Generalized Finite Element Method

Uday Banerjee

Dept. of Mathematics
Syracuse University

IMA
University of Minnesota
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Outline

- Introduction
- Description of GFEM
- Approximation results
- Examples of local approximation spaces based on available information
- “Computing” the local approximation space
- Condition number
- Open problems

Generalized Finite Element Method (GFEM):

The GFEM is a flexible *Galerkin method*. The main attributes of GFEM are:

• It allows to incorporate special "features" of the unknown solution into the *trial space*, based on the available information. If such information is not available, it could be "extracted" by solving certain local problems. Thus, unlike FEM, the functions in trial space of GFEM may not be piecewise polynomials.

• Trial spaces consisting of smooth functions (e.g., a subspace of $H^2(\Omega)$) can be constructed easily; they can be used to approximate solutions of higher order problems.

• A simple mesh can be used; the mesh does not have to conform to the geometry of the underlying domain. *But this comes at a cost* - we will discuss it later.

Contributions in various aspects of GFEM

• In 1983, Babuška and Osborn (SINUM) proposed a *Generalized Finite Element method*, where "the shape of trial functions is governed by the differential equation." Some of the ideas were related to Ritz method based on *L*-splines.

• In 1994, Babuška, Caloz, and Osborn (SINUM) proposed a *Special Finite Element Method-III*, where the local approximations, incorporating the "features" of the unknown solution (uni-directional composite), were obtained. The local approximations were then “glued together” using a particular Partition of Unity (PU).

• These ideas of using local approximations and a Partition of Unity were later developed, elaborated, and formalized by Babuska and Melenk in 1996-97 (CMAME & IJNME); the method was called the *Partition of Unity Method*.

Contributions in various aspects of GFEM

• In a series of papers ('00,'02,'09), Griebel & Schweitzer addressed several implementational issues, i.e., efficient cover construction, multilevel solver, h-p adaptivity and parallelization.

• Babuška, B., Osborn investigated the superconvergence in GFEM ('07).

• A lot of work has been done on the implementation in the engineering community: Oden & Duarte (*h-p Clouds*); Belytschko, Dolbow, Moes, Sukumar, Fries, Bordas .... (*XFEM*) especially for crack propagation, Bathe & De (*Method of Spheres*); Babuška, Strouboulis, Copps, Zhang (*GFEM*).

• Similar ideas, but not the same - Hou, Enquist, E., Effendiev (*Multiscale FEM*)

The idea is not new!!

The idea of augmenting the standard FE space with non-polynomial functions is old.

• Augmenting the FE space with singular functions can be found in “Strang-Fix” book.

• Fix, Gulati, Wakoff (’73) showed that using singular functions yielded much better approximation (of singular solutions) compared to FEM based on higher order polynomials. Similar observations were made also for the approximation of eigenvalue problems (with singular eigenfunctions). But they also reported that the condition number increases.

• The idea was also investigated by Byskov (’70, ’73); Rao, Raju, Murty (’71), Yamamoto, Tokuda (’73);,Benzley (’74); Blum, Dobrowolski (’82)

• Use of singular solutions, in the context of integral equations, were investigated by Wendland, Stephan, Hsiao (’79); Stephan, Wendland (’83).

Al Schatz

DESCRIPTION OF GFEM

Model Problem

- Let $\Omega \subset \mathbb{R}^2$ with piecewise smooth boundary $\Gamma = \Gamma_D \cup \Gamma_N$. Ω may have reentrant corners, voids, or cracks.

- Let $H_D^1(\Omega) = \{v \in H^1(\Omega) \,:\, v = 0 \text{ on } \Gamma_D\}$.

$$\text{Find } u \in H_D^1(\Omega) \text{ such that}$$
$$B(u,v) = F(v), \quad \forall\, v \in H_D^1(\Omega)$$

$$\text{where } B(u,v) \equiv \int_\Omega a \nabla u \cdot \nabla v \, dx \; ; \quad F(v) \equiv \int_\Omega fv \, dx + \int_{\Gamma_N} gv \, ds$$

- We assume $a(x) \in L^\infty(\Omega)$ and $0 < a_0 \le a(x) \le a_1$. Note that $a(x)$ could also be replaced by matrix $A(x)$.

If $\Gamma_D = \emptyset$, then f, g must satisfy the compatibility condition $F(1) = 0$.

The trial space for the GFEM

Main components:

1. Patches: Let $\{\omega_i\}_{i=0}^{N}$ be subdomains of Ω, with diam$(\omega_i) = \mathcal{O}(h)$, such that $\bigcup_{i=0}^{N} \omega_i = \Omega$.

Any $x \in \Omega$ belongs to at most κ sets ω_i.

2. Partition of Unity: Let $\{\phi_i\}_{i=0}^{N}$ be functions in $C^0(\Omega)$, such that

$$\begin{aligned} \phi_i(x) &= 0, \quad \forall\, x \in \Omega \backslash \omega_i \\ \sum_{i=0}^{N} \phi_i(x) &= 1, \quad \forall\, x \in \Omega \\ \|\nabla \phi_i(x)\|_{L^\infty(\Omega)} &\leq \frac{C}{h}. \end{aligned}$$

$\{\phi_i\}_{i=0}^{N}$ is a partition of unity subordinate to the patches $\{\omega_j\}_{i=0}^{N}$.

