Optimal Transceiver Design for Multi-Access Communication*

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Main Points

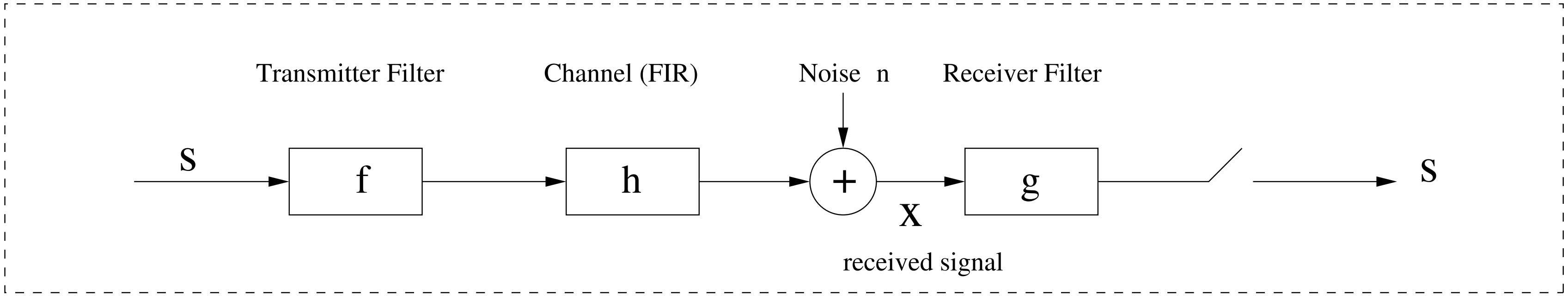
- An important problem in the management of communication networks: *resource allocation*
- Frequency, transmitting power; Goal: high data rate, low bit error rate
- Transmitter + receiver for multi-user high speed broadband Digital Subscribe Line application
- Direct formulation is nonconvex; equivalent formulation is SDP (thus convex);
- Further simplification to SOCP, and to combinatorial polynomial time algorithm
- Valuable guidelines and insights for optimal practical transceiver design

Content

- Elements of data communication: OFDM, subcarriers, power loading, precoding/equalization.
- Transceiver (Transmitter + Receiver) design for the two user multi-access channel:
  - SDP formulation
  - SOCP formulation
  - $O(n^3)$ strongly polynomial algorithm
- General multi-access channel
- Extensions to vector broadcasting channel

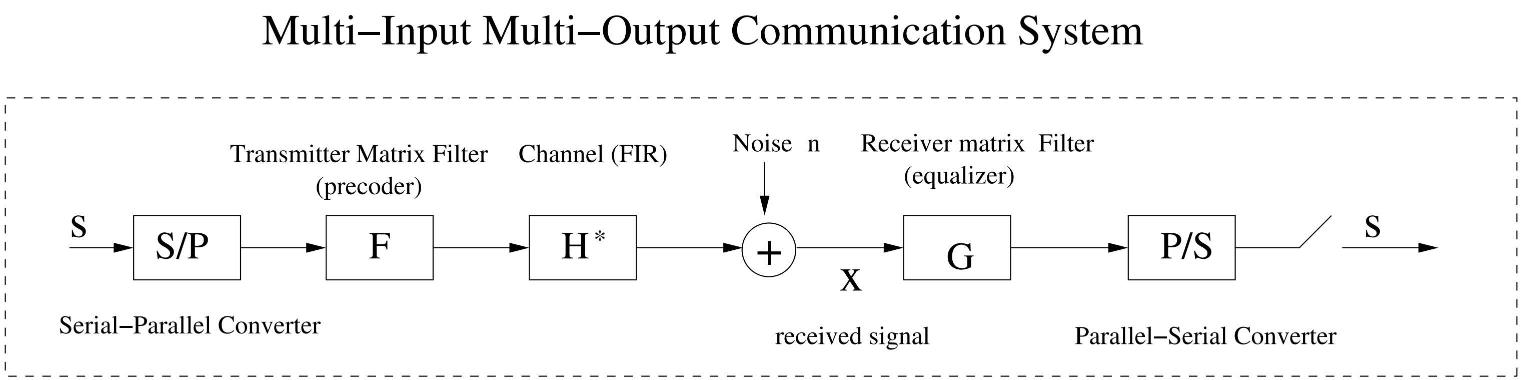
Single User SISO Communication System

Single Input Single Output Communication System

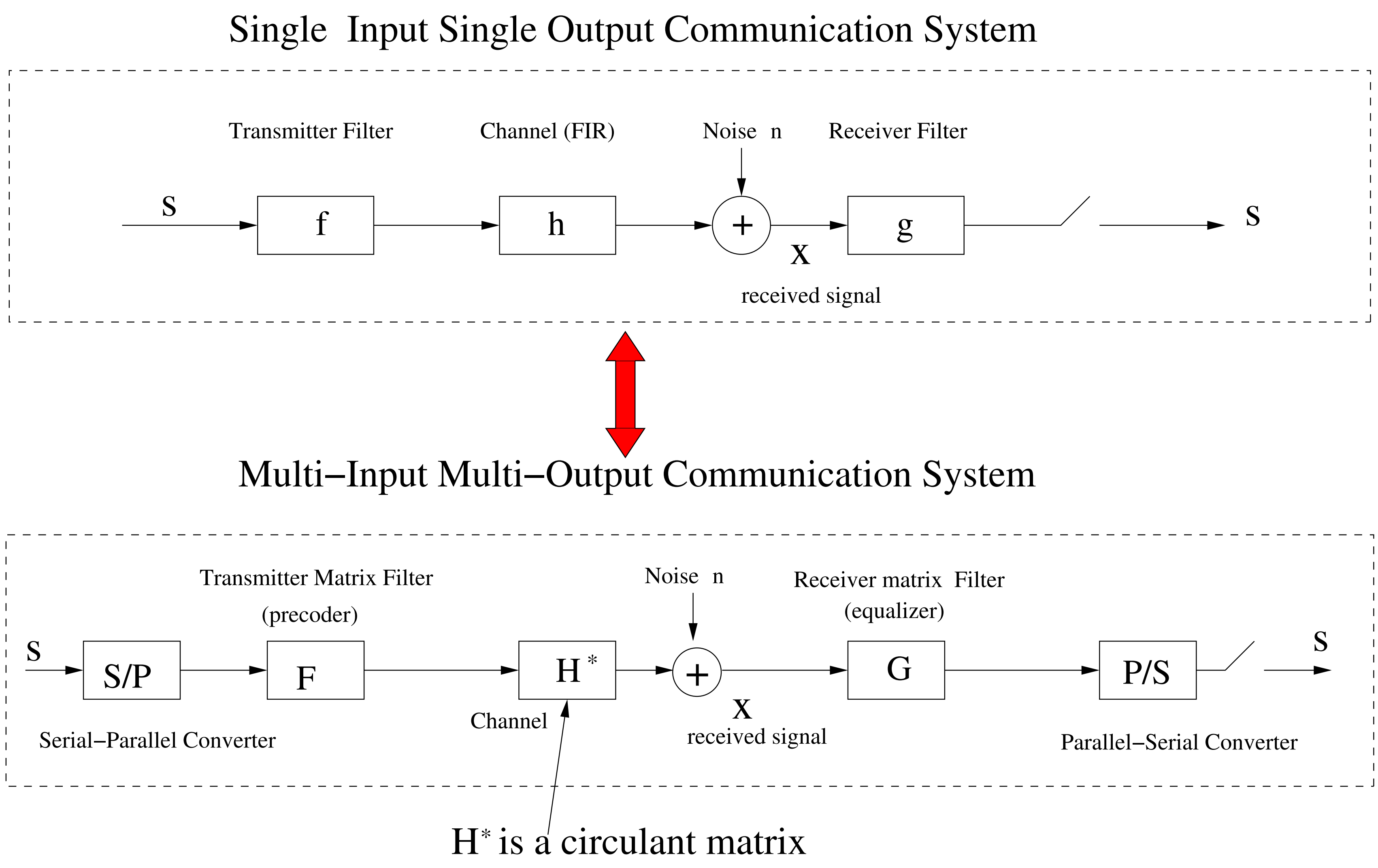


- s is input signal (assumed statistically white)
- $h \in \Re^k$ is a linear, time-invariant channel (assumed known)
- f, g are transmitter filter and equalizer filter respectively
- n is the additive (Gaussian) noise

