

SOURCE TYPE OF OPTIMIZATION FOR INVERSE SCATTERING

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Abstract *This paper reviews the contrast source inversion method as a robust algorithm to solve the nonlinear inverse scattering problem, where the discrepancy between the measured and predicted scattered field data is minimized. The algorithm is based on a source type of integral representation of the scattered field, where the integral operator acts on a contrast source, being the product of the interior field and the material contrast of the scattering object. In particular, we discuss the effective implementation of a multiplicative constraint that minimizes the spatial variations of the contrast.*

1. Introduction

Inverse scattering concerns the determination of the shape, location and constitutive parameters of a local inhomogeneity from measurements of the scattered wavefield, when a number of waves are generated such that they successively illuminate the domain of interest. Most of the methods utilize the domain integral representation of the scattered field to minimize the discrepancy between the measured data and predicted data as a function of the material contrast. This contrast is defined as the difference between the constitutive parameters of the target (or object) under consideration and those of a known background medium. We assume that the unknown scatterer is contained in a domain D (see Fig. 1) and that this object is irradiated successively by a number of known incident fields $u_j^{\text{inc}}(\mathbf{p})$, $j = 1, \dots, J$, originating from difference source positions. For each incident field the total field will be denoted by $u_j(\mathbf{p})$ in D and by $u_j(\mathbf{p}) = u_j^{\text{inc}}(\mathbf{p}) + u_j^{\text{sct}}(\mathbf{p})$ exterior to D . Here, u stands either for the acoustic pressure in an acoustic scattering problem, for the electric-field vector in electromagnetic scattering, or for the particle-velocity vector in elastodynamic scattering. The total field and the scattered field satisfy the following domain integral representations

$$u_j^{\text{sct}}(\mathbf{p}) = \int_D g(\mathbf{p}, \mathbf{q}) \chi(\mathbf{q}) u_j(\mathbf{q}) \, d\mathbf{v}(\mathbf{q}), \quad \mathbf{p} \in \text{ext } D, \quad (1)$$

$$u_j(\mathbf{p}) = u_j^{\text{inc}}(\mathbf{p}) + \int_D g(\mathbf{p}, \mathbf{q}) \chi(\mathbf{q}) u_j(\mathbf{q}) \, d\mathbf{v}(\mathbf{q}), \quad \mathbf{p} \in \text{int } D, \quad (2)$$

where $\chi(\mathbf{q})$ denotes the contrast of the object with respect to its background medium, and $g(\mathbf{p}, \mathbf{q})$ denotes the Green function, being the fundamental point source solution in the background medium. In some surface S outside D , the integral representation of Eq. 1 for points exterior to D is written symbolically as the *data equation*,

$$f_j = G_S \chi u_j, \quad \mathbf{p} \in S, \quad (3)$$

while the integral equation of Eq. 2 is written symbolically as the *object equation*,

$$u_j^{\text{inc}} = u_j - G_D \chi u_j, \quad \mathbf{p} \in D, \quad (4)$$

where the operator G_S is an operator mapping from $L^2(D)$ into $L^2(S)$ and G_D is an operator mapping $L^2(D)$ into $L^2(D)$. Let us further denote the norm in $L^2(S)$ and $L^2(D)$ as $\|\cdot\|_S$ and $\|\cdot\|_D$, respectively. The inverse scattering problem consists of determining $\chi(\mathbf{p})$ from a knowledge of the incident fields, $u_j^{\text{inc}}(\mathbf{p})$, on D and the scattered fields, $f_j(\mathbf{p})$, on S . This problem is both nonlinear and highly ill-posed.

