The Importance of Admissible Maps between Sequential Dynamical Systems

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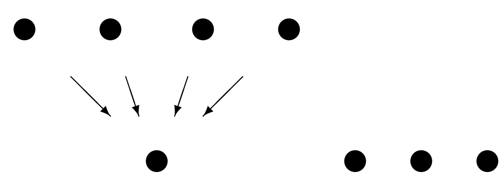
Monomial systems: (as examples of agent based dynamical systems)

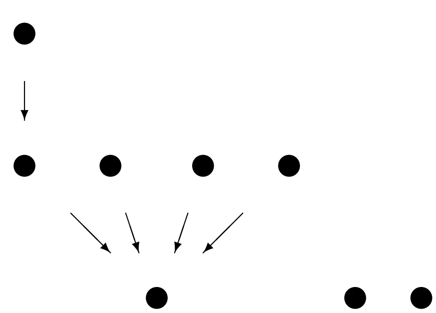
$(f_1, \dots, f_n) : k^n \longrightarrow k^n$, where $k = \{0, 1\}$ and $f_i = \alpha_i x_1^{e_{1i}} \dots x_n^{e_{ni}} : k^n \longrightarrow k$ with $\alpha, e_1, \dots, e_n \in k$ (k can be replaced by a prime field, then number theoretic tools have to be applied).

We want to study the dynamic behaviour and identify fixed point systems. The following examples are a fixed point systems:

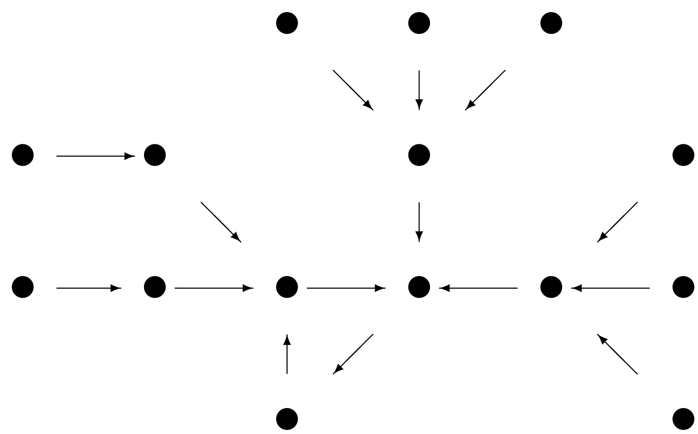
Example 1: $(xy, 1, yz) : k^3 \longrightarrow k^3$ also written as $(x, y, z) \mapsto (xy, 1, yz)$

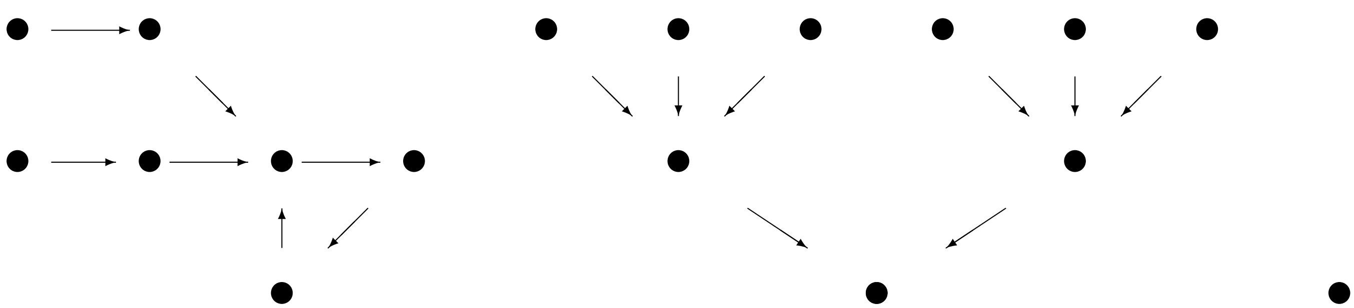


Example 2: $(xz, 1, yz) : k^3 \longrightarrow k^3$ also written as $(x, y, z) \mapsto (xz, 1, yz)$



Example 3: $(x_2, x_3x_4, x_5, x_1, x_2) : k^5 \longrightarrow k^5$





Theorem: (design principles for fixed point systems)
1) Triangular monomial systems (f_i contains only the variables $x_1, \ldots, x_i$) are fixed point systems.