An Equivalent MIMO System



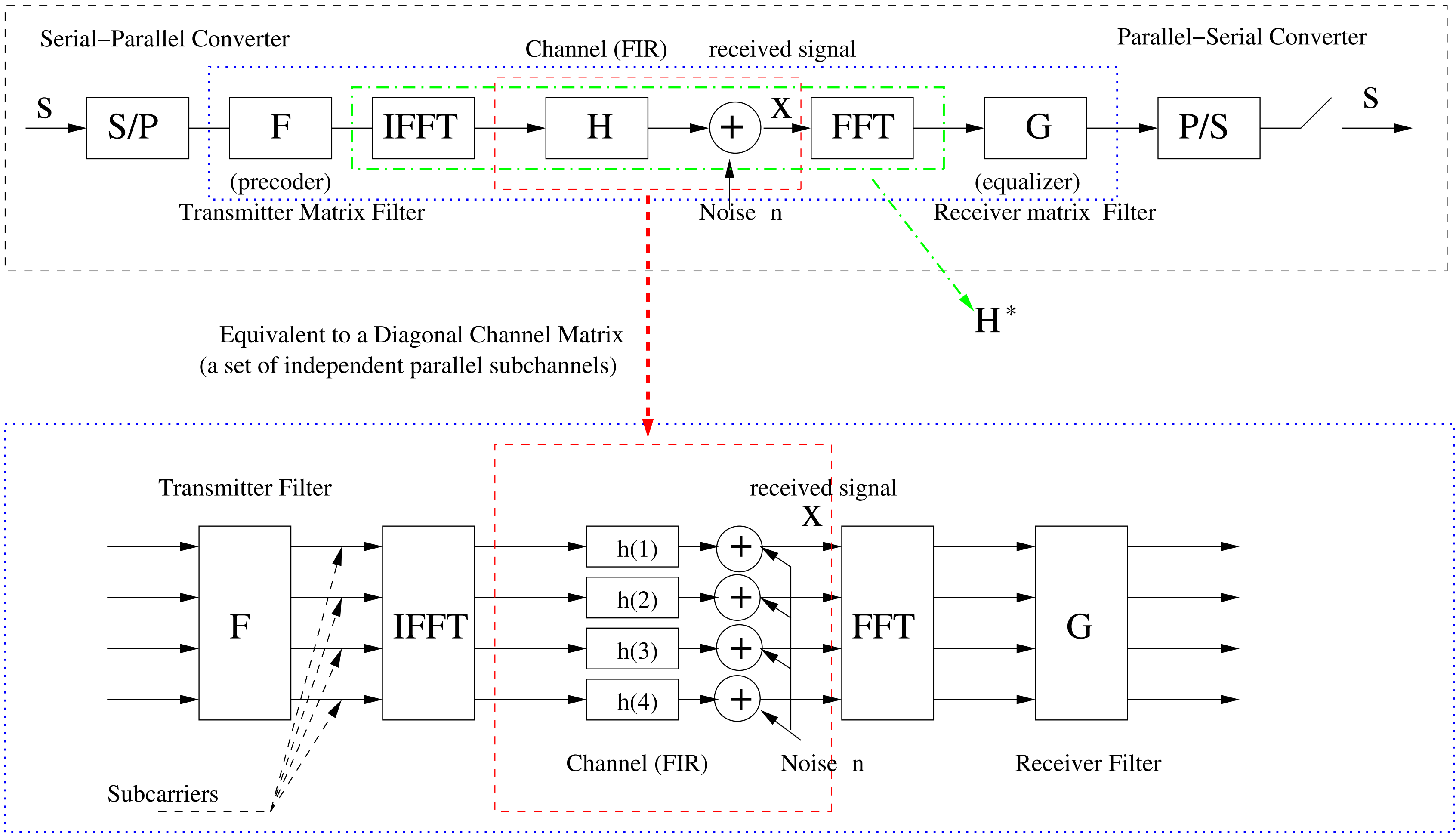
- S/P: serial-parallel converter (with cyclic prefixing); P/S: parallel-serial
- $\mathbf{F}$: $n \times \ell$ transmitter matrix filter (or precoder), obtained from f; data rate $= \ell/n$
- $\mathbf{G}$: $\ell \times n$ receiver equalizer matrix (obtained from g)
- $\mathbf{H}$: channel matrix (obtained from h); $\mathbf{n}$: noise
- $\mathbf{x} = \mathbf{H}^* \mathbf{F}\mathbf{s} + \mathbf{n}$, $\mathbf{H}^*$ is circulant.


Single Input Single Output Communication System
Transmitter Filter
Channel (FIR)
Noise n
Receiver Filter
S
f
h
+
g
X
S
received signal
Multi–Input Multi–Output Communication System
Transmitter Matrix Filter
(precoder)
Noise n
Receiver matrix Filter
(equalizer)
S
S/P
F
H*
+
G
P/S
S
X
Channel
received signal
Serial–Parallel Converter
Parallel–Serial Converter
H* is a circulant matrix

OFDM System

- The circulant channel matrix $\mathbf{H}^*$ can be diagonalized via IFFT/FFT: $\mathbf{H}^* = \mathbf{D}^\dagger \mathbf{H} \mathbf{D}$, where $\mathbf{D}$ is the standard discrete Fourier transform matrix, $\mathbf{H}$ is diagonal
- The diagonalized channel becomes a set of independent subchannels
- Each subchannel corresponds to a subcarrier with a particular frequency (from FFT)
- This diagonalization is not channel dependent
- Orthogonal Frequency Division Multiplexing System employs IFFT/FFT to decompose the channel $\mathbf{H}^*$

Multi–Input Multi–Output Communication System

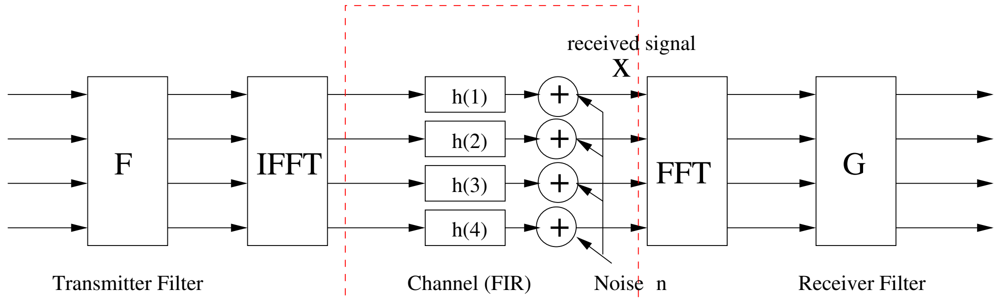


Orthogonal Frequence Division Multiplexing (OFDM) System

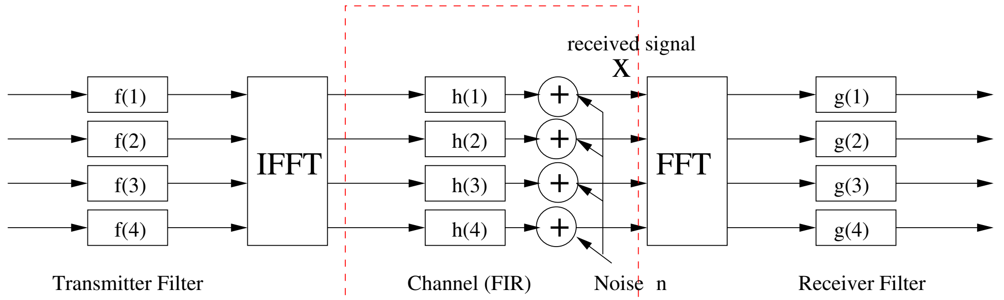
Linearly Precoded/Power Loaded OFDM

- The precoder $\mathbf{F}$ can be a general $n \times n$ matrix, subject to power constraint $\mathrm{tr}(\mathbf{F}\mathbf{F}^{\dagger}) \leq p$.
- The optimized design $\mathbf{F}$ will have a rank $\ell \leq n$, resulting in an optimal data rate ℓ/n.
- A special, and popular, linear precoder is the so called *power loading precoder*: $\mathbf{F}$ is diagonal.

