Matrix Cube Theorems and Tight Tractable Approximations of Semi-Infinite LMIs

Arkadi Nemirovski,
Technion – Israel Institute of Technology
nemirovs@ie.technion.ac.il

joint research with A. Ben-Tal and C. Roos

1. Semi-infinite LMIs with structured norm-bounded uncertainty
   - Motivation and examples
   - Complexity status
   - Approximations
2. Approximating semi-infinite LMIs with structured norm-bounded uncertainty

Semi-infinite LMIs

♣ **A** **semi-infinite LMI** **is an infinite system of LMIs of the form**

$$A_0 + \sum_{i=1}^{n} x_i A_i \succeq 0 \quad \forall [A_0, ..., A_n] \in \mathcal{U} \qquad \textbf{(S)}$$

where

- x **is the design vector**
- $\mathcal{U}$ **is a set in the space** $(\mathbf{S}^m)^{n+1}$ **of** $(n+1)$**-tuples of** $m \times m$ **symmetric matrices** $A_0, ..., A_n$**.**

♣ **In applications,** $\mathcal{U}$ **usually arises in the form**

$$\begin{array}{rl}\mathcal{U} = \mathcal{U}_\rho = & \{[A_0, ..., A_n] = [A_0^*, ..., A_n^*] \\ & \quad + \sum_{k=1}^{K} \delta_k [A_0^k, ..., A_n^k] : \delta \in \rho\mathbf{\Delta}\},\end{array} \qquad (*)$$

where

- $[A_0^*, ..., A_n^*]$ **is the "nominal data",**
- $[A_0^k, ..., A_n^k]$ **and** δ_k **are the directions and the magnitudes of "basic perturbations",**
- $\mathbf{\Delta} \subset \mathbf{R}^K$ **is the set of "basic perturbations of magnitude** ≤ 1**, which is a convex compact symmetric w.r.t. the origin,**
- $\rho \geq 0$ **is the "uncertainty level".**

$$A_0 + \sum_{i=1}^{n} x_i A_i \succeq 0 \quad \forall [A_0, ..., A_n] \in \mathcal{U} \qquad \textbf{(S)}$$

$$\begin{array}{rl} \mathcal{U} = & \{[A_0, ..., A_n] = [A_0^*, ..., A_n^*] \\ & \quad + \sum_{k=1}^{K} \delta_k [A_0^k, ..., A_n^k] : \delta \in \rho \mathbf{\Delta}\} \end{array} \qquad (*)$$

♣ **In the case of** $(*)$**, (S) reads**

$$\begin{array}{c} \mathcal{A}_0(x) + \rho \sum_{k=1}^{K} \delta_k \mathcal{A}_k(x) \succeq 0 \quad \forall \delta \in \mathbf{\Delta} \\ \mathcal{A}_k(x) = A_0^k + \sum_{i=1}^{n} x_i A_i^k, \ k = 0, 1, ..., K. \end{array}$$

♣ **Sources of semi-infinite LMIs:**

- **Robust Counterparts of uncertain LMIs with affine data uncertainty**
- **Robust Control**
- **Some problems of maximizing convex functions over convex sets**

♣ **Example:** Consider uncertain Lyapunov LMI

$$\begin{array}{c}(A_* + \rho\Xi)^T X + X(A_* + \rho\Xi) \preceq -I \\ [A, \Xi \in \mathbf{R}^{m\times m}, X \in \mathbf{S}^m]\end{array} \qquad \textbf{(L)}$$

with perturbation Ξ **running through a given compact set** $\mathbf{\Delta}$**.**

♠ **The Robust Counterpart of (L) is the semi-infinite LMI**

$$[-I - A_*^T X - XA_*] - \rho[\Xi^T X + X\Xi] \succeq 0 \quad \forall \Xi \in \mathbf{\Delta} \quad \textbf{(R)}$$

♠ **(R) is of direct interest for Control: solutions** $X \succ 0$ **to (R) are exactly the Lyapunov certificates for the stability of the uncertain time-varying dynamical system**

$$\dot{z}(t) = A(t)z(t), \quad A(t) \in A_* + \rho\mathbf{\Delta}$$

♠ **(R) is related to the problem of maximizing a convex quadratic form over the unit cube, which is a NP-complete combinatorial problem.**