2) The following operations on monomial systems preserve fixed point systems:
a) Multiplication of a monomial system with a permutation $\sigma \in S_n$:

$$(f_1, \ldots, f_n) \mapsto \sigma(f_1, \ldots, f_n) :=$$
$$(f_{\sigma^{-1}(1)}(x_{\sigma(1)}, \ldots, x_{\sigma(n)}), \ldots, f_{\sigma^{-1}(1)}(x_{\sigma(1)}, \ldots, x_{\sigma(n)})).$$

b) Multiplication of a monomial system with a monomial $m = \alpha x_1^{e_1} \ldots x_n^{e_n}$:

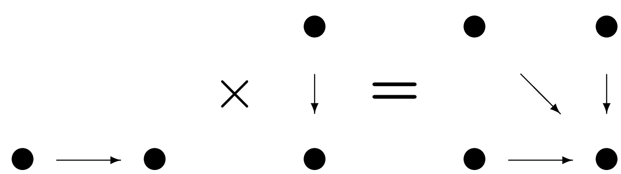
$$(f_1, \ldots, f_n) \mapsto (m \cdot f_1, \ldots, m \cdot f_n).$$

c) Union of two monomial systems:

$$(f_1, \ldots, f_n) \cup (g_1, \ldots, g_m) = (f_1, \ldots, f_n, g_1, \ldots, g_m).$$

This system has as state space graph the product of the two original state space graphs.

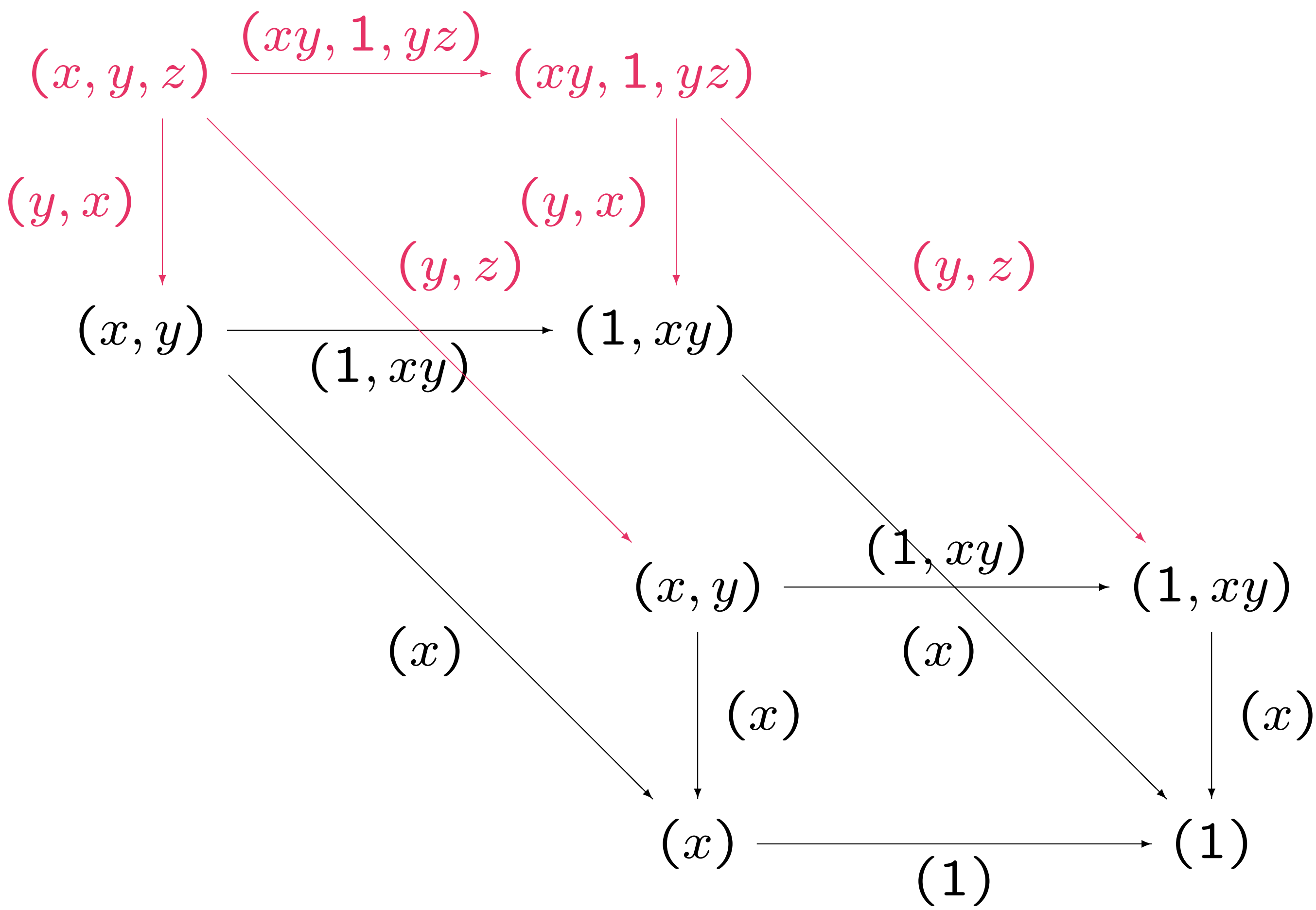
Example: The union $(1) \cup (1) = (1,1)$ has as state (phase) space graph the product