Linearly Precoded/Power Loaded OFDM



General Linearly Precoded OFDM System



Power–Loaded OFDM System

Two-User Multi-Access Communication Channel

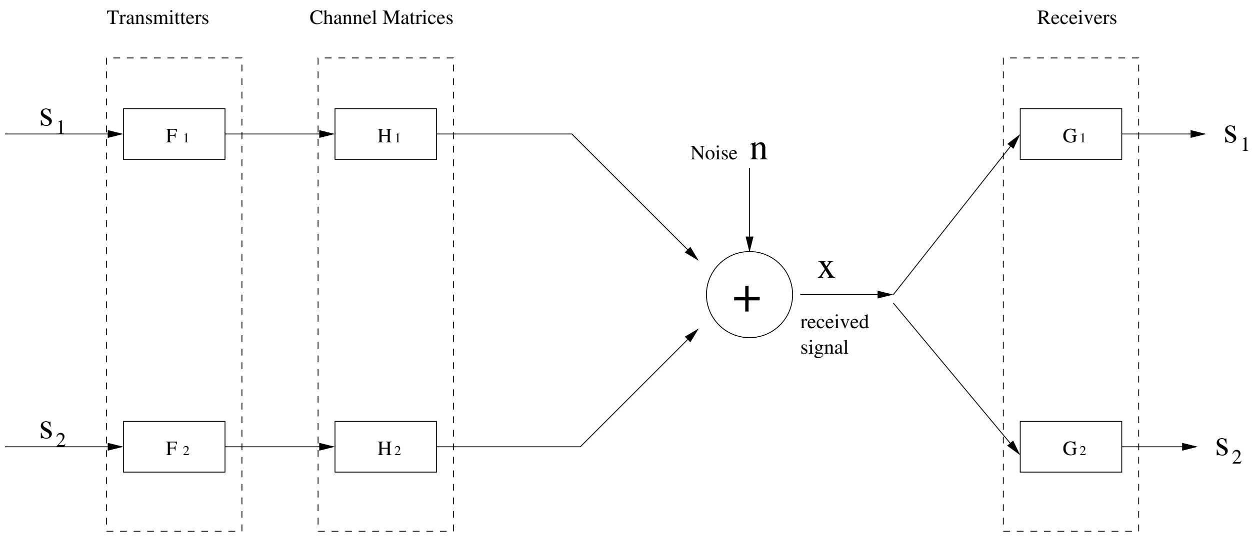


Diagram of Two–user Communication System

Mathematical model: $\mathbf{x} = \mathbf{H}_1\mathbf{F}_1\mathbf{s}_1 + \mathbf{H}_2\mathbf{F}_2\mathbf{s}_2 + \rho\mathbf{n}, \quad \rho > 0.$

Detection: $\mathbf{s}_i = \mathrm{sign}\left(\mathbf{G}_i\mathbf{x}\right), \ i = 1, 2.$

Goal: Given the channel matrices, $\mathbf{H}_1$, $\mathbf{H}_2$, design transceivers $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{G}_1$, $\mathbf{G}_2$.

Applications and Previous Work

- Applications include the current and future systems of DSL, DAB, DVB.
- Equalizer (or receiver) design for a fixed transmitter has been researched extensively in the last decade.
- The joint transmitter and receiver (transceiver) design was considered recently, but only for the single user case (Giannakis' group, UMN; Cioffi's group, Stanford; ...).
- In the single user transceiver design work, the design criteria used include:
  - Minimum Mean Square Error,
  - maximum information rate,
  - channel capacity
- The last two require complex receiver structures.

Mean Square Error

- The detection with receiver (equalizer) $\mathbf{G}_i$: $\hat{\mathbf{s}}_i = \text{sign}\left(\mathbf{G}_i\mathbf{x}\right)$.
- Let $\mathbf{e}_i$ denote the error vector (before making the hard decision) for user i, $i = 1, 2$. Then

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{G}_1\mathbf{x} - \mathbf{s}_1 = \mathbf{G}_1(\mathbf{H}_1\mathbf{F}_1\mathbf{s}_1 + \mathbf{H}_2\mathbf{F}_2\mathbf{s}_2 + \rho\mathbf{n}) - \mathbf{s}_1 \\ &= \left(\mathbf{G}_1\mathbf{H}_1\mathbf{F}_1 - \mathbf{I}\right)\mathbf{s}_1 + \mathbf{G}_1\mathbf{H}_2\mathbf{F}_2\mathbf{s}_2 + \rho\mathbf{G}_1\mathbf{n}. \end{aligned}$$

- This further implies

$$E(\mathbf{e}_1\mathbf{e}_1^\dagger) = \left(\mathbf{G}_1\mathbf{H}_1\mathbf{F}_1 - \mathbf{I}\right)\left(\mathbf{G}_1\mathbf{H}_1\mathbf{F}_1 - \mathbf{I}\right)^\dagger + \left(\mathbf{G}_1\mathbf{H}_2\mathbf{F}_2\right)\left(\mathbf{G}_1\mathbf{H}_2\mathbf{F}_2\right)^\dagger + \rho^2\mathbf{G}_1\mathbf{G}_1^\dagger$$

- Similarly, we have

$$E(\mathbf{e}_2\mathbf{e}_2^\dagger) = \left(\mathbf{G}_2\mathbf{H}_2\mathbf{F}_2 - \mathbf{I}\right)\left(\mathbf{G}_2\mathbf{H}_2\mathbf{F}_2 - \mathbf{I}\right)^\dagger + \left(\mathbf{G}_2\mathbf{H}_1\mathbf{F}_1\right)\left(\mathbf{G}_2\mathbf{H}_1\mathbf{F}_1\right)^\dagger + \rho^2\mathbf{G}_2\mathbf{G}_2^\dagger.$$

Formulation: MMSE Equalizer Case

- Our goal is to design a set of transmitting matrix filters $\mathbf{F}_i$ and a set of matrix equalizers $\mathbf{G}_i$ such that the total mean squared error

$$\mathrm{MSE} = \mathrm{tr}(E(\mathbf{e}_1\mathbf{e}_1^\dagger)) + \mathrm{tr}(E(\mathbf{e}_2\mathbf{e}_2^\dagger))$$

  is minimized.

- As is always the case in practice, there are power constraints on the transmitting matrix filters:

$$\mathrm{tr}(\mathbf{F}_1\mathbf{F}_1^\dagger) \le p_1, \quad \mathrm{tr}(\mathbf{F}_2\mathbf{F}_2^\dagger) \le p_2$$

- The above is nonconvex.
- We first eliminate the variables $\mathbf{G}_1$, $\mathbf{G}_2$: the MMSE equalizers.

Formulation: MMSE Equalizer Case

- By minimizing $E(\mathbf{e}_1\mathbf{e}_1^\dagger)$ with respect to $\mathbf{G}_1$, we obtain the following MMSE equalizer for user 1: $\boxed{\mathbf{G}_1 = \mathbf{F}_1^\dagger \mathbf{H}_1^\dagger \mathbf{W},}$ where

$$\boxed{\mathbf{W} = \left(\mathbf{H}_1\mathbf{F}_1\mathbf{F}_1^\dagger\mathbf{H}_1^\dagger + \mathbf{H}_2\mathbf{F}_2\mathbf{F}_2^\dagger\mathbf{H}_2^\dagger + \rho^2\mathbf{I}\right)^{-1}.}$$

- Substituting this into $E(\mathbf{e}_1\mathbf{e}_1^\dagger)$ gives:

$$E(\mathbf{e}_1\mathbf{e}_1^\dagger) = -\mathbf{F}_1^\dagger\mathbf{H}_1^\dagger\mathbf{W}\mathbf{H}_1\mathbf{F}_1 + \mathbf{I}.$$

- Similarly, the MMSE equalizer $\mathbf{G}_2$ for user 2 is given by $\boxed{\mathbf{G}_2 = \mathbf{F}_2^\dagger\mathbf{H}_2^\dagger\mathbf{W}}$ and resulting minimized (with respect to $\mathbf{G}_2$) mean square error for user 2 is given by:

$$E(\mathbf{e}_2\mathbf{e}_2^\dagger) = -\mathbf{F}_2^\dagger\mathbf{H}_2^\dagger\mathbf{W}\mathbf{H}_2\mathbf{F}_2 + \mathbf{I}.$$

Total MSE

Substituting into the above expression gives rise to

$$\begin{aligned}
\text{MSE} &= \text{tr}(E(\mathbf{e}_1\mathbf{e}_1^\dagger)) + \text{tr}(E(\mathbf{e}_2\mathbf{e}_2^\dagger)) \\
&= -\text{tr}\left(\mathbf{F}_1^\dagger\mathbf{H}_1^\dagger\mathbf{W}\mathbf{H}_1\mathbf{F}_1\right) - \text{tr}\left(\mathbf{F}_2^\dagger\mathbf{H}_2^\dagger\mathbf{W}\mathbf{H}_2\mathbf{F}_2\right) + 2n \\
&= -\text{tr}\left(\mathbf{W}\mathbf{H}_1\mathbf{F}_1\mathbf{F}_1^\dagger\mathbf{H}_1^\dagger\right) - \text{tr}\left(\mathbf{W}\mathbf{H}_2\mathbf{F}_2\mathbf{F}_2^\dagger\mathbf{H}_2^\dagger\right) + 2n \\
&= -\text{tr}\left(\mathbf{W}(\mathbf{H}_1\mathbf{F}_1\mathbf{F}_1^\dagger\mathbf{H}_1^\dagger + \mathbf{H}_2\mathbf{F}_2\mathbf{F}_2^\dagger\mathbf{H}_2^\dagger)\right) + 2n \\
&= \rho^2\text{tr}\left(\mathbf{W}\right) + n,
\end{aligned}$$

where the last step follows from the definition of $\mathbf{W}$.