Given $G \succ 0$**, let us choose** A_* **and** $\mathbf{\Delta}$ **as**

$$-I - A_*^T - A_* = G^{-1}; \quad \mathbf{\Delta} = \{\Xi : |\Xi|_{ij} \leq 1/2\}.$$

In this case $X = I$ **is feasible for (R) iff**

$$G^{-1} \succeq \rho B \quad \forall (B \in \mathbf{S}^n : |B_{ij}| \leq 1)$$

$$
\begin{array}{c}
G^{-1} \succeq \rho B \quad \forall (B \in \mathbf{S}^n : |B_{ij}| \leq 1) \\
\Updownarrow \\
\xi^T G^{-1} \xi \geq \rho \max\limits_{B=B^T : |B_{ij}| \leq 1} \xi^T B \xi \quad \forall \xi \\
\Updownarrow \\
\xi^T G^{-1} \xi \geq \rho \|\xi\|_1^2 \quad \forall \xi \\
\Updownarrow \quad \left[\begin{array}{l}\textbf{passing to}\\ \textbf{conjugate}\\ \textbf{norms}\end{array}\right] \\
\eta^T G \eta \leq \rho^{-1} \|\eta\|_\infty^2 \quad \forall \eta \\
\Updownarrow \\
\max\limits_{\eta : \|\eta\|_\infty \leq 1} \eta^T G \eta \leq \rho^{-1}
\end{array}
$$

♠ **Thus, checking whether the maximum of a given convex quadratic form over the unit cube is** $\leq \rho^{-1}$ **is equivalent to checking whether** $X = I$ **is a solution to a specific semi-infinite LMI – the Robust Counterpart**

$$-I - A_*^T X - X A_* - \rho \left[\Xi^T X + X \Xi\right] \succeq 0 \quad \forall \Xi \in \mathbf{\Delta}$$

of the uncertain Lyapunov LMI

$$[A + \rho \Xi]^T X + X [A + \rho \Xi] \preceq -I$$

affected by simple-looking *interval uncertainty***:**

$$\Xi \in \mathbf{\Delta} = \{\Xi : |\Xi_{ij}| \leq 1\}.$$

♣ **Good news: semi-infinite LMIs are important**
♣ **Bad news: Already simple-looking semi-infinite LMIs, like the Robust Counterpart of the uncertain Lyapunov LMI with interval uncertainty**

$$\begin{array}{l}\mathcal{A}(X)+\rho[\Xi^T X+X\Xi]\succeq 0\\ \forall(\Xi:|\Xi_{ij}|\leq D_{ij},i,j=1,...,m)\end{array}\qquad\textbf{(RL)}$$

are NP-hard.
♣ **Conclusion: When handling intractable semi-infinite LMIs, a natural course of actions is to look for their** *tight tractable approximations.*

♠ **Definition.** **We say that an LMI**

$$\mathcal{S}_\rho(x,u) \succeq 0 \qquad (\mathbf{A}[\rho])$$

is an *approximation* **of the semi-infinite LMI**

$$\mathcal{A}_0(x) + \rho \sum_{k=1}^{K} \delta_k \mathcal{A}_k(x) \succeq 0 \quad \forall \delta \in \mathbf{\Delta} \qquad (\mathbf{L}[\rho])$$

if the projection $\mathcal{Y}[\rho]$ of the solution set of $(\mathbf{A}[\rho])$ on the space of x-variables is contained in the solution set $\mathcal{X}[\rho]$ of $(\mathbf{L}[\rho])$.

Approximation is called *tight within factor $\theta \geq 1$*, **if**

$$\mathcal{X}[\theta\rho] \subset \mathcal{Y}[\rho] \subset \mathcal{X}[\rho],$$

or, equivalently:

(i) whenever x can be extended to a feasible solution of the approximation, x is feasible for the semi-infinite LMI of interest at the uncertainty level ρ;

(i) whenever x can<u>not</u> be extended to a feasible solution of approximation, x is <u>not</u> feasible for the LMI of interest *with increased by factor θ uncertainty level*.

$$\underbrace{\mathcal{A}_0(x)+\rho\sum_{k=1}^{K}\delta_k\mathcal{A}_k(x)}_{\mathcal{A}(x,\delta)}\succeq 0\quad\forall\delta\in\mathbf{\Delta}$$

♣ **Possibilities to build tight, within moderate factors, approximations of semi-infinite LMIs depend on the structure of the LMI and the uncertainty set.**