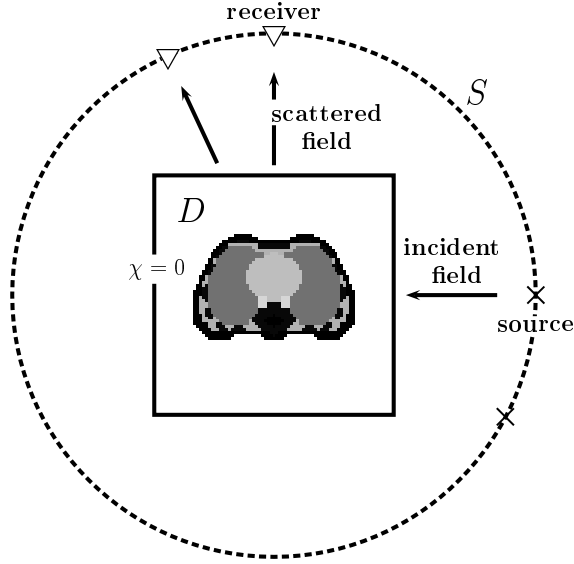


Figure 1: Scattering configuration enclosed in a test domain D .

2. The optimization problem

With considerable success, the inverse scattering problem has been cast as an optimization problem of finding χ to minimize the mismatch in the data equation of Eq. 3 subject to the constraint that the object equation of Eq. 4 is satisfied in some sense. The existence of a minimizer can be guaranteed by a suitable choice of the class of admissible values of χ . However, whether this minimizer is related to a local or global minimum remains an ongoing concern. Standardly, the fields and the contrast are considered as fundamental unknowns. Since the inverse scattering problem is nonlinear, inversion methods are iterative in nature and each iteration requires the solution of a forward problem, where for given contrast the pertaining field solution is computed. This complication is avoided in the modified gradient method [1] where the unknown fields and contrast are updated iteratively. Defining in each iteration ($n = 1, 2, \dots$) two sequences of updates, viz. $u_{j,n}$ for the fields u_j and χ_n for the contrast χ , then the modified gradient optimizes a cost functional being the superposition of the mismatch of the measured field data and the predicted data and the error in satisfying consistency in the interior of the object, viz.,

$$F_{S+D}(u_j, \chi) = \frac{\sum_{j=1}^J \|f_j - G_S \chi u_j\|_S^2}{\sum_{j=1}^J \|f_j\|_S^2} + \frac{\sum_{j=1}^J \|u_j^{\text{inc}} - u_j + G_D \chi u_j\|_D^2}{\sum_{j=1}^J \|u_j^{\text{inc}}\|_D^2}. \quad (5)$$

In these relations integral operators act on contrast sources being the products of the unknown fields and the unknown contrast. Therefore, a more efficient and versatile algorithm is developed, in which the contrast sources and the contrast are considered as fundamental unknowns.

3. The contrast source inversion method

In the so-called contrast source inversion (CSI) method [2], the updates for the contrast sources $w_{j,n} = \chi_n u_{j,n}$ and the contrast χ_n are obtained by an iterative minimization of the two-terms cost functional

$$F_{S+D}(w_j, \chi) = \frac{\sum_{j=1}^J \|f_j - G_S w_j\|_S^2}{\sum_{j=1}^J \|f_j\|_S^2} + \frac{\sum_{j=1}^J \|\chi u_j^{\text{inc}} - w_j + \chi G_D w_j\|_D^2}{\sum_{j=1}^J \|\chi_{n-1} u_j^{\text{inc}}\|_D^2}. \quad (6)$$

This minimization is carried in three alternate steps.

(i) By letting the contrast to be the one obtained in the previous iteration, i.e. $\chi = \chi_{n-1}$, the cost functional is quadratic in terms of the contrast sources, and these contrast sources are updated via a single conjugate gradient step with a gradient direction of the two-terms functional

$$F_{S+D}(w_j, \chi_{n-1}) = \frac{\sum_{j=1}^J \|f_j - G_S w_j\|_S^2}{\sum_{j=1}^J \|f_j\|_S^2} + \frac{\sum_{j=1}^J \|\chi_{n-1} u_j^{\text{inc}} - w_j + \chi_{n-1} G_D w_j\|_D^2}{\sum_{j=1}^J \|\chi_{n-1} u_j^{\text{inc}}\|_D^2}. \quad (7)$$