Formulation: MMSE Equalizer Case

- Introduce matrix variables: $\mathbf{U}_1 = \mathbf{F}_1\mathbf{F}_1^\dagger, \quad \mathbf{U}_2 = \mathbf{F}_2\mathbf{F}_2^\dagger.$
- Then the MMSE transceiver design problem becomes

$$\begin{array}{ll} \text{minimize}_{\mathbf{U}_1,\mathbf{U}_2} & \mathrm{tr}\left((\mathbf{H}_1\mathbf{U}_1\mathbf{H}_1^\dagger + \mathbf{H}_2\mathbf{U}_2\mathbf{H}_2^\dagger + \rho^2\mathbf{I})^{-1}\right) \\ \text{subject to} & \mathrm{tr}(\mathbf{U}_1) \le p_1, \quad \mathrm{tr}(\mathbf{U}_2) \le p_2, \\ & \mathbf{U}_1 \succeq \mathbf{0}, \quad \mathbf{U}_2 \succeq \mathbf{0}. \end{array}$$

- Reformulate using the auxiliary matrix variable $\mathbf{W}$:

$$\begin{array}{ll} \text{minimize}_{\mathbf{W},\mathbf{U}_1,\mathbf{U}_2} & \mathrm{tr}\,(\mathbf{W}) \\ \text{subject to} & \mathrm{tr}(\mathbf{U}_1) \le p_1, \quad \mathrm{tr}(\mathbf{U}_2) \le p_2, \\ & \mathbf{W} \succeq (\mathbf{H}_1\mathbf{U}_1\mathbf{H}_1^\dagger + \mathbf{H}_2\mathbf{U}_2\mathbf{H}_2^\dagger + \rho^2\mathbf{I})^{-1} \\ & \mathbf{U}_1 \succeq \mathbf{0}, \quad \mathbf{U}_2 \succeq \mathbf{0}. \end{array}$$

SDP Formulation

- The constraint $\mathbf{W} \succeq (\mathbf{H}_1\mathbf{U}_1\mathbf{H}_1^\dagger + \mathbf{H}_2\mathbf{U}_2\mathbf{H}_2^\dagger + \rho^2\mathbf{I})^{-1}$ is equivalent to LMI:

$$\begin{bmatrix} \mathbf{W} & \mathbf{I} \\ \mathbf{I} & \mathbf{H}_1\mathbf{U}_1\mathbf{H}_1^\dagger + \mathbf{H}_2\mathbf{U}_2\mathbf{H}_2^\dagger + \rho^2\mathbf{I} \end{bmatrix} \succeq 0. \qquad (3)$$

- We obtain an SDP formulation:

$$\begin{array}{ll} \text{minimize}_{\mathbf{W},\mathbf{U}_1,\mathbf{U}_2} & \mathrm{tr}\,(\mathbf{W}) \\ \text{subject to} & \mathrm{tr}(\mathbf{U}_1) \le p_1, \quad \mathrm{tr}(\mathbf{U}_2) \le p_2, \\ & \mathbf{W} \text{ satisfies (3)}, \\ & \mathbf{U}_1 \succeq \mathbf{0}, \quad \mathbf{U}_2 \succeq \mathbf{0}. \end{array}$$

- Interior point method with arithmetic complexity $O(n^{6.5}\log(1/\epsilon))$, $\epsilon > 0$ is the solution accuracy.

OFDM: Diagonal Designs are Optimal!

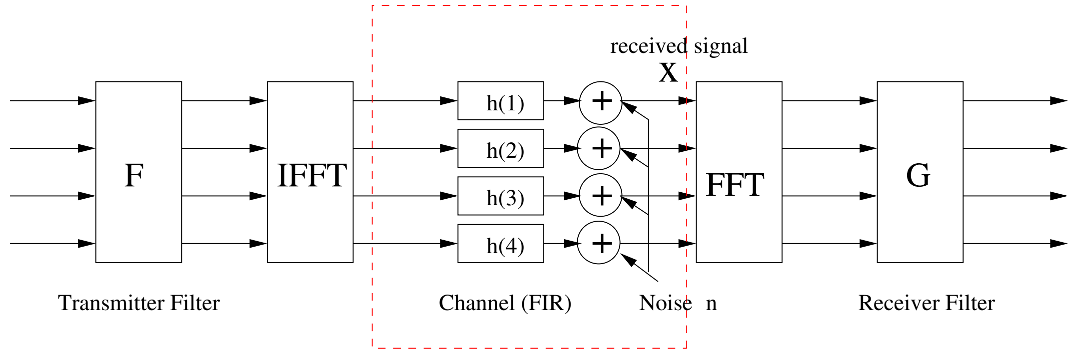
Result

If $\mathbf{H}_1$ and $\mathbf{H}_2$ are diagonal, as in the OFDM systems, then the optimal transmitters are also diagonal.

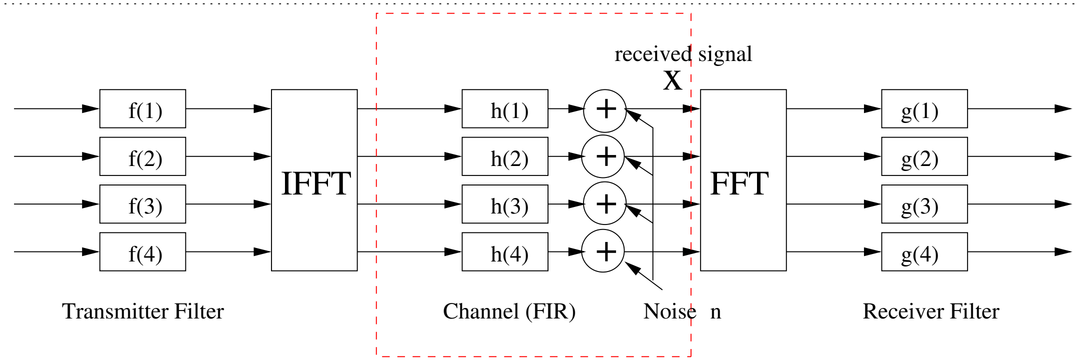
Implication

The MMSE transceivers for an multi-user OFDM system can be implemented by optimally setting the data rates and allocating power to each subcarrier for all the users.

Linearly Precoded/Power Loaded OFDM



General Linearly Precoded OFDM System



Power–Loaded OFDM System

From SDP to SOCP Formulation

- Restricting to diagonal designs, the SDP becomes SOCP:

$$\begin{array}{ll} \text{minimize}_{\mathbf{w},\mathbf{u}_1,\mathbf{u}_2} & \displaystyle\sum_{i=1}^{n} \mathbf{w}_i \\ \text{subject to} & \displaystyle\sum_{i=1}^{n} \mathbf{u}_1(i) \le p_1, \quad \sum_{i=1}^{n} \mathbf{u}_2(i) \le p_2, \\ & \mathbf{w}_i \left(|\mathbf{h}_1(i)|^2 \mathbf{u}_1(i) + |\mathbf{h}_2(i)|^2 \mathbf{u}_2(i) + \rho^2 \right) \ge 1, \\ & \mathbf{u}_1(i) \ge 0, \quad \mathbf{u}_2(i) \ge 0, \quad i = 1, 2, ..., n. \end{array}$$

- There exist highly efficient (general purpose) interior point methods to solve the above second order cone program.
- Arithmetic complexity $O(n^{3.5} \log(1/\epsilon))$, $\epsilon > 0$ is the accuracy.