The trial space of GFEM

Main components:

3. Local approximation spaces: To each patch ω_i, we associate a $(m_i + 1)$-dimensional space V_i of functions defined on $\bar{\omega}_i$ as

$$V_i = \text{span}\{\xi_{ij}\}_{j=0}^{m_i}; \quad \xi_{ij} \in H^1(\omega_i) \cap C(\bar{\omega}_i),$$

• We require that V_i s contain *constants* when $\omega_i \cap \Gamma_D = \emptyset$. The functions in V_i, for $\omega_i \cap \Gamma_D \neq \emptyset$, must satisfy the essential boundary condition.

• *V_i's are chosen carefully based on the available information on the unknown solution u; they mimic u locally in ω_i ensuring good local approximation. ξ_{ij} are also called **enrichments***.

For example, if u is smooth in ω_i, then we choose $V_i = \mathcal{P}^{p_i}(\omega_i)$.

But if u is not smooth in ω_i, polynomials may not be the best choice for approximation, and V_i may contain additional "suitable" (non-polynomial) functions.

Main components:

4. Trial space:

$$\mathcal{S} = \sum_{i=0}^{N} \phi_i V_i = \text{span}\{\phi_i \xi_{ij}\},\ 1 \leq j \leq m_i,\ 0 \leq i \leq N$$

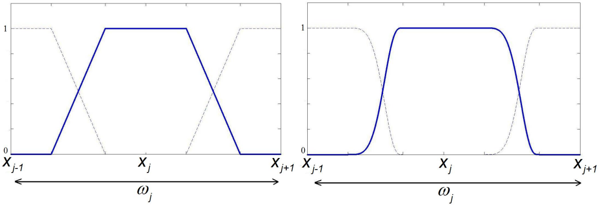
Since $\xi_{ij} \in H^1(\omega_i) \cap C(\bar{\omega}_i)$ and the PU ϕ_j has compact support, we have $\mathcal{S} \subset H^1(\Omega)$

• In certain applications, e.g., in crack problems, we will allow V_i to include discontinuous functions

Partition of Unity:

Selection of the Partition of Unity is flexible.

• ‘*“flat-top” functions:*



• *Standard hat functions*; in fact, they are used extensively in practice. The patch ω_i is the standard FE star.

Partition of Unity:

• *Shepard functions:* Define $\tilde{\phi}_i(x)$ such that $\tilde{\phi}_i(x) = 0, \quad \text{for } x \in \Omega \backslash \omega_j$, and $\|D\phi_i\|_{L^\infty(\omega_i)} \leq C[\text{diam}\omega_i]^{-1}$. Set

$$\phi_i(x) = \frac{\tilde{\phi}_i(x)}{\sum_j \tilde{\phi}_j(x)}$$

The patches ω_i could be circular, polygonal, or of any desired shape.

• Shape functions used in meshless methods could serve as PU.

A simple example of the trial space:

Let $\Omega = (0,1),\ x_i = ih,\ 0 \le i \le N$ with $Nh = 1$. Consider the patches $\omega_i = \Omega \cap (x_{i-1}, x_{i+1}),\ 0 \le i \le N$.

• Suppose the unknown solution is smooth. Since solutions are well-approximated by polynomials, we consider

$$V_i = \text{span}\,\{1, (x - x_i)/h\}$$

• Then the trial space is

$$\mathcal{S} = \sum_{i=0}^{N} \phi_i V_i = \left\{ v \in H^1 : v = \sum_{i=0}^{N} \phi_i(x)[\alpha_i + \beta_i(x - x_i)/h] \right\}$$

Note that if the functions ϕ_i are smooth, then functions in $\mathcal{S}$ are also globally smooth.

Hat functions as PU

Recall $\mathcal{S} = \sum_i \phi_i V_i$, where $V_i = \text{span}\{1, \xi_{ij}\}$

Suppose ϕ_i is the *standard hat function* (centered at x_i).

• If $V_i = \text{span}\{1\}$ (constant enrichment), then $\mathcal{S}$ is the standard FE space of piecewise linears.

• If $V_i = \text{span}\{1, (x - x_i)/h\}$ (linear enrichment), Then then $\mathcal{S}$ is the standard FE space of C^0 piecewise quadratics.

