeta_n)$, where $\zeta_n = \zeta_n(X_n)$ is such that $\mathbf{P}$-a.s. $\zeta_n > 0$ and

$$\limsup_{t\uparrow\zeta_n} |X_n(t)|_B = \infty$$

on $\{\zeta_n < \infty\}$. For $M > 0, T > 0, n, n'$, let $\mathcal{T}_n^{M,T}$ be the set of all stopping times $\tau \le T$ such that $\sup_{s\le\tau}|X_n(s)|_B \le M + |X_n(0)|_B$, $\mathcal{T}_{n,n'}^{M,T} = \mathcal{T}_n^{M,T}\cap\mathcal{T}_{n'}^{M,T}$.

Lemma 19. *a) Let $\mathbf{P}$-a.s. $\zeta_n = \infty$ for all n. Assume that for each M, T*

$$\lim_{n\to\infty}\sup_{n'\ge n,\tau\in\mathcal{T}_{n,n'}^{M,T}}\mathbf{E}\sup_{s\le\tau}|X_n(s) - X_{n'}(s)|_B = 0,$$

$$\sup_n \mathbf{E}\sup_{s\le T}|X_n(s)|_B < \infty.$$

Then there is a B-valued continuous process X and a subsequence n_k such that $\mathbf{P}$-a.s. for each T

$$\sup_{s\le T}|X_{n_k}(s) - X(s)|_B \to 0.$$

b) Assume that for some $M_0 > 1, T_0 > 0$

$$\lim_{n\to\infty}\sup_{n'\ge n,\tau\in\mathcal{T}_{n,n'}^{M_0,T_0}}\mathbf{E}\sup_{s\le\tau}|X_n(s) - X_{n'}(s)|_B = 0,$$

and

$$\lim_{T\to 0}\sup_{n,\tau\in\mathcal{T}_n^{M_0,T_0}}\mathbf{P}(\sup_{s\le\tau\wedge T}|X_n(s)|_B > |X_n(0)|_B + M_0 - 1) = 0.$$

Then there is a bounded stopping time τ such that $\mathbf{P}(\tau > 0) = 1$ and a B-valued continuous process X on $[0,\tau]$ and a subsequence n_k such that $\mathbf{P}$-a.s.

$$\sup_{s\le\tau}|X_{n_k}(s) - X(s)|_B \to 0.$$

Moreover, if $\sup_n \mathbf{E}|X_n(0)|_B^p < \infty, p \ge 1$, then $\mathbf{E}\sup_{t\le\tau}|X(t)|_B^p < \infty$.

Proof. a) Obviously, there are $n_k \uparrow \infty, T_k \uparrow \infty, M_k \uparrow \infty$ such that

$$\sup_{n' \geq n_k} \mathbf{E} \sup_{s \leq \tau \in \mathcal{T}^k_{n,n'}} |X_{n_k}(s) - X_{n'}(s)|_B \leq 2^{-2k},$$

where $\mathcal{T}^k_{n,n'} = \mathcal{T}^{M_k,T_k}_{n,n'}$. Fix $T > 0$ and $M > 0$. Let $\tau_k = \inf(t : |X_{n_k}(t)|_B > |X_{n_k}(0)|_B + M + 2^{-k}) \wedge T$. Then, for k so that $M_k > M + 2^{-k}$, $T_k > T$, we have

$$\mathbf{P}(\sup_{s \leq \tau_k \wedge \tau_{k+1}} |X_{n_k}(s) - X_{n_{k+1}}(s)|_B > 2^{-k}/4)$$

$$\leq 4 \cdot 2^k \mathbf{E} \sup_{s \leq \tau_k \wedge \tau_{k+1}} |X_{n_k}(s) - X_{n_{k+1}}(s)|_B \leq 4 \cdot 2^{-k}.$$

By Borelli-Cantelli lemma,

$$\sup_{s \leq \tau_k \wedge \tau_{k+1}} |X_{n_k}(s) - X_{n_{k+1}}(s)|_B \leq 2^{-k}/4$$

or $\tau_{k+1} \leq \tau_k$ for sufficiently large k. Let $\tau = \lim_k \tau_k$. Then $X_{n_k}(t)$ converges to some process $X(t)$ on $[0, \tau]$. Also,

$$\mathbf{P}(\tau < T) = \lim \mathbf{P}(\tau_k < T) \leq M^{-1} \mathbf{E} \sup_{s \leq T} |X_{n_k}(s)|_B.$$

b) Obviously, there is $n_k \uparrow \infty$ such that

$$\sup_{n' \geq n_k} \mathbf{E} \sup_{s \leq \tau \in \mathcal{T}_{n,n'}} |X_{n_k}(s) - X_{n'}(s)|_B \leq 2^{-2k},$$

where $\mathcal{T}_{n,n'} = \mathcal{T}^{M_0,T_0}_{n,n'}$. Let $\tau_k = \inf(t : |X_{n_k}(t)|_B > |X_{n_k}(0)|_B + M_0 - 1 + 2^{-k}) \wedge T_0$. Then

$$\mathbf{P}(\sup_{s \leq \tau_k \wedge \tau_{k+1}} |X_{n_k}(s) - X_{n_{k+1}}(s)|_B > 2^{-k}/4)$$

$$\leq 4 \cdot 2^k \mathbf{E} \sup_{s \leq \tau_k \wedge \tau_{k+1}} |X_{n_k}(s) - X_{n_{k+1}}(s)|_B \leq 4 \cdot 2^{-k}.$$