Subcarrier Allocation

- OFDM channel with optimal transceivers:

$$\mathbf{x} = \mathbf{H}_1\mathbf{F}_1\mathbf{s}_1 + \mathbf{H}_2\mathbf{F}_2\mathbf{s}_2 + \rho\mathbf{n}, \text{ with } \mathbf{H}_i,\ \mathbf{F}_i \text{ diagonal.}$$

- Consider the two subcarrier case and assuming equal power p to the two subcarriers:

$$\begin{aligned}\mathbf{x}(1) &= 0.01e^{0.2j}\mathbf{f}_1(1)\mathbf{s}_1(1) + 0.7e^{2.2j}\mathbf{f}_2(1)\mathbf{s}_2(1) + \rho^2\mathbf{n}(1),\\ \mathbf{x}(2) &= 0.8e^{1.7j}\mathbf{f}_1(2)\mathbf{s}_1(2) + 0.5e^{0.7j}\mathbf{f}_2(2)\mathbf{s}_2(2) + \rho^2\mathbf{n}(2),\end{aligned}$$

  – if subcarrier 1 is allocated to user 1, subcarrier 2 is allocated to user 2, then $\mathbf{f}_2(1) = \mathbf{f}_1(2) = 0,\ \mathbf{f}_1(1) = \mathbf{f}_2(2) = \sqrt{p}$

$$\text{SINR for user 1} = \frac{0.01^2 p}{\rho^2}, \quad \text{SINR for user 1} = \frac{0.5^2 p}{\rho^2}$$

  – if subcarrier 1 is allocated to user 2, subcarrier 2 is allocated to user 1, then $\mathbf{f}_1(1) = \mathbf{f}_2(2) = 0,\ \mathbf{f}_2(1) = \mathbf{f}_1(2) = \sqrt{p}$

$$\text{SINR for user 1} = \frac{0.8^2 p}{\rho^2}, \quad \text{SINR for user 1} = \frac{0.7^2 p}{\rho^2}$$

Properties of Optimal MMSE Transceiver

- Let $\mathbf{u}_1^* \geq 0$, $\mathbf{u}_2^* \geq 0$ be the optimal transceivers. Define:

$$\begin{cases} I_1 = \{i \mid \mathbf{u}_1^*(i) > 0,\ \mathbf{u}_2^*(i) = 0\}, & I_2 = \{i \mid \mathbf{u}_1^*(i) = 0,\ \mathbf{u}_2^*(i) > 0\}, \\ I_s = \{i \mid \mathbf{u}_1^*(i) > 0,\ \mathbf{u}_2^*(i) > 0\}, & I_u = \{i \mid \mathbf{u}_1^*(i) = 0,\ \mathbf{u}_2^*(i) = 0\}. \end{cases}$$

- I_1, I_2: subcarriers allocated to user 1 and user 2;
  I_s and I_u: subcarriers *shared* and *unused*;
  data rates: $(|I_1| + |I_s|)/n$, $(|I_2| + |I_s|)/n$

-
  - For each $i \in I_1$ and $j \in I_2$, we have $\boxed{\dfrac{|\mathbf{h}_1(i)|^2}{|\mathbf{h}_2(i)|^2} \geq \dfrac{|\mathbf{h}_1(j)|^2}{|\mathbf{h}_2(j)|^2}.}$
  - For all $i, j \in I_s$, we have $\boxed{\dfrac{|\mathbf{h}_1(i)|^2}{|\mathbf{h}_2(i)|^2} = \dfrac{|\mathbf{h}_1(j)|^2}{|\mathbf{h}_2(j)|^2}.}$
  - For any $i \in I_u$ and any $j \in I_1 \cup I_s$, we have $|\mathbf{h}_1(i)|^2 < |\mathbf{h}_1(j)|^2$. Similarly, for any $i \in I_u$ and any $j \in I_2 \cup I_s$, we have $|\mathbf{h}_2(i)|^2 < |\mathbf{h}_2(j)|^2$.

Intuitive Interpretation

- $\mathbf{x} = \mathbf{H}_1\mathbf{F}_1\mathbf{s}_1 + \mathbf{H}_2\mathbf{F}_2\mathbf{s}_2 + \rho\mathbf{n}$, with $\mathbf{H}_i$, $\mathbf{F}_i$ diagonal;
  $\mathbf{x}(i) = \mathbf{h}_1(i)\mathbf{f}_1(i)\mathbf{s}_1(i) + \mathbf{h}_2(i)\mathbf{f}_2(i)\mathbf{s}_2(i) + \rho^2\mathbf{n}(i)$.
- In a fading environment, the path gains $|\mathbf{h}_1(i)|^2$, $|\mathbf{h}_2(i)|^2$ are random,

  $\Rightarrow$ the probability of having two equal path gains is zero.

  $\Rightarrow$ I_s is singleton: *at most one subcarrier should be shared by the two users.*
- The remaining subcarriers are allocated to the two users according to the path gain ratios: subcarrier i to user 1 and subcarrier j to user 2 only if

$$\boxed{\frac{|\mathbf{h}_1(i)|^2}{|\mathbf{h}_2(i)|^2} \geq \frac{|\mathbf{h}_1(j)|^2}{|\mathbf{h}_2(j)|^2}.}$$

- The subcarriers in I_u have small path gains for both users (i.e., both $|\mathbf{h}_1(i)|^2$ and $|\mathbf{h}_2(i)|^2$ are small), and they should not be used by either user, i.e., they are useless subcarriers!

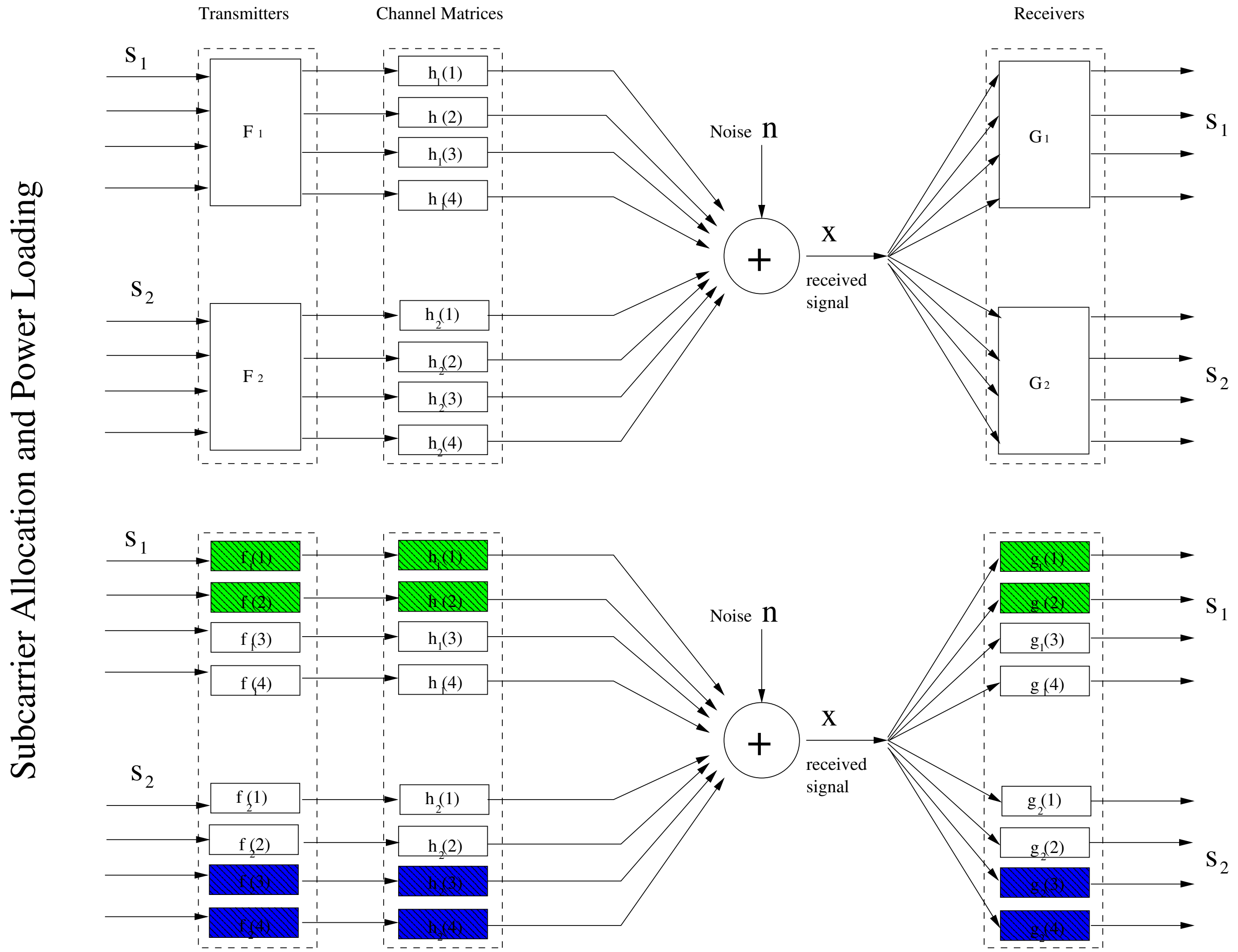
A Strongly Polynomial Time Algorithm

- The properties of optimal MMSE transceivers can be used to design a *combinatorial* algorithm.
- Assume