• Let $s(x)$ be a global function (could be a singular function). Let $V_i = \text{span}\{1, s(x)\}$, then

$$\mathcal{S} = \sum_i \phi_i V_i = \sum_i a_i \phi_i + \sum_i b_i \phi_i s(x)$$

If we restrict $b_i = b$, then

$$\begin{aligned} \mathcal{S} &= \sum_i a_i \phi_i + \sum_i b_i \phi_i s(x) \\ &= \sum_i a_i \phi_i + bs(x) \sum_i \phi_i = \sum_i a_i \phi_i + bs(x) \end{aligned}$$

APPROXIMATION RESULTS IN GFEM

Approximation result:

Babuška, Melenk, Caloz, Osborn '94, '96, '97

Theorem: Let $u \in H^1_D(\Omega)$. Suppose u can be approximated on the patch ω_i by $\xi_i^u \in V_i$ such that

$$\|u - \xi_i^u\|_{L^2(\omega_i)} \leq \epsilon_{1,i} \;\text{ and }\; \|u - \xi_i^u\|_{\mathcal{E}(\omega_i)} \leq \epsilon_{2,i}$$

Then $\xi^u = \sum_{i=1}^N \phi_i \xi_i^u \in \mathcal{S}$ satisfies

$$\|u - \xi^u\|_{L^2(\Omega)} \leq C\Big[\sum_{i=0}^N (\epsilon_{1,i})^2\Big]^{1/2}$$

$$\&\qquad \|u - \xi^u\|_{\mathcal{E}(\Omega)} \leq C\Big[\sum_{i=0}^N \frac{(\epsilon_{1,i})^2}{h^2} + \sum_{i=0}^N (\epsilon_{2,i})^2\Big]^{1/2}$$

Here $\|\cdot\|_{\mathcal{E}(\Omega)} = B(\cdot,\cdot)^{1/2}$; the energy norm,

Convergence result:

Theorem: Let $u \in H^1_D(\Omega)$. Suppose $\{\omega_i\}$ and $\{V_i\}$ satisfy a *uniform Poincare inequality*, namely,

(a) For $\overline{\omega}_i \cap \Gamma_D = \emptyset$, V_i's contain constants and

$$\|v\|_{L^2(\omega_i)} \le Ch\,|v|_{H^1(\omega_i)}, \quad \forall\, v \in H^1(\omega_i) \text{ satisfying } \int_{\omega_i} v\,dx = 0$$

(b) For $|\overline{\omega}_i \cap \Gamma_D| > 0$,

$$\|v\|_{L^2(\omega_i)} \le Ch\,|v|_{H^1(\omega_i)},\ \forall\, v \in H^1(\omega_i) \text{ satisfying } v\big|_{\overline{\omega}_i \cap \Gamma_D} = 0$$

Then there exists $\tilde{\xi}^u_i \in V_i$ such that $\tilde{\xi}^u = \sum_i \phi_i\, \tilde{\xi}^u_i \in \mathcal{S}$ satisfies

$$\|u - \tilde{\xi}^u\|_{\mathcal{E}(\Omega)} \le C\Big[\sum_{i=0}^{N} (\epsilon_{2,i})^2\Big]^{1/2}$$

Remark

- The result can also be stated as

$$\inf_{v\in\mathcal{S}} \|u - v\|_{\mathcal{E}(\Omega)} \leq C\Big[\sum_{i=0}^{N} \inf_{\xi_i\in V_i} \|u - \xi_i\|^2_{\mathcal{E}(\omega_i)}\Big]^{1/2}$$

The convergence is given in terms of $\mathrm{diam}\{\omega_i\} = h_i$ and the $\dim\{V_i\} = m_i$.

This is just like the *h* and *p* extension in FEM.

Role of PU in GFEM

Recall $\inf_{v \in \mathcal{S}} \|u - v\|_{\mathcal{E}(\Omega)} \leq C \big[\sum_{i=0}^{N} \inf_{\xi_i \in V_i} \|u - \xi_i\|^2_{\mathcal{E}(\omega_i)} \big]^{1/2}$

In GFEM, the role of the partition of unity $\{\phi_i\}$ is to "glue-together" the local approximation spaces V_i to obtain a globally H^1 trial space $\mathcal{S}$, i.e., $\mathcal{S} = \sum_i \phi_i V_i$. The accuracy comes from V_i.

• Griebel & Schweitzer have taken this point of view, and they use the so called "flat-top" PU.

• But the PU $\{\phi_i\}$ may have global approximation property, e.g., hat functions, and it increases the accuracy of GFEM (next slide). Thus one may use PU with global approximation property to enhance accuracy.

• It is easier to modify a FE code to incorporate the local enrichments; this yields GFEM with hat functions as the PU. This approach has become very popular in the engineering community, especially the XFEM community. This is another reason for using the hat functions as the PU in GFEM.

Non optimality of the result

Recall $\inf_{v\in\mathcal{S}} \|u-v\|_{\mathcal{E}(\Omega)} \le C\big[\sum_{i=0}^{N} \inf_{\xi_i\in V_i} \|u-\xi_i\|^2_{\mathcal{E}(\omega_i)}\big]^{1/2}$

• If the PU functions have “global approximation properties", then the GFEM approximation result is not optimal for smooth solutions.

• Let $\Omega=(0,1),\ x_i=ih,\ 0\le i\le N;\ \omega_i=\Omega\cap(x_{i-1},x_{i+1})$.
Let ϕ_i be the hat-function centered at x_i and consider $V_i=\mathcal{P}^1(\omega_i)$.