This is best described by the definition of products with projections.

The two examples $(xy, 1, yz) : k^3 \longrightarrow k^3$ and $(xz, 1, yz) : k^3 \longrightarrow k^3$ cannot be reduced into products of smaller systems. They are "irreducible".

One of the examples can, however, be written as a pullback (= fiber product) of triangular systems of lower dimension:


(x, y, z)
(xy, 1, yz)
(xy, 1, yz)
(y, x)
(y, x)
(y, z)
(y, z)
(x, y)
(1, xy)
(1, xy)
(x, y)
(1, xy)
(1, xy)
(x)
(x)
(x)
(x)
(x)
(1)
(1)

Fiber Products:

$$\begin{array}{ccc} \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} & \xrightarrow{\text{proj}} & \mathcal{Y} \\ \Big\downarrow{\scriptstyle \text{proj}} & & \Big\downarrow{\scriptstyle g} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Z} \end{array}$$

need for their construction the definition of admissible maps.

This is another example for a design principle, a decomposition and a dimension reduction:

Theorem: *A fibre product of monomial fixed point systems $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, f, g$ is a fixed point system.*

Mathematical Problems, that involve admissible maps between agent based systems:

- What is a simulation (of an SDS by an SDS)? An admissible map between SDSs.
- Find techniques to reduce the dimension of a system. Construct a fibre product decomposition.
- Find an optimal simulation (of a parallel dynamical system (PDS)) by a sequential dynamical system (SDS). Certain limits or colimits defined with admissible maps.

Definition: A *sequential dynamical system (SDS)* consists of

1. a *dependency graph* $F = (V_F, E_F)$,
2. a family of *state sets* $(k[a] \big| a \in V_F)$,
3. a collection of *local update functions*

   $(f_a : k[a_1] \times \ldots \times k[a_n] \longrightarrow k[a_1] \times \ldots \times k[a_n] \big| a \in V_F)$

   or $(f_a : k^r \longrightarrow k^r)$,
4. an *update schedule* $\alpha : \{1, \ldots, t\} \longrightarrow V_F$, determining the order in which the vertices (agents) a update their state by f_a.