$$\frac{|\mathbf{h}_1(1)|^2}{|\mathbf{h}_2(1)|^2} > \frac{|\mathbf{h}_1(2)|^2}{|\mathbf{h}_2(2)|^2} > \cdots > \frac{|\mathbf{h}_1(n-1)|^2}{|\mathbf{h}_2(n-1)|^2} > \frac{|\mathbf{h}_1(n)|^2}{|\mathbf{h}_2(n)|^2}.$$

- Then $I_1 \subseteq \{1, ..., i\}$ and $I_2 \subseteq \{i, ..., n\}$ for some i.
- Leads to an $O(n^3)$ strongly polynomial time (combinatorial) algorithm (vs. $O(n^{3.5} \log 1/\epsilon)$ interior point algorithm for SOCP).

Practical Implications



General m-User Case

- Mathematical model:

$$\mathbf{x} = \mathbf{H}_1\mathbf{F}_1\mathbf{s}_1 + \mathbf{H}_2\mathbf{F}_2\mathbf{s}_2 + \cdots + \mathbf{H}_m\mathbf{F}_m\mathbf{s}_m + \rho\mathbf{n}.$$

- Let $\mathbf{G}_i$ be the linear MMSE matrix equalizer at the i-th receiver. Then the total MSE is given by

$$\rho^2 \mathrm{tr}\left((\mathbf{H}_1\mathbf{F}_1\mathbf{F}_1^\dagger\mathbf{H}_1^\dagger + \cdots + \mathbf{H}_i\mathbf{F}_i\mathbf{F}_i^\dagger\mathbf{H}_i^\dagger + \rho^2\mathbf{I})^{-1} \right) + (m-1)n.$$

- Let $\mathbf{U}_i = \mathbf{F}_i\mathbf{F}_i^\dagger$. Then the power constrained optimal MMSE transmitter design problem can be described as:

$$\begin{array}{ll} \text{minimize}_{\mathbf{U}_1,\ldots,\mathbf{U}_m} & \mathrm{tr}\left((\mathbf{H}_1\mathbf{U}_1\mathbf{H}_1^\dagger + \cdots + \mathbf{H}_m\mathbf{U}_m\mathbf{H}_m^\dagger + \rho^2\mathbf{I})^{-1} \right) \\ \text{subject to} & \mathrm{tr}(\mathbf{U}_i) \le p_i, \quad \mathbf{U}_i \succeq \mathbf{0}, \quad i = 1, \ldots, m. \end{array}$$

SDP/SOCP Formulation

SDP formulation

$$\begin{aligned} \text{minimize} \quad & \mathrm{tr}\,(\mathbf{W}) \\ \text{subject to} \quad & \mathrm{tr}(\mathbf{U}_i) \le p_i, \quad \mathbf{U}_i \succeq \mathbf{0}, \quad i = 1, 2, ..., m, \\ & \begin{bmatrix} \mathbf{W} & \mathbf{I} \\ \mathbf{I} & \mathbf{H}_1\mathbf{U}_1\mathbf{H}_1^\dagger + \cdots + \mathbf{H}_m\mathbf{U}_m\mathbf{H}_m^\dagger + \rho^2\mathbf{I} \end{bmatrix} \succeq 0. \end{aligned}$$

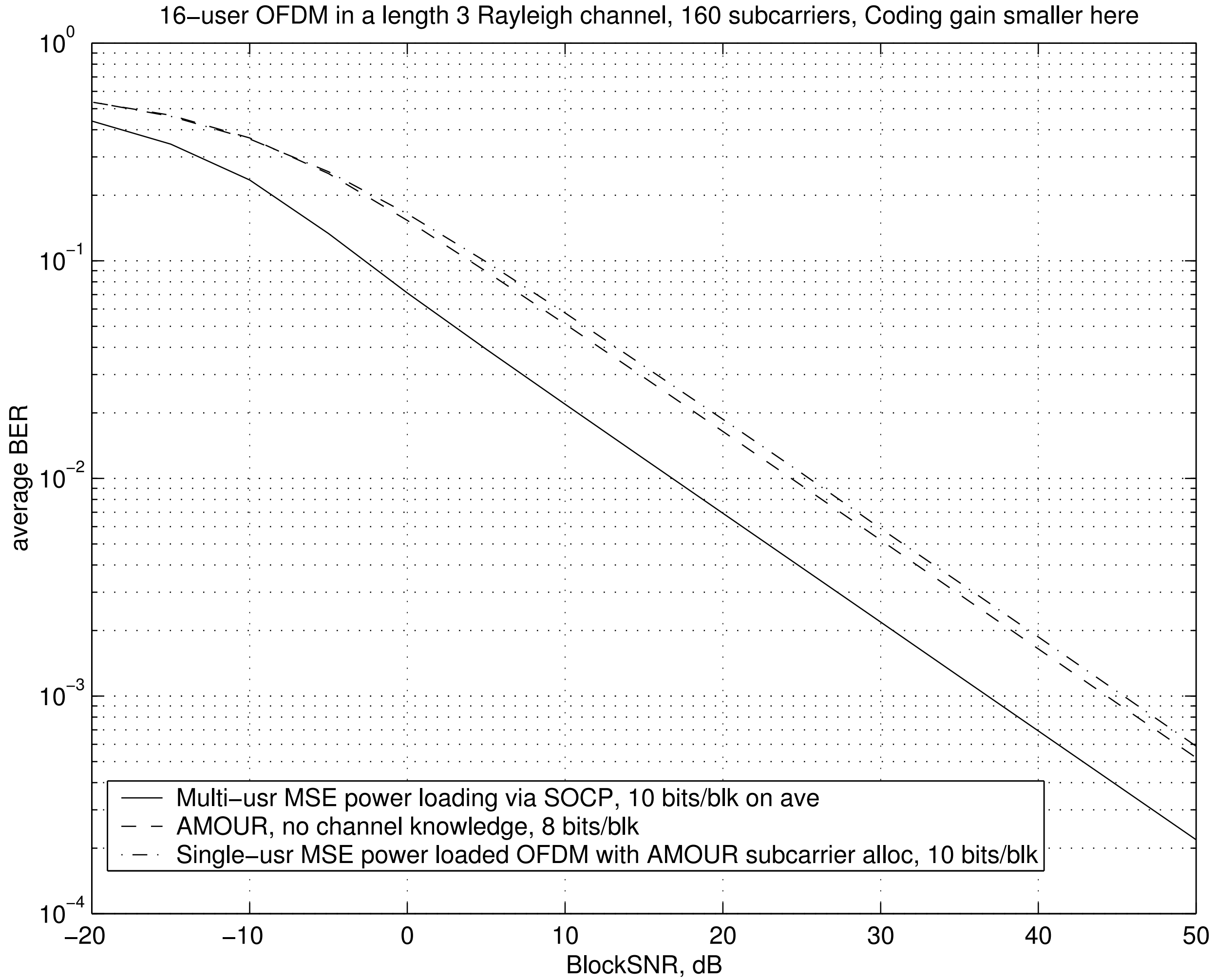
SOCP formulation

$$\begin{aligned} \text{minimize}_{\mathbf{w},\mathbf{u}_1,\ldots,\mathbf{u}_m} \quad & \sum_{i=1}^n \mathbf{w}_i \\ \text{subject to} \quad & \sum_{i=1}^n \mathbf{u}_j(i) \le p_j, \quad j = 1, 2, ..., m, \\ & \mathbf{w}_i \left(|\mathbf{h}_1(i)|^2 \mathbf{u}_1(i) + \cdots + |\mathbf{h}_m(i)|^2 \mathbf{u}_m(i) + \rho^2 \right) \ge 1, \\ & \mathbf{u}_j(i) \ge 0, \quad i = 1, 2, ..., n, \; j = 1, 2, ..., m. \end{aligned}$$