Then from the GFEM approximation result, we have

$$\begin{aligned}\inf_{\xi\in\mathcal{S}} \|u-\xi\|_{\mathcal{E}(\Omega)} &\le C\big[\sum_{i=0}^{N} \inf_{\xi_i\in V_i} \|u-\xi_i\|^2_{\mathcal{E}(\omega_i)}\big]^{1/2} \\ &\le C\big[\sum_{i=0}^{N} h^2|u|^2_{H^2(\omega_i)}\big]^{1/2} = Ch|u|_{H^2(\Omega)}\end{aligned}$$

But $\mathcal{S}=\sum_{i=0}^{N}\phi_i V_i$ is the space of C^0 piecewise quadratics, and it is well known that

$$\inf_{\xi\in\mathcal{S}} \|u-\xi\|_{\mathcal{E}(\Omega)} \le Ch^2|u|_{H^3(\Omega)}$$

PU with global approximation properties

The problem was partially addressed in Anitescu-B.'10.

• Consider scattered set of nodes $\{x_i\}$ in $\mathbb{R}^2$.

• Associated with each x_i, suppose there is a C^0 function $\phi_i(x)$ with piecewise continuous derivatives, with compact support $\overline{\omega}_i$;
$C_1 h \leq \text{diam}(\omega_i) \leq C_2 h$.

• $\bigcup_i \omega_i$ cover Ω, the domain of interest. Also an $x \in \Omega$ belongs to at most κ open sets ω_i.

• $|D^\alpha \phi_i|_\infty \leq C h^{-|\alpha|}$ for $0 \leq |\alpha| \leq s$.

• The functions $\{\phi_i\}$ (such that $\omega_i \cap \Omega \neq \emptyset$) **reproduce polynomials of degree** l with respect to $\{x_i\}$, i.e.

$$\sum_{\omega_i \cap \Omega \neq \emptyset} x_i^p \phi_i(x) = x^p, \quad \text{for all } 0 \leq |p| \leq l, \;\; x \in \Omega$$

Substituting $|p| = 0$, it is clear that $\{\phi_i\}$ (such that $\omega_i \cap \Omega \neq \emptyset$) form a Partition of Unity.

PU with global approximation properties

The functions $\{\phi_i\}$, which form a Partition of Unity, have global approximation property.

Theorem: *[Strang-Fix'73, Han-Meng'01]* Let v be smooth in Ω and let $\{\phi_i\}$ reproduce polynomials of degree l. Then there exists Φ, a linear combination of $\{\phi_i\}$, such that

$$\|v - \Phi\|_{H^t(\Omega)} \leq Ch^{l+1-t}$$

Note: This result requires *a shape restriction* on the patches ω_j.

PU with global approximation property: GFEM

Consider the GFEM with

- a partition of unity $\{\phi_i\}$ that reproduces polynomials of degree l
- local approximation space $V_i = \mathcal{P}^k(\omega_i)$; polynomials of degree k.
- $\mathcal{S} = \sum_i \phi_i V_i$

Theorem: [Anitescu-B'10] Let u be a smooth function. Then

$$\inf_{v\in\mathcal{S}} \|u - v\|_{\mathcal{E}(\Omega)} \leq Ch^{l+k}\|u\|_{H^{k+l+1}(\Omega)}$$

The general result:

Theorem: Suppose $u \in W_q^s(\Omega)$ with $q \geq 1$ and let $\phi_i \in W_\infty^M(\Omega)$. Then for $0 \leq t \leq \min\{k + l + 1, M\}$

$$\inf_{v\in\mathcal{S}} \|u - v\|_{W_q^t(\Omega)} \leq Ch^{\min\{s,k+l+1-t\}}\|u\|_{W_q^{\min\{s,k+l+1\}}(\Omega)}$$

LOCAL APPROXIMATION USING AVAILABLE INFORMATION:

SOME EXAMPLES

Examples of local approximation spaces V_i

• Suppose, the only available information on the solution u is that it is smooth, i.e., $u \in H^k(\omega_i)$ for $k = 1, 2, 3, \ldots$. Then we choose

$$V_i = \mathcal{P}^p(\omega_i)$$

where $\mathcal{P}^p(\omega_i)$ is the space of polynomials of degree p.

• It can be shown that this choice is *almost uniformly optimal* with respect to the given information. (Babuška, B.,Osborn '02)

Any other information, e.g., conditions on $\partial\omega_i$, may result into loss of almost optimality of V_i.

Examples of local approximation spaces V_i

- Suppose u is the solution of

$$\triangle u = 0 \text{ in } \Omega; \;\; u = 0 \text{ on } \Gamma_D, \;\; \partial_n u = g \text{ on } \Gamma_N$$

Consider a patch ω_i away from the boundary. Then the choice is

$$\begin{aligned} V_i &= \{v \in \mathcal{P}^p(\omega_i) \; : \; v \text{ is harmonic}\} \\ &= \text{span } \{\Re(z^j), \Im(z^j) \,|\, j = 0, 1, \cdots p\} \end{aligned}$$

Thus V_i is the span of first $2p + 1$ harmonic polynomials.

The optimality of V_i, for a circular patch ω_i was elaborated in Melenk, Babuška '97 and Babuška, B., Osborn '03

Examples of local approximation spaces V_i

- Let $\Omega = (0,1) \times (0,1)$ and consider the problem

$$B(u,v) = F(v), \quad \forall\, v \in H_0^1(\Omega)$$

with $a(x,y) = a(x) \in L^\infty(0,1)$ such that $0 < \alpha \leq a(x) \leq \beta$.

