Exit Problems for system with Perturbations Approximating White Noise

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Consider a system

$$\dot{X}_t = b(X_t), \;\; X_0 = x \in R^n.$$

where $b(x)$ is a smooth vector field in R^n which has an asymptotically stable equilibrium at a point $O \in R^n$ such that : $\lim_{t\to\infty} X_t = O$ for each trajectory $\dot{X}_t = b(X_t), X_0 = x \in R^n$.

Now consider the system with a small perturbation :

$$\dot{X}_t^\varepsilon = b(X_t^\varepsilon) + \varepsilon \dot{W}_t,\; X_0^\varepsilon = x \in R^n, 0 < \varepsilon \ll 1. \tag{1}$$

where $\dot{W}_t$ is a standard n-dimensional white noise.

let $D \subset R^n$ be a bounded domain with smooth boundary ∂D. The bounded domain D is attracted to $O \in D$. Let $b(x) \cdot n(x)|_{x\in\partial D} < 0$, where $n(x)$ is the exterior normal to ∂D. Let the initial point $X_0^\varepsilon = x \in D$.

Denote by $\tau^\varepsilon = \min\{t : X_t^\varepsilon \notin D\}$ the first exit time from D for the process X_t^ε. It is well know that X_t^ε is a Markov Process. The exit problem for the perturbed system X_t^ε , which describes the asymptotic behaviour of τ^ε and $X_{\tau^\varepsilon}^\varepsilon$, is studied in [1]. The main result is given in terms of action functionals [1]. The meaning of the action functional $\varepsilon^{-1} S_{0T}(\varphi)$ is that $exp\{\varepsilon^{-1} S_{0T}(\varphi)\}$ is, roughly speaking, the main term of the probability that $X_t^\varepsilon, 0 \le t \le T$, belong to a small neighborhood of a function $\varphi : [0, T] \to R^n$ when $\varepsilon \downarrow 0$. Under the uniform topology in the space C_{0T} of continuous function $[0, T] \to R^n$, the action functional for the family X_t^ε when $\varepsilon \downarrow 0$, in the space of C_{0T}, is equal $\frac{1}{\varepsilon^2} S_{0T}(\varphi)$, where

$$S_{oT}(\varphi) = \begin{cases} \frac{1}{2}\int_0^T |\dot{\varphi}_s - b(\varphi_s)|^2 ds, \; if \varphi \; is \; abs. \; cont., \; \varphi_0 = X_0^\varepsilon = x, \\ +\infty, \;\; for \; the \; rest \; of \; C_{0T}. \end{cases}$$

If the perturbation is not white noise, the action function, in general, will be different. Furthermore, the Markovian property of the process X_t^ε, as a rule, does not hold. We will consider the action functions and the exit problem for a perturbed system where the perturbation differs from white noise, but is, in a sense, close to it.

Three cases are considered.

Case 1: Let W_t be replaced by a random walk ξ_t^δ with time step δ and space step h, where h is a random variable and has the Gaussian distribution $N(0,\delta)$, $\delta = Var(h)$. ξ_t^δ converges to W_t when $\delta \downarrow 0$ and it can be shown that if $\mu^2 := \frac{\delta}{\varepsilon} = O(1)$, the family $\varepsilon\xi_t^\delta$ has the same action functional as the family εW_t as $\varepsilon \downarrow 0$ under the uniform topology.

Case 2: For fixed n, let W_t be replaced by a random walk $\xi_t^{\delta,n}$ with time step δ and space step $0, \pm\sqrt{\delta}, \ldots, \pm n\sqrt{\delta}$, such that $\xi_{t+\delta}^{\delta,n} - \xi_t^{\delta,n} = \pm i\sqrt{\delta}$ with probability $\frac{1}{2}P_i, i = 1, \ldots, n$ and $P_0 + P_1 + \ldots + P_n = 1$. $\xi_t^{\delta,n}$ converges to W_t when $\delta \downarrow 0$ and $\sum_{i=1}^n i^2 P_i = 1$. However, the large deviations in this case are different from those in the case of white noise. It turns out that if $\mu^2 := \frac{\delta}{\varepsilon} = O(1)$ and $P_0, \ldots, P_n$ satisfies $\sum_{i=1}^n P_i i^{2k} = (2k-1)!!,\ k = 1, \ldots, n$, the action functional for the family $\varepsilon\xi_t^{\delta,n}$ in the space C_{0T} when $\varepsilon \downarrow 0$ is $\varepsilon^{-2} S_{oT}^\mu(\varphi)$ where

$$S_{oT}^\mu(\varphi) = \begin{cases} \frac{1}{2}\int_0^T (\dot\varphi_s)^2 ds + O(\mu^{2n}), & if \int_0^T |\dot\varphi_s|^2 < \infty. \\ +\infty, & otherwise. \end{cases}$$

Based on the action functional and the Markovian property of the random walk, exit problem for case1 and 2 can be described following the same idea as Chapter4 [1].

Case 3: Let $\dot W_t$ be replaced by η_t^δ where η_t^δ is a n-dimensional mean-zero Gaussian process with the correlation matrix such that $E\eta_s^\delta\eta_t^\delta = \frac{1}{\delta^n}\rho(\frac{t-s}{\delta})$ where $\rho(t) = 0$ when $|t| > 1$. After the replacement, equation (**??**) becomes $\dot X_t^{\delta,\varepsilon} = b(X_t^{\delta,\varepsilon}) + \varepsilon\eta_t^\delta,\ X_0^\varepsilon = x \in R^n, 0 < \varepsilon \ll 1$. The action functional for the family $\varepsilon X_t^{\delta,\varepsilon}$ will be discussed as well as the exit problem for the system $X_t^{\delta,\varepsilon}$ when both δ and ε tend to 0. Notice that the process X_t^ε is not Markovian any more.

References

[1] M. Freidlin and A. Wentzell, *Random Perturbatons of Dynamical Systems.* , 2nd Edition, Springer, 1998.