Simulation Scenario

- Uplink with 16 active users and 160 available subcarriers
- Each user “sees” its own Rayleigh channel (complex-valued)
- Three schemes:
  1. AMOUR – No channel knowledge; Each user uses 10 subcarriers, spreads 8 bits over these carriers using a DFT-type spreading.
  2. Individually MMSE power-loaded OFDM – Same subcarrier allocation as AMOUR. Each user sends 1 bit per subcarrier, i.e 10 bits per block; knows its allocated channels and does MMSE power loading for these bits.
  3. Multi-user MMSE power loaded OFDM – Using the SOCP formulation. In this case the subcarrier allocations and the number of bits per block vary from block to block, but the average number of bits per block remains 10.

Simulation Results

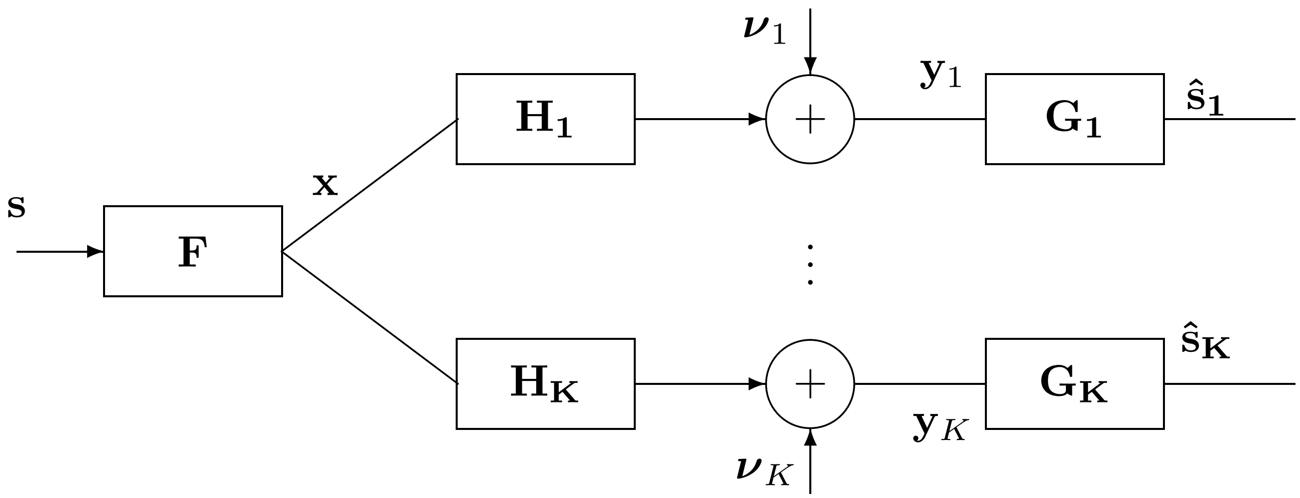


Efficiency of the Design Approach

On a PIII 600Mhz PC,

- Two users, 2 symbols per block, length 3 channel;
  - SDP $\sim$ 0.65 *secs*
  - SOCP $\sim$ 0.13 *secs*
- 16 users, 10 symbols per block, length 3 channel
  - SOCP $\sim$ 0.65 *secs*

Extensions to Vector Broadcasting Channel



$$\mathbf{y}_i = \mathbf{H}_i \mathbf{F} \mathbf{s} + \boldsymbol{\nu}_i, \quad i = 1, 2, ..., K$$

where $\boldsymbol{\nu}_i$ is the zero-mean Gaussian noise associated with the i^{th} receiver, which has a known covariance matrix $\mathbf{R}_i$.

Optimal Transceiver for Vector Broadcasting Channel

- The equalizer output is

$$\hat{\mathbf{s}}_i = \mathbf{G}_i \mathbf{H}_i \mathbf{F} \mathbf{s} + \mathbf{G}_i \boldsymbol{\nu}_i, \quad i = 1, \ldots, K. \tag{1}$$

- Let $\mathbf{e}_i$ denote the error vector associated with the i^{th} receiver,

$$\mathbf{e}_i = \mathbf{s} - \hat{\mathbf{s}}_i.$$

- The weighted MSE is given by

$$\mathrm{WMSE} = \sum_{i=1}^{K} \alpha_i \,\mathrm{Tr}(\mathrm{E}\{\mathbf{e}_i \mathbf{e}_i^H\}), \tag{2}$$

where the α_i's are non-negative weights.

Optimal Transceiver for Vector Broadcasting Channel

- The covariance matrix of $\mathbf{e}_i$ is

$$\mathrm{E}\{\mathbf{e}_i\mathbf{e}_i^H\} = (\mathbf{I} - \mathbf{G}_i\mathbf{H}_i\mathbf{F})(\mathbf{I} - \mathbf{G}_i\mathbf{H}_i\mathbf{F})^H + \mathbf{G}_i\mathbf{R}_i\mathbf{G}_i^H$$

  where the covariance matrices of the signal and noise are $\mathrm{E}\{\mathbf{s}\mathbf{s}^H\} = \mathbf{I}$ and $\mathrm{E}\{\boldsymbol{\nu}_i\boldsymbol{\nu}_i^H\} = \mathbf{R}_i$ respectively.

- The problem of designing $\mathbf{F}$ and $\mathbf{G}_i$ so as to minimize the weighted MSE subject to a bound on the transmitted power can be cast as the following optimization problem:

$$\min_{\mathbf{F},\mathbf{G}_1,\ldots,\mathbf{G}_K} \quad \sum_{i=1}^{K} \alpha_i \,\mathrm{Tr}(\mathrm{E}\{\mathbf{e}_i\mathbf{e}_i^H\}) \tag{3a}$$

$$\text{subject to} \quad \mathrm{Tr}(\mathbf{F}\mathbf{F}^H) \le p. \tag{3b}$$

Optimal Transceiver for Vector Broadcasting Channel

- Since the $\mathbf{G}_i$'s are unconstrained variables, they can be eliminated from (3) by first minimizing the weighted MSE with respect to $\mathbf{G}_i$. This results in the MMSE equalizers,

$$\mathbf{G}_i = \mathbf{F}^H \mathbf{H}_i^H \big(\mathbf{H}_i \mathbf{F}\mathbf{F}^H \mathbf{H}_i^H + \mathbf{R}_i\big)^{-1} = \mathbf{F}^H \mathbf{H}_i^H \mathbf{W}_i, \qquad (4)$$

where

$$\mathbf{W}_i = \big(\mathbf{H}_i \mathbf{F}\mathbf{F}^H \mathbf{H}_i^H + \mathbf{R}_i\big)^{-1}, \quad i = 1, \dots, K. \qquad (5)$$

Substituting (4) into $(3a)$ yields

$$\begin{aligned} \mathrm{WMSE} &= n \sum_{i=1}^{K} \alpha_i - \mathrm{Tr}\Big(\sum_{i=1}^{K} \alpha_i \mathbf{F}^H \mathbf{H}_i^H \mathbf{W}_i \mathbf{H}_i \mathbf{F}\Big) \\ &= \mathrm{Tr}\Big(\sum_{i=1}^{K} \alpha_i \mathbf{W}_i \mathbf{R}_i\Big). \end{aligned}$$

Optimal Transceiver for Vector Broadcasting Channel

- Noting that $\mathbf{R}_i$ and $\mathbf{W}_i$ are positive definite for all $i = 1, \ldots, K$, and letting $\mathbf{U} = \mathbf{F}\mathbf{F}^H$, the optimal MMSE transceiver design problem can be cast as:

$$\begin{aligned}
&\min_{\mathbf{U}, \mathbf{W}_i,\, i=1,\ldots,K} && \mathrm{Tr}\Big(\sum_{i=1}^{K} \alpha_i \mathbf{W}_i \mathbf{R}_i\Big) && \text{(6a)}\\
&\text{subject to} && \mathbf{W}_i \succeq (\mathbf{H}_i \mathbf{U} \mathbf{H}_i^H + \mathbf{R}_i)^{-1}, \quad \forall i && \text{(6b)}\\
& && \mathrm{Tr}(\mathbf{U}) \le p, && \text{(6c)}\\
& && \mathbf{U} \succeq \mathbf{0}. && \text{(6d)}
\end{aligned}$$