Consider a uniform triangulation of Ω and suppose the piecewise linear hat functions serve as the partition of unity. In this case, the standard "finite element star" centered at the node (x_i, y_i) serve as the patch ω_i .
Let

$$V_i = \text{span}\,\{1, \int_{x_i}^{x} \frac{dt}{a(t)}\,, y - y_i\}$$

Then one can show that (Babuška, Caloz, Osborn '94)

$$inf_{\xi \in \mathcal{S}} \|u - \xi\|_{\mathcal{E}(\Omega)} \leq Ch\|f\|_{L^2(\Omega)}$$

Examples of V_i (domain $\Omega \in \mathbb{R}^2$ with a crack)

• Let a, f, and g (the Neumann data) of the model problem be smooth, and suppose the crack is a straight line.

We have to consider three types of patches ω_i.

(i) patches ω_i away from the crack

(ii) patches ω_i that do not contain the crack-tip, but the crack "cuts" ω_i into two parts

(iii) patches ω_i that contain the crack-tip

(i) For ω_i away from the crack, the solution u is smooth, and we choose

$$V_i = \mathcal{P}^p(\omega_i)$$

Clearly, there exists $\xi_i \in V_i$ such that

$$\|u - \xi_i\|_{\mathcal{E}(\omega_i)} \text{ is small}$$

Examples of V_i (domain in $\mathbb{R}^2$ with a crack)

(ii) The crack "cuts" the patch ω_i into two parts – $\omega_i^{(1)}$ and $\omega_i^{(2)}$, such that $\omega_i = \omega_i^{(1)} \cup \omega_i^{(2)}$ and $\omega_i^{(1)} \cap \omega_i^{(2)} = \emptyset$.

Suppose the solution u is smooth on $\omega_i^{(1)}$ and $\omega_i^{(2)}$ (but not on ω_i).

We consider the local approximation space

$$V_i = \begin{cases} \mathcal{P}^{p_1}(\omega_i^{(1)}) & \text{on } \omega_i^{(1)} \\ \mathcal{P}^{p_2}(\omega_i^{(2)}) & \text{on } \omega_i^{(2)} \end{cases}$$

There exists $\xi_i = (\xi_i^{(1)}, \xi_i^{(2)}) \in V_i$ such that

$$\|u - \xi_i^{(1)}\|^2_{\mathcal{E}(\omega_i^{(1)})} + \|u - \xi_i^{(2)}\|^2_{\mathcal{E}(\omega_i^{(2)})} \quad \text{is small.}$$

Note that the function $(\chi_i^{(1)}, \chi_i^{(2)})$*, where* $\chi_i^{(j)}$ *is the characteristic function for* $\omega_i^{(j)}$*,* $j = 1, 2$*, serves as the "constant" on* ω_i*.*

Examples of V_i (domain in $\mathbb{R}^2$ with a crack)

(iii) The patches ω_i contain the crack-tip:

Let (r, θ) be the polar-coordinates of the points in $\mathbb{R}^2$ with the crack-tip as the origin. It is well known that near the crack-tip,

$$u(r,\theta) = \sum_{k=0}^{s} a_k r^{\lambda_k} [\log r]^{\mu_k} \psi_k(\theta) + \zeta(r,\theta)$$

where $\lambda_{k+1} \geq \lambda_k$, $\mu_{k+1} \geq \mu_k$, $\psi_k(\theta)$ is discontinuous at the crack, and $\zeta(r,\theta)$ is smoother than any of the terms in the sum.

We choose

$$V_i = \text{span}\left\{ r^{\lambda_k} [\log r]^{\mu_k} \psi_k(\theta) \right\}_{k=0}^{s} + \mathcal{P}^{s+1}(\omega_i)$$

There exists $\xi_i \in V_i$ such that

$$\|u - \xi_i\|_{\mathcal{E}(\omega_i)} \text{ is small}$$

EXTRACTING INFORMATION FOR LOCAL APPROXIMATION: LOCAL PROBLEMS

Towards a general approach to construct V_i

• Babuška and Lipton have recently (2010) proposed an idea to create local approximating spaces, when $a(x, y) \in L^\infty(\Omega)$. They have *proved* that these local approximating functions *converge exponentially* in the energy norm to the exact solution in a patch.

Towards a general approach to construct V_i

• Suppose $\omega \subset \Omega$ be a square patch. For an integer $N > 0$, consider concentric squares

$$\omega \subset \omega_M \subset \omega_{M-1} \subset \cdots \subset \omega_1 \subset \Omega$$

• On each ω_k, $1 \leq k \leq M$, we solve two problems:

(1) Solve an eigenvalue problem

$$\phi_{kj} \in H^1(\omega_k) \neq 0,$$
$$\int_{\omega_k} a\nabla\phi_{kj} \cdot \nabla z \, dx = \lambda_j \int_{\omega_k} \phi_{kj} \, z \, dx, \quad \forall \, z \in H^1(\omega_k)$$

for $j = 1, 2, \ldots, n$; $\lambda_j \neq 0$.