♣ **A "good case" here is given by** *structured norm-bounded perturbations***:**

$$\mathcal{A}(x,\delta)\equiv\mathcal{A}_0(x)+\rho\sum_{k=1}^{K}\left[L_k^T\Delta_kR_k(x)+R_k^T(x)\Delta_k^TL_k\right]$$
$$\Delta=\mathrm{Diag}\left\{\Delta_1,...,\Delta_K\right\}\in\mathbf{\Delta}$$

where

- **The** $m\times m$ **matrix** $\mathcal{A}_0(x)$ **is symmetric and affine in** x**;**
- **The matrices** L_k**,** $R_k(x)$ **are** $d_k\times m$**, and** $R_k(x)$ **are affine in** x**;**
- **perturbations** $\Delta=\mathrm{Diag}\{\Delta_1,...,\Delta_k\}$ **are block-diagonal matrices with** K **diagonal blocks of the sizes** d_1**,...,**d_K**;**
- $\mathbf{\Delta}$ **is comprised of all** $\Delta=\mathrm{Diag}\{\Delta_1,...,\Delta_k\}$ **such that**

$$\|\Delta_k\|\leq 1,\ k=1,...,K;\quad \Delta_k=\delta_kI_{d_k},k\in\mathcal{I}_{\mathrm{s}}.$$

$$\mathcal{A}_0(x) + \rho \sum_{k=1}^{K} \left[L_k^T \Delta_k R_k(x) + R_k^T(x) \Delta_k^T L_k\right] \succeq 0$$
$$\forall \left(\Delta = \{\Delta_k\} : \begin{array}{c} \|\Delta_k\| \leq 1,\, k \leq K \\ \Delta_k = \delta_k I_{d_k},\, k \in \mathcal{I}_\mathrm{s} \end{array} \right)$$

• **Example: Semi-infinite LMI with "interval uncertainty"**

$$\mathcal{A}_0(x) + \rho \sum_{k=1}^{K} \delta_k \mathcal{A}_k(x) \succeq 0 \quad \forall (\delta \in \mathbf{R} : \|\delta\|_\infty \leq 1)$$

can be rewritten equivalently as

$$\mathcal{A}_0(x) + \rho \sum_{k=1}^{K} [L_k^T \Delta_k R_k(x) + R_k^T(x) \Delta_k^T L_k] \succeq 0 \ \forall \Delta \in \boldsymbol{\Delta}$$

where $\boldsymbol{\Delta}$ corresponds to "repeated scalar perturbations" ($\mathcal{I}_\mathrm{s} = \{1, ..., K\}$) and L_k, $R_k(x)$ are given by

$$\mathcal{A}_k(x) = L_k^T R_k(x) + R_k^T(x) L_k.$$

• <u>**Special case**</u>**: The semi-infinite Lyapunov LMI with interval uncertainty**

$$\underbrace{[-I - A_*^T X - X A_*]}_{\mathcal{A}_0(x)} + \rho[\Xi^T X + X\Xi] \succeq 0$$
$$\forall(\Xi : |\Xi_{ij}| \leq D_{ij}, i, j = 1, ..., m)$$
$$\Updownarrow$$
$$\mathcal{A}_0(x) + \rho \sum_{i,j} \delta_{ij} D_{ij} [e_j e_i^T X + X e_i e_j^T] \succeq 0$$
$$\forall(\delta = \{\delta_{ij}\} : |\delta_{ij}| \leq 1)$$
$$\Updownarrow$$

$$\boxed{\begin{array}{c} \mathcal{A}_0(X) + \rho \sum_{i,j} \left[L_{ij} \delta_{ij} R_{ij}(X) + R_{ij}^T(X) \delta_{ij} L_{ij} \right] \succeq 0 \\ \forall \Delta = \mathrm{Diag}\{\delta_{ij}\} \in \mathbf{\Delta} \\ \left[\begin{array}{c} L_{ij} = D_{ij} e_j^T, R_{ij}(X) = e_i^T X \\ d_{ij} = 1,\ i, j = 1, ..., m, \\ \mathcal{I}_\mathrm{s} = \{(i, j) : 1 \leq i, j \leq m\} \end{array} \right] \end{array}}$$

Remark: In the description of structured norm-bounded uncertainty

$$\mathbf{\Delta} = \left\{ \mathrm{Diag}\{\Delta_1, ..., \Delta_K\} : \begin{array}{c} \Delta_k \in \mathbf{R}^{d_k \times d_k} \\ \|\Delta_k\| \leq 1,\ k \leq K \\ \Delta_k = \delta_k I_{d_k},\ k \in \mathcal{I}_{\mathrm{s}} \end{array} \right\}$$