By Borelli-Cantelli lemma,

$$\sup_{s \leq \tau_k \wedge \tau_{k+1}} |X_{n_k}(s) - X_{n_{k+1}}(s)|_B \leq 2^{-k}/4$$

or $\tau_{k+1} \leq \tau_k$ for sufficiently large k. Let $\tau = \lim_k \tau_k$. Then $X_{n_k}(s)$ converges to some process $X(s)$ on $[0, \tau]$. Also,

$$\mathbf{P}(\tau < \varepsilon) = \lim \mathbf{P}(\tau_k < \varepsilon) \leq \lim \sup_k \mathbf{P}(\sup_{s \leq \tau_k \wedge \varepsilon} |X_{n_k}(s)|_B > |X_{n_k}(0)|_B + M_0 - 1) \to 0,$$

as $\varepsilon \to 0$, i. e. $\mathbf{P}(\tau = 0) = 0$.

Since

$$\sup_k \mathbf{E} \sup_{t \leq \tau} |X_{n_k}(t)|^p_B < \infty,$$

if $\sup_n \mathbf{E}|X_n(0)|^p_B < \infty$, the last assertion follows by Fatou lemma. ∎

5.3. **Estimates of gradient projection.** In this section, for the sake of convenience, we summarize some basic estimates for gradient projections proved in [37].

Lemma 20. *(see Lemma 2.13 in* [37]*) Assume* $\mathbf{v} \in \mathbb{H}_p^{s+1}(Y), p \in (1, \infty)$. *Then*

$$\mathcal{G}(\partial_l \mathbf{v}) = \partial_l \mathcal{G}(\mathbf{v}) = -\,(1-\Delta)^{-s/2}\mathbf{R}R_l((1-\Delta)^{s/2} div\,\mathbf{v}). \tag{5.2}$$

There is a constant C *such that for all* $\mathbf{v} \in \mathbb{H}_p^{s+1}(Y)$,

$$||\partial\mathcal{G}(\mathbf{v})||_{s,p} \leq C||div\,\mathbf{v}||_{s,p},$$

and for all $\mathbf{v} \in \mathbb{H}_p^{s}(Y)$,

$$||\mathcal{G}(\mathbf{v})||_{s,p} \leq C||div\,\mathbf{v}||_{s-1,p} + ||\mathbf{v}||_{s-1,p}.$$

We need L_p-estimates of the function $\mathcal{G}(\mathbf{h})$ where $\mathbf{h} = c^j\partial_j\mathbf{v}$.

Lemma 21. *(see Lemma 2.14 in* [37]*) Let* $\mathbf{h} = c^j(x)\partial_j\mathbf{v}(x)$, *where* $c = (c^j)$ *is a measurable d-vector of Hilbert space* Y*-valued functions,* $\mathbf{v} \in \mathbb{H}_p^{s+1}, div\,\mathbf{v} = 0$, $\varepsilon \in (0,1)$. *Assume*

$$||c||_{B^{|s|}} < \infty, \text{ if } s \geq 1,$$

$$||c||_{B^1} < \infty, \text{ if } s \in (-1,1),$$

$$||c||_{B^{-s+\varepsilon}} < \infty, \text{ if } s \leq -1.$$

Then

$$||\mathcal{G}(\mathbf{h})||_{s,p} \leq \begin{cases} C(||\partial_l c^j\partial_j\mathbf{v}||_{s-1,p} + ||c^j\partial_j\mathbf{v}||_{s-1,p}) & \text{if } s > 0, \\ C(||\partial_l c^j v^l||_{s,p} + ||(div\,c)\,\mathbf{v}||_{s,p}], & \text{if } s \leq 0. \end{cases}$$

Also, we need L_p-estimates of the function $\mathcal{G}(\mathbf{h})$ where $\mathbf{h} = \partial_i(c^{ij}(x)\partial_j\mathbf{v})$.