(2) Solve a Dirichlet problem

$$v_{kj} \in H^1(\omega_k) \text{ with } v_{kj}\big|_{\partial\omega_k} = \phi_{kj}$$
$$\int_{\omega_k} a\nabla v_{kj} \cdot \nabla z \, dx = 0, \quad \forall \, z \in H^1_0(\omega_k)$$

Towards a general approach to construct V_i

- For each k, define

$$\mathcal{F}_n(\omega, \omega_k) \equiv \text{span}\{v_{kj}\big|_{\omega}\}_{j=1}^{n}$$

- Let $m = n \times M$ and define

$$\mathcal{T}(m, \omega, \omega^*) \equiv \mathcal{F}_n(\omega, \omega^*) + \mathcal{F}_n(\omega, \omega_1) + \cdots + \mathcal{F}_n(\omega, \omega_M)$$

Theorem: Choose $0 < \gamma < \frac{1}{d}$ where $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. Let N be the largest integer $\leq n^\gamma$. Then there exists $z_u \in \mathcal{T}(m, \omega, \omega^*)$ such that

$$\|u - z_u\|_{\mathcal{E}(\omega_i)} \leq Ce^{-m^{\frac{\gamma}{\gamma+1}}}$$

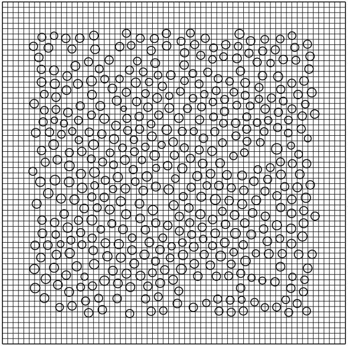
Towards a general approach to construct V_i

Computational experience: Babuška, Strouboulis, Copps '00, '01

Suppose u is the solution of

$$\triangle u = 0 \text{ in } \Omega;$$
$$\partial_n u = 0 \text{ on } \Gamma_I,$$
$$\partial_n u = g \text{ on } \Gamma_O$$

Γ_O: Outer boundary
Γ_I: Union of the boundary of the circular holes.



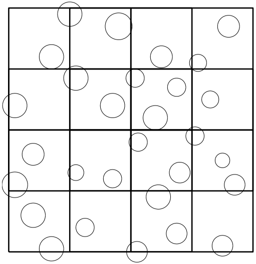
Towards a general approach to construct V_i

Rectangular patches associated with uniform mesh was considered.

Standard bilinear “hat-functions” were taken as the PU.

Consider a portion of the interior of the domain.

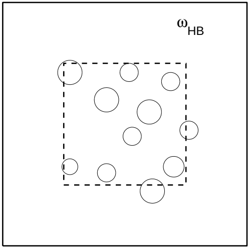
Let ω be a patch, where we seek to find the local approximation space V.



Towards a general approach to construct V_i

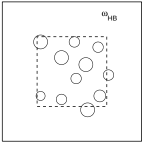
- Corresponding to the patch ω, we define a domain $\tilde{\omega}_{HB}$, called the *Handbook domain*.

$\tilde{\omega}_{HB}$ is the bigger square, obtained by eliminating the holes (of Ω) that intersect $\partial\tilde{\omega}_{HB}$.



Towards a general approach to construct V_i

The functions ξ_j, $j = 1, 2, \ldots, 2p+1$ in the local approximation space $V(\omega)$ are obtained by solving the following problem:



$$
\begin{aligned}
\triangle \xi_j &= 0 \quad \text{in } \tilde{\omega}_{HB}; \\
\frac{\partial}{\partial n} \xi_j &= 0 \quad \text{on } \tilde{\omega}^O_{HB}; \\
\frac{\partial}{\partial n} \xi_j &= \begin{cases} \partial_n \left[\Re(z^p)\right] & \text{for } j = 2p-1 \\ \partial_n \left[\Im(z^p)\right] & \text{for } j = 2p \end{cases} \quad \text{on } \tilde{\omega}^I_{HB};
\end{aligned}
$$

$\tilde{\omega}^O_{HB}$: Outer boundary of $\tilde{\omega}_{HB}$

$\tilde{\omega}^I_{HB}$: Union of the boundaries of the holes.

Towards a general approach to construct V_i

Let the local approximation space be $V = \text{span}\{1, \xi_j; 1 \leq j \leq 2p+1\}$.

It was observed that

$$\inf_{v \in V} \|u - v\|_{\mathcal{E}(\omega)} \approx O(e^{-0.48(2p+1)})$$

Thus the the exact solution u is accurately approximated by the functions in V on the patch ω.

CONDITION NUMBER

On the condition number

The following ongoing work is joint with I. Babuška, T. Fries

• The condition number of the stiffness matrix of GFEM, $\mathfrak{K}(A)$, can be *extremely large*, based on the choice of PU functions as well as the choice of local approximating function space V_i.

• Suppose the PU does not have any global approximation property, e.g., "flat-top" PU, Then if $V_i = \mathcal{P}^p(\omega_i)$, then (H. Li, 2010)

$$\mathfrak{K}(A) \approx Ch^{-2}$$

where $h \approx \text{diam}\{\omega_i\}$; the constant C depend on the gradient of the PU functions and can be big.