(ii) With the update for the contrast sources defined as $w_{j,n}$, we immediately obtain the field update using the integral representation

$$u_{j,n} = u_j^{\text{inc}} + G_D w_{j,n}. \quad (8)$$

(iii) For given contrast sources $w_{j,n}$ and fields $u_{j,n}$ the contrast update χ is found by a direct minimization of the second term of Eq. 6,

$$F_D(\chi) = \frac{\sum_{j=1}^J \|\chi u_{j,n} - w_{j,n}\|_D^2}{\sum_{j=1}^J \|\chi_{n-1} u_j^{\text{inc}}\|_D^2}, \quad (9)$$

which is equivalent to find the contrast χ as the least-square fit of the constitutive relation $w_{j,n} = \chi u_{j,n}$ between the contrast sources and the fields. The update of the contrast is found analytically as

$$\chi_n^{\text{analyt}} = \frac{\sum_{j=1}^J w_{j,n} \overline{u_{j,n}}}{\sum_{j=1}^J |u_{j,n}|^2}, \quad (10)$$

where the overline denotes complex conjugate. In principle, the computational complexity of one iteration of the CSI method is approximately equal to two iterations for solving the linear forward problem with a conjugate gradient method. This enables us to attack a full vectorial three-dimensional nonlinear inversion problems with the present-day computer power.

It is noted that, in view of the presence of the second term in the cost functional of Eq. 6, the problem of updating the contrast sources is well-posed, while the analytical expression for the update of the contrast avoids the ill-posedness of the inverse problem. In most inversion algorithms an operator acts on the contrast and a numerical inversion of the ill-posed operator may lead to numerical instabilities.

4. Extra regularization

When the number of data is limited and a significant noise level is present in the data, one way to operate is to limit the number of unknown parameters in the contrast, or in other words to limit the spatial variation of the contrast. At the other hand, the question is how to restore the band limitation of the images we have obtained with an inversion method. In image processing, there are a number of methods to enhance an image by minimizing its variation. Our strategy is to combine one and another by minimization of the spatial variation of the contrast, during the whole inversion process. We therefore define some norm $F_n^R(\nabla\chi)$, where $\nabla\chi$ is the gradient of the contrast. The standard way is to include such a norm in the cost functional by introducing an extra penalty function, viz.,

$$\mathcal{F}_n(w_{j,n}, \chi_n) = F_{S+D}(w_{j,n}, \chi_n) + \gamma^2 F_n^R(\nabla\chi_n). \quad (11)$$

As known in the literature the addition of the regularization term F_n^R to the cost functional has a very positive effect on the quality of the reconstruction. The total variation as L^1 norm has been used by adding it as a penalty function to the modified gradient algorithm [3] and the contrast-source-inversion method [4], resulting in a substantial improvement of the performance of the reconstruction methods, both for ‘blocky’ and smooth contrast configurations. The drawback is the presence of the positive weighting parameter γ^2 in the cost functional, which, with the present knowledge, can only be determined through considerable numerical experimentation and *a priori* information of the desired reconstruction. Furthermore, the complexity of a full three-dimensional problem makes the approach of extensive numerical experimentation not feasible.

Numerical experiments have shown that the results improve when we let the parameter γ^2 decrease with increasing number of iterations. In fact, a good choice seems to take this parameter proportional to the value of the cost functional F_{n-1} of the previous iteration. This numerical experimentation has led us to the idea of multiplicative regularization technique [4], viz.,

$$\mathcal{F}_n(w_{j,n}, \chi_n) = F_n(w_{j,n}, \chi_n) \times F_n^R(\nabla\chi_n). \quad (12)$$

Minimization of this functional with respect to changes in the contrast will change the minimizer χ_n^{analyt} given in Eq. 10 to

$$\chi_n = \chi_n^{\text{analyt}} + \beta_n d_n, \quad (13)$$

where d_n is the conjugate gradient direction of the cost functional of Eq. 12 with respect to changes of the contrast around the point $\chi = \chi_n^{\text{analyt}}$. In view of the previous analytical minimization step, the gradient with respect to the contrast contains only a contribution of the regularization additionally imposed [5]. This simplifies the algorithm. In general, the real parameter β_n is found from a line minimization as minimizer of the cost functional of Eq. 12.