1×1 **perturbation blocks** Δ_k **can be considered both as scalar** $(k \in \mathcal{I}_{\mathrm{s}})$ **and as full** $(k \notin \mathcal{I}_{\mathrm{s}})$**. It is convenient to treat these blocks as full. Thus, from now on**

$$k \in \mathcal{I}_{\mathrm{s}} \Rightarrow d_k \geq 2.$$

In particular, from now on we treat the Lyapunov LMI with interval uncertainty as the semi-infinite LMI

$$\begin{array}{c} \mathcal{A}_0(X) + \rho \sum\limits_{i,j} \left[L_{ij} \delta_{ij} R_{ij}(X) + R_{ij}^T(X) \delta_{ij} L_{ij} \right] \succeq 0 \\ \forall \Delta = \mathrm{Diag}\{\delta_{ij}\} \in \mathbf{\Delta} \\ \left[\begin{array}{c} L_{ij} = D_{ij} e_j^T, R_{ij}(X) = e_i^T X \\ d_{ij} = 1,\ i,j = 1, ..., m, \\ \mathcal{I}_{\mathrm{s}} = \emptyset \end{array} \right] \end{array}$$

with full 1×1 **perturbation blocks.**

♣ <u>Matrix Cube Theorem</u> [Ben-Tal, Nemirovski, Roos, 2001]. **Consider a semi-infinite LMI with structured norm-bounded uncertainty**

$$\begin{array}{l}\mathcal{A}_0(x) + \rho\sum\limits_{k=1}^{K}[L_k\Delta_k R_k(x) + R_k^T(x)\Delta_k^T L_k] \succeq 0\\ \qquad\forall\left(\Delta=\{\Delta_k\}: \begin{array}{c}\Delta_k\in\mathbf{R}^{d_k\times d_k}\\ \|\Delta_k\|\leq 1,\, k\leq K\\ \Delta_k=\delta_k I_{d_k},\, k\in\mathcal{I}_{\rm s}\end{array}\right)\end{array} \qquad (\mathbf{R}[\rho])$$

The system of LMIs in variables x**,** $X_k\in\mathbf{S}^m$**,** $\lambda_k\in\mathbf{R}$**,** $k\not\in\mathcal{I}_{\rm s}$**:**

$$\begin{array}{l}X_k\succeq\pm[L_kR_k(x)+R_k^T(x)L_k],\ k\in\mathcal{I}_{\rm s}\\ \left[\begin{array}{cc}X_k-\lambda_kL_k^TL_k & R_k^T(x)\\ R_k(x) & \lambda_kI_{d_k}\end{array}\right]\succeq 0,\quad k\not\in\mathcal{I}_{\rm s}\\ \mathcal{A}_0(x)\succeq\rho\sum\limits_{k=1}^{K}X_k\end{array} \qquad (\mathbf{A}[\rho])$$

is an approximation of $(\mathbf{R}[\rho])$ **tight within the factor**

$$\vartheta\left(\max_{k\in\mathcal{I}_{\rm s}} d_k\right) \qquad [\max_{k\in\emptyset} d_k]=1$$

Here $\vartheta(\mu)$ **is a universal function of** μ **such that**

$$\vartheta(1)=\frac{\pi}{2}=1.57...,\ \vartheta(2)=2,\ \vartheta(\mu)\leq\sqrt{2\pi\mu}.$$

Besides this, in the case $K=1$ **of a single perturbation block,** $(\mathbf{A}[\rho])$ **is equivalent to** $(\mathbf{R}[\rho])$**.**

♠ **Sketch of the proof.**

A. A simple sufficient condition for the validity of the LMI

$$\mathcal{A}_0(x) + \rho \sum_{k=1}^{K} [L_k^T \Delta_k R_k(x) + R_k^T(x) \Delta_k^T L_k] \succeq 0$$

for all perturbations Δ_k satisfying $\|\Delta_k\| \leq 1$, $\Delta_k = \delta_k I_{d_k}$, $k \in \mathcal{I}_\mathrm{s}$, is the existence of matrices X_k such that

$$\begin{array}{ll} (a) & X_k \succeq L_k^T \Delta_k R_k(x) + R_k^T(x) \Delta_k^T L