Condition number: Hat functions as PU

• In engineering practice, the standard hat functions are used extensively as the PU by the XFEM community, especially in crack propagation problems; the finite element stars serve as the patches ω_i.

• With hat functions serving as the PU and $V_i = \mathcal{P}^p(\omega_i)$, Babuška & Melenk ('96, '97) showed that the stiffness matrix could have zero eigenvalues.

• **Another example (1-d):** Let $V_i = \text{span}\{1, e^{-\alpha x}\}$, where α is large. One can show that

$$\mathfrak{K}(A) \geq \frac{e^{\alpha}}{\alpha^2 h^4}$$

Thus $\mathfrak{K}(A)$ is extremely large for large values of α.

On the condition number

• Recall $V_i = \text{span}\{\xi_{ij}\}_{j=0}^{m_i}$ where $\xi_{i0} = 1$. Then the trial space

$$\mathcal{S} = \sum \phi_i V_i \equiv \mathcal{S}^0 + \mathcal{S}^1$$

where

$$\mathcal{S}^0 = \text{span}\{\phi_i\}_{i=0}^{N} \qquad \mathcal{S}^1 = \text{span}\{\phi_i \xi_{ij}, 1 \leq j \leq m_i\}_{i=0}^{N}$$

$\mathcal{S}^0$ is the piecewise linear standard FE space. Thus $\mathcal{S}$ could be viewed as an extension of FE space.

• With $\mathcal{S}$ as the trial and test space, the stiffness matrix of GFEM is of the form

$$A = \begin{bmatrix} B(\mathcal{S}^0, \mathcal{S}^0) & B(\mathcal{S}^1, \mathcal{S}^0) \\ B(\mathcal{S}^0, \mathcal{S}^1) & B(\mathcal{S}^1, \mathcal{S}^1) \end{bmatrix}$$

Note that $B(\mathcal{S}^0, \mathcal{S}^0)$ is the standard FE stiffness matrix $[B(\phi_i, \phi_j)]$.

On the condition number

Assumption 1:

$\mathcal{S}^0$ and $\mathcal{S}^1$ are *almost orthogonal* with respect to the inner product $B(\cdot,\cdot)$, i.e., there exist $L_1 > 0$, U_1 such that

$$\begin{aligned} L_1\left[\|\zeta_1\|^2_{\mathcal{E}(\Omega)} + \|\zeta_2\|^2_{\mathcal{E}(\Omega)}\right] &\leq B(\zeta_1+\zeta_2, \zeta_1+\zeta_2) \\ &\leq U_1\left[\|\zeta_1\|^2_{\mathcal{E}(\Omega)} + \|\zeta_2\|^2_{\mathcal{E}(\Omega)}\right] \\ & \forall\, \zeta_1 \in \mathcal{S}^0,\ \zeta_2 \in \mathcal{S}^1 \end{aligned}$$

On the condition number

• For an element τ_i, let $A^{(i)}$ be the *element matrix* for $B(\mathcal{S}^1, \mathcal{S}^1)$, i.e., for $\eta \in \mathcal{S}^1$,

$$\int_{\tau_i} a\nabla\eta \cdot \nabla\eta dx = [\eta]_{(i)}^T A^{(i)} [\eta]_{(i)}$$

The matrix $B(\mathcal{S}^1, \mathcal{S}^1)$ can be assembled using $A^{(i)}$.

• Let $D^{(i)}$ be the diagonal matrix such that the diagonal elements of $\widetilde{A}^{(i)} = D^{(i)} A^{(i)} D^{(i)}$ are of $O(1)$.

Assumption 2:

There exist $L_2 > 0$, $U_2 > 0$, independent of i, such that

$$L_2 \| \left[D^{(i)}\right]^{-1} \mathbf{x} \|^2 \leq \mathbf{x}^T A^{(i)} \mathbf{x} \leq U_2 \| \left[D^{(i)}\right]^{-1} \mathbf{x} \|^2 \quad \forall \mathbf{x}$$

or

$$L_2 \|\mathbf{y}\|^2 \leq \mathbf{y}^T \widetilde{A}^{(i)} \mathbf{y} \leq U_2 \| \mathbf{y} \|^2, \quad \forall\, \mathbf{y}$$

On the condition number

Theorem: Let A be stiffness matrix. Let D be a diagonal matrix such that the diagonal elements of $\widetilde{A} = D\,A\,D$ are $O(1)$. Suppose $\mathcal{S}_h^0,\ \mathcal{S}_h^1$ satisfy assumptions 1 and 2. Then

$$\mathfrak{K}(\widetilde{A}) \approx \mathfrak{K}(\widetilde{A}_{FE})$$

where $A_{FE} = B(\mathcal{S}^0, \mathcal{S}^0)$.

Stable GFEM – SGFEM

• Let $V_i = \text{span}\{\xi_{ij}\}_{j=0}^{m_i}$, where $\xi_{i0} = 1$ be the local approximating space that accurately approximates the exact solution locally in the patch ω_i (FE star – union of elements).