The structure of our multiplicative regularization procedure is such that it will minimize the regularization factor with a large weighting parameter in the beginning of the optimization process, because the value of F_n is still large, and that it will gradually minimize more and more the error in the data and object equations when the regularization factor F_n^R remains a nearly constant value. If noise is present in the data, the data error term will remain at a large value during the optimization and therefore, the weight of the regularization factor will be more significant. Hence, the noise will, at all times, be suppressed in the reconstruction process and we automatically fulfill the need of a larger regularization when the data contains noise.

5. Normalized regularization factor

Inspired by the edge-preserving algorithms in image restoration [7] and in inverse scattering [8], an efficient regularization factor in the form of a weighted L^2 norm has been developed [6], such that the weighting favors flat parts and non-flat parts of the contrast profile almost equally, viz.,

$$F_n^R(\nabla\chi) = \frac{1}{V} \int_D \frac{|\nabla\chi(\mathbf{p})|^2 + \delta_n^2}{|\nabla\chi_{n-1}(\mathbf{p})|^2 + \delta_n^2} dv(\mathbf{p}), \quad (14)$$

where $V = \int_D dv(\mathbf{p})$ denotes the volume of the test domain D . The regularization factor is equal to one when the contrast χ is equal to the contrast χ_{n-1} of the previous iteration. The positive parameter δ_n^2 in the numerator takes care that the regularization factor does not vanish. Further, it also controls the strength of regularization. We therefore have chosen to increase the regularization as a function of the number of iterations by decreasing this parameter δ_n^2 . Since the object error term will decrease as a function of the number of iterations, we choose

$$\delta_n^2 = F_D(\chi_{n-1}) \tilde{\Delta}^2, \quad (15)$$

where $F_D(\chi_{n-1})$ denotes the object error Eq. 9 and $\tilde{\Delta}$ denotes the reciprocal mesh size of the discretized domain D .

Since the regularization factor is equal to one when $\chi = \chi_{n-1}$, the updating of the contrast sources in the inversion algorithm is not changed. Only the updating of the contrast is changed, see Eq. 13, where the parameter β_n follows from minimization of the multiplicative cost functional of Eq. 12. Since we deal with a product of two L^2 -norms the minimization of the multiplicative cost functional of Eq. 12 can be performed analytically. The cost functional is a fourth-degree polynomial in β_n . Then, differentiation with respect to β_n yields a cubic equation with one real root and two complex conjugate roots. The real root is the desired minimizer β_n . Further, a careful analysis [5] of this polynomial provides sufficient conditions for convexity of this minimization problem in updating the contrast. In almost all numerical experiments we have observed that the choice of Eq. 15 satisfies these conditions of convexity. The contrast source inversion method with this multiplicative regularization factor is denoted shortly as the MR-CSI method.

6. Examples and Conclusions

We present a number of reconstruction examples, both from synthetic and experimental scattered field data, with applications sub-surface sensing [9], microwave tomography [10] and to medical imaging citeAbubakarMallorqui.

Although there are many fundamental problems to solve, both theoretically as numerically, with these numerical examples and results we have shown that a well-designed optimization method, e.g., the "Multiplicative Regularized Contrast Source Inversion" method, makes the 'blind' reconstruction of realistic configurations from scattered-field data feasible. In view of the efficiency and robustness of the present algorithms we conclude that in the near future nonlinear inversion methods will have a prominent position to handle full vectorial 3D scattered field data.

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