The *global update function* is

$$f = f_{\alpha(t)} \circ \ldots \circ f_{\alpha(1)} : k[a_1] \times \ldots \times k[a_n] \longrightarrow k[a_1] \times \ldots \times k[a_n].$$

Theorem: *Let F be a graph together with an update schedule $\alpha : \{1,\dots,t\} \longrightarrow V_F$. Then there are maps*

$$\{1,\dots,t\} \overset{\gamma}{\longleftarrow} \mathcal{O}_F \overset{\beta}{\longrightarrow} V_F$$

with

- *a finite poset $\mathcal{O}_F$,*
- *a bijective order preserving map $\gamma : \mathcal{O}_F \longrightarrow \{1,\dots,t\}$ and*
- *a map $\beta : \mathcal{O}_F \longrightarrow V_F$*

such that

1. $\forall i,j \in \mathcal{O}_F : i \rhd j \Rightarrow \{\beta(i),\beta(j)\} \in E_F$,
2. $\forall i,j \in \mathcal{O}_F : \{\beta(i),\beta(j)\} \in E_F \Rightarrow i \geq j \vee j \geq i$,
3. $\alpha = \beta\gamma^{-1}$.

$\mathcal{O}_F, \gamma_F$, and β_F are uniquely determined by α and F.

Theorem: *Let $\mathcal{F}$ be an SDS. Then the global update function $f_\alpha : k^r \longrightarrow k^r$ of $\mathcal{F}$ depends only on $\beta_F : \mathcal{O}_F \longrightarrow V_F$. Any map $\beta_F : \mathcal{O}_F \longrightarrow V_F$ with a poset $\mathcal{O}_F$ satisfying conditions 1. and 2. of the previous Theorem defines a unique global update function.*

Definition: Let $\mathcal{F} = \Big(F, (k[a]), (f_a), \alpha\Big)$ and $\mathcal{G} = \Big(G, (k[b]), (f_b), \alpha'\Big)$ be SDSs. An *admissible map of sequential dynamical systems* (morphism) $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ consists of

- a graph morphism $\varphi_g : G \longrightarrow F$ (reverse direction!),
- a family of maps $(\varphi_s[b] : k[\varphi_g(b)] \longrightarrow k[b] \mid b \in V_G, \varphi_s[b] \in Z)$,
- a morphism of posets $\varphi_o : \mathcal{O}_G \longrightarrow \mathcal{O}_F$

such that the diagrams

$$\begin{array}{ccc} k^r & \xrightarrow{\quad f_a \quad} & k^r \\ {\scriptstyle \varphi^*}\downarrow & & \downarrow{\scriptstyle \varphi^*} \\ k^s & \xrightarrow{\prod_{i\in(\varphi_g\beta_G)^{-1}(a)} g_{\beta_G(i)}} & k^s \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathcal{O}_G & \xrightarrow{\quad \beta_G \quad} & V_G \\ {\scriptstyle \varphi_o}\downarrow & & \downarrow{\scriptstyle \varphi_g} \\ \mathcal{O}_F & \xrightarrow{\quad \beta_F \quad} & V_F \end{array}$$

commute, where

- $k^r := \prod_{a\in V_F} k[a]$,
- $\varphi^*(x[a_1], \ldots, x[a_r]) :=$ $(\varphi_s[b_1](x[\varphi_g(b_1)]), \ldots, \varphi_s[b_s](x[\varphi_g(b_s)]))$, and
- the product $\prod_{i\in(\varphi_g\beta_G)^{-1}(a)} g_{\beta_G(i)}$ is taken in the order of elements i in $\mathcal{O}_G$ in the partial order of $\mathcal{O}_G$.

Theorem: *Any admissible map* $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ *induces a map (morphism) of global update functions* $\Phi : (X, f) \longrightarrow (Y, g)$*, i.e. a map* $\Phi : X \longrightarrow Y$ *such that* $\Phi f = g\Phi$*.*

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \Phi \downarrow & & \downarrow \Phi \\ Y & \xrightarrow{g} & Y \end{array}$$

In particular

- Sequential dynamical systems form a category,
- the construction of the global update function is a functor,
- the category of SDS has colimits (but no limits),
- admissible maps (simulations) $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ preserve fixed points,
- they map limit cycles onto limit cycles.