Corollary 6. *(see Corollary 2.15 in* [37]*) Let* $\mathbf{h} = \partial_i(c^{ij}(x)\partial_j\mathbf{v})$, *where* $c = (c^{ij})$ *is a measurable function,* $\mathbf{v} \in \mathbb{H}_p^{s+1}, div\,\mathbf{v} = 0$, $\varepsilon \in (0,1)$. *Assume*

$$|c|_{B^{|s|}} < \infty, \text{ if } s \geq 1,$$

$$|c|_{B^1} < \infty, \text{ if } s \in (-1,1),$$

$$|c|_{B^{-s+\varepsilon}} < \infty, \text{ if } s \leq -1.$$

Then

$$|\mathcal{G}(\mathbf{h})|_{s-1,p} \leq \begin{cases} C(|\partial_l c^{ij}\partial_j v^l|_{s-1,p} + |c^{ij}\partial_j v^l|_{s-1,p}) & \text{if } s > 0, \\ C(|\partial_l c^{ij} v^j|_{s,p} + |\partial_j c^{ij}\,\mathbf{v}|_{s,p}], & \text{if } s \leq 0. \end{cases}$$

5.4. **Biot-Savart law in $\mathbf{R}^d$.** Biot-Savart law is usually discussed only in dimensions $d = 2, 3$. In this subsection we introduce a slightly more general construction for any d.

Definition 4. *a) Given two vectors* $\mathbf{a} = (a^1, \ldots, a^d), \mathbf{b} = (b^1, \ldots, b^d)$ *in* $\mathbf{R}^d$, *we define their product*

$$\mathbf{a} \times \mathbf{b} = \left(\varepsilon_{lk}(a^k b^l - a^l b^k)\right)_{1 \leq k < l \leq d} \in \mathbf{R}^{d(d-1)/2},$$

where $\varepsilon_{lk} = (-1)^{l+k-1}$;

(In standard notations, we could use also a matrix $\sqrt{2}\mathbf{a}\wedge\mathbf{b} = \left(a^kb^l - a^lb^k\right)_{1\le k,l\le d}$*).*

b) Given a vector field $\mathbf{v}(x) = (v^1(x),\dots,v^d(x))$*, we define a new vector field*

$$curl\mathbf{v} = \boldsymbol{\nabla}\times\mathbf{v} = \left(\varepsilon_{lk}(\partial_k v^l - \partial_l v^k)\right)_{1\le k<l\le d} \in \mathbf{R}^{d(d-1)/2}.$$

Remark 9. *a) Given a scalar function* a*, we have*

$$curl\,(a\mathbf{v}) = a\,curl\,(\mathbf{v}) + (\nabla a)\times\mathbf{v}. \tag{5.3}$$

b) If $\mathbf{v} = \boldsymbol{\nabla}p$ *(*p *is a scalar function), then* $curl\mathbf{v} = \boldsymbol{\nabla}\times\mathbf{v} = \mathbf{0}$.

Proposition 8. *For each* $\mathbf{v}\in\mathbb{H}_p^1$

$$\partial_m\mathcal{S}(\mathbf{v}) = -\sum_j R_mR_j(\partial_j\mathbf{v} - \boldsymbol{\nabla}v^j).$$

There is a constant C *so that for all* $\mathbf{v}\in\mathbb{H}_p^1$

$$|\partial\mathcal{S}(\mathbf{v})|_p \le C|curl\mathbf{v}|_p.$$

In general, there is a constant C *so that for all* $\mathbf{v}\in\mathbb{H}_p^{s+1}$,

$$|\partial\mathcal{S}(\mathbf{v})|_{s,p} = |\mathcal{S}(\partial\mathbf{v})|_{s,p} \le C|curl\mathbf{v}|_{s,p},$$

$$|\mathcal{S}(\mathbf{v})|_{s+1,p} \le C(|curl\mathbf{v}|_{s,p} + |\mathbf{v}|_{s,p}),$$

or for all $\mathbf{v}\in\mathbb{H}_p^s$

$$|\mathcal{S}(\mathbf{v})|_{s,p} \le C(|curl\mathbf{v}|_{s-1,p} + |\mathbf{v}|_{s-1,p}).$$

Proof. Indeed, considering Fourier transforms, we easily find that for all $\mathbf{v}\in\mathbb{C}_0^\infty$,

$$\mathcal{S}(\mathbf{v}) = \mathbf{v} - \mathcal{G}(\mathbf{v}). = -\sum_j (R_jR_j\mathbf{v} - R_jRv^j) = -R(R\wedge\mathbf{v}),$$

$$\partial_m\mathcal{S}(\mathbf{v}) = -\sum_j R_mR_j(\partial_j\mathbf{v} - \boldsymbol{\nabla}v^j).$$

Also,

$$\partial_mJ_s\mathcal{S}(\mathbf{v}) = -\sum_j R_mR_j(\partial_jJ_s\mathbf{v} - \boldsymbol{\nabla}J_sv^j)$$

$$= -\sum_j R_mR_jJ_s(\partial_j\mathbf{v} - \boldsymbol{\nabla}v^j),$$

where $J_s = (1-\Delta)^{s/2}$. Since Riesz transform is bounded in L_p and

$$|\mathbf{v}|_{s,p} + |\partial\mathbf{v}|_{s,p} \sim |\mathbf{v}|_{s+1,p},$$

the statement follows. ∎

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