• We define a modified space $\bar{V}_i = \text{span}\{1, \bar{\xi}_{ij}\}_{j=1}^{m_i}$, where

$$\bar{\xi}_{ij} = \xi_{ij} - \mathcal{I}_{\omega_i}\xi_{ij}$$

and $\mathcal{I}_{\omega_i}\xi_{ij}$ is the piecewise linear interpolant of ξ_{ij} on ω_i.

Example: Let $\omega_i = (x_{i-1}, x_{i+1})$. Then

$$\mathcal{I}_{\omega_i}\xi_{ij} = \xi_{ij}(x_{j-1})\phi_{i-1} + \xi_{ij}(x_j)\phi_i + \xi_{ij}(x_{j+1})\phi_{i+1}$$

Note $\bar{\xi}_{ij}(x_k) = 0$ for $k = i - 1,\ i,\ i + 1$; it is a “double bubble".

We refer the GFEM with modified local approximation space $\bar{V}_i$ as SGFEM.

Accuracy of SGFEM

- Modifying the local approximation spaces do not affect accuracy.

Recall that $\mathcal{S}^1 = \text{span}\{\phi_i \xi_{ij}, 1 \leq j \leq m_i\}_{i=0}^N$.

Define $\bar{\mathcal{S}}^1 = \text{span}\{\phi_i \bar{\xi}_{ij}, 1 \leq j \leq m_i\}_{i=0}^N$

Theorem: Suppose for $0 \leq i \leq N$, there exists $\xi_i \in V_i$ such that $\|u - \xi_i\|_{\mathcal{E}(\omega_i)} \leq \epsilon_i$, where ϵ_i is small. Then there exists $v \in \bar{\mathcal{S}} \equiv \mathcal{S}^0 + \bar{\mathcal{S}}^1$, such that

$$\|u - v\|_{\mathcal{E}(\Omega)} \leq \Big[\sum_i \epsilon_i^2 \Big]^{1/2}$$

Do $\mathcal{S}^0$ and $\mathcal{S}^1$ satisfy the assumptions?

• $\mathcal{S}^0$ and $\bar{\mathcal{S}}^1$ are always almost orthogonal in 1-d; thus they satisfy Assumption 1

It is trivial to note that

$$\int_{x_i}^{x_{i+1}} \phi_i' \, (\phi_i \bar{\xi}_{ij})' \, dx = -\frac{1}{h} \int_{x_i}^{x_{i+1}} (\phi_i \bar{\xi}_{ij})' \, dx = 0$$

Using this idea it is easy to show that for $\zeta_1 \in \mathcal{S}_h^0$ and $\zeta_2 \in \bar{\mathcal{S}}_h^0$,

$$\frac{\alpha}{\beta} \left[\|\zeta_1\|^2_{\mathcal{E}(\Omega)} + \|\zeta_2\|^2_{\mathcal{E}(\Omega)} \right] \leq B(\zeta_1 + \zeta_2, \zeta_1 + \zeta_2)$$

• In higher dimensions, this property has to be checked.

Do $\mathcal{S}^0$ and $\mathcal{S}^1$ satisfy the assumptions?

Interface problems:

$$a(x) = a_2(x) = \begin{cases} 1, & 0 \le x < b_1^* \\ \frac{1}{2}, & b_1^* \le x < b_2^* \\ 1, & b_2^* \le x \le 1 \end{cases}$$

Use the uniform mesh $x_i = ih$.

Suppose $b_1^* = x_{m-1} + h/2$ and $b_2^* = x_m + \beta h/2$ with $0 < \beta < 1$.

$V_i = \text{span}\{1, \xi_{i1}\}$ where $\xi_{i1} = \int_{x_i}^{x} \frac{1}{a(t)}\, dt$

$\bar{V}_i = \text{span}\{1\}$ for $i \neq m-1, m, m+1$

and consequently

$$\bar{\mathcal{S}}^1 = \text{span}\{\phi_i \xi_{i1}\}_{i=0}^{N} = \text{span}\{\phi_i \xi_{i1}\}_{i=m-1}^{m+1}$$

Do $\mathcal{S}^0$ and $\mathcal{S}^1$ satisfy the assumptions?

Interface Problems:
The matrix $B(\mathcal{S}^1, \mathcal{S}^1)$ is 3×3 assembled from element matrices $A^{(m)}$ and $A^{(m+1)}$. It can be shown that

$$\frac{1}{32}\|\mathbf{y}\|^2 \leq \mathbf{y}^T\widetilde{A}^{(m)}\mathbf{y} \leq \frac{3}{32}\|y\|^2$$
$$\frac{1}{6}\|\mathbf{y}\|^2 \leq \mathbf{y}^T\widetilde{A}^{(m+1)}\mathbf{y} \leq \|y\|^2$$

The bounds are independent of β. Thus the Assumption 2 is satisfied independent of β.

Major open problems

• NUMERICAL INTEGRATION !!!!
This the price one pays for using meshes that does not conform to the geometry of the problem.

• Computation of enrichment functions and the associated mathematical analysis.

• Adaptivity and aposteriori error estimation.

• Imposition of Dirichlet boundary conditions.