- Using the Schur complement, the constraint (6b) can be re-written in a linear matrix inequality (LMI) form as:

$$\begin{bmatrix} \mathbf{W}_i & \mathbf{I} \\ \mathbf{I} & \mathbf{H}_i \mathbf{U} \mathbf{H}_i^H + \mathbf{R}_i \end{bmatrix} \succeq \mathbf{0}, \qquad i = 1, \ldots, K. \tag{7}$$

SDP and SOCP formulations

- Therefore, the formulation in (6) can be rewritten as:

$$\begin{aligned} \min_{\mathbf{U},\mathbf{W}_i} \quad & \mathrm{Tr}\Big(\sum_{i=1}^{K} \alpha_i \mathbf{W}_i \mathbf{R}_i\Big) && \text{(8a)} \\ \text{subject to} \quad & (7),\ \mathrm{Tr}(\mathbf{U}) \le p, \text{ and } \mathbf{U} \succeq \mathbf{0} && \text{(8b)} \end{aligned}$$

- If the channel matrices $\mathbf{H}_i$ are diagonal, then the optimal matrices $\mathbf{W}_i$ and $\mathbf{U}$ are also diagonal.
- We can replace $\mathbf{W}_i$ in (8) by vectors $\mathbf{w}_i$ and $\mathbf{u}$ that represent the diagonal elements of $\mathbf{W}_i$ and $\mathbf{U}$ respectively to obtain

$$\begin{aligned} \min_{\mathbf{u},\mathbf{w}_i} \quad & \sum_{i=1}^{K}\sum_{k=1}^{n} \alpha_i \sigma_i^2(k)\mathbf{w}_i(k) && \text{(9a)} \\ \text{subject to} \quad & \mathbf{w}_i(k)\big(\mathbf{u}(k)|H_i(k)|^2 + \sigma_i^2(k)\big) \ge 1, \quad \forall i && \text{(9b)} \\ & \sum_k \mathbf{u}(k) \le p, \ \ \mathbf{u}(k) \ge 0, \quad \forall k = 1, \ldots, n. && \text{(9c)} \end{aligned}$$

A Simulation Example

- Consider a two user broadcast system which employs cyclic prefix based multicarrier modulation with 32 subcarriers.
- Each user's channel is a three-tap FIR filter. The frequency responses of these filters are plotted in Figure 1, along with the optimal power allocation. This figure shows that optimal power loading, in the MMSE sense, at an SNR of 10 dB, assigns power in a way similar to the water-filling principle.
- In Figure 2, we consider the same channels as in Figure 1. We compare the bit error rate performance when optimal and uniform power loadings are used. We observe that for a bit error rate of 2×10^{-2}, a gain of about 3 dB is obtained via optimal power loading.

Simulation Result

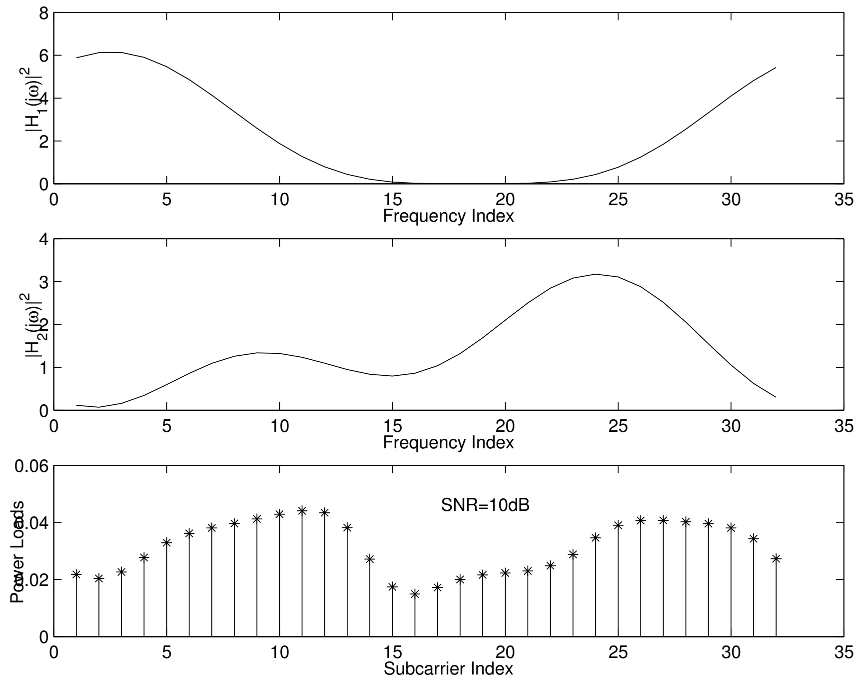


Figure 1: Optimal power loading in the MMSE sense at a block SNR of 10 dB. The frequency responses of the two users' channels are shown.

Simulation Result

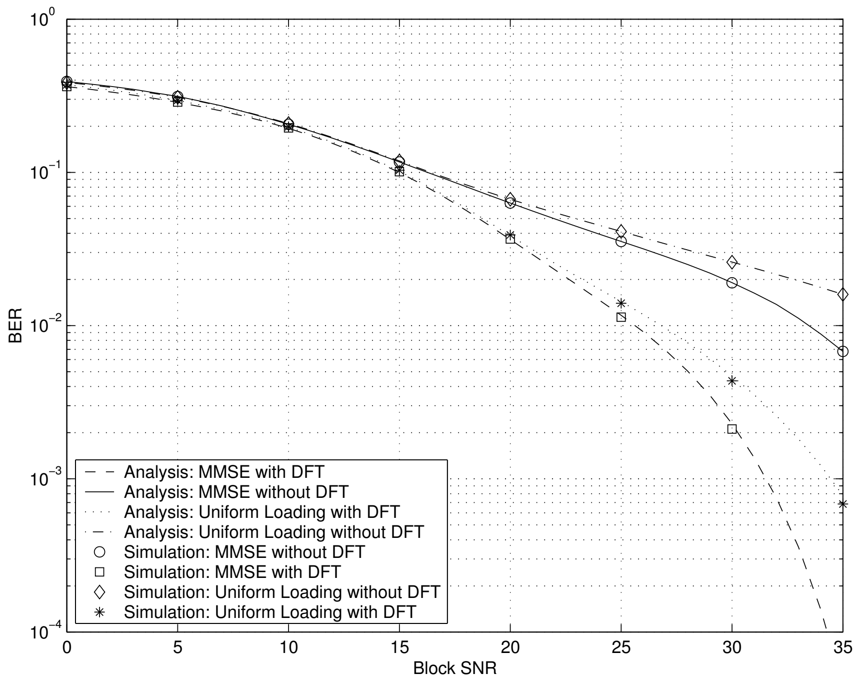


Figure 2: A comparison of the bit error rate performance of various systems: Analytical and simulation results.

Summary

- So far we have
  - Presented various SDP/SOCP formulations and algorithms for the optimal transceiver design problems
  - Studied the properties of the optimal transceiver designs.
  - Demonstrated the potential of SDP/SOCP/interior point methods in digital communication.
  - Results provide valuable guidelines and insights for the practical system design.
- Future work
  - More simulation results
  - Incorporating QoS and other receiver structures in the formulation.
  - Extensions involving sum rate maximization.
  - Dynamic re-optimization when users enter/drop from the system.

Closing Words

- The recent advances in conic/robust optimization have recently started to significantly impact various fields of applied sciences and engineering, e.g., digital signal processing and communication.
- In the past, the widely used optimization methods in both fields had been the gradient descent or least squares methods.
- The opportunity is ripe to use the newly developed interior point optimization techniques and highly efficient software tools to help advance the fields of signal processing and digital communication.
- The successful applications include adaptive filtering, robust beamforming, design and analysis of multi-user communication system, channel equalization, decoding and detection, ...
  - giving powerful new modeling and computational tools to solve previously considered intractable problems.
  - providing insight and understanding of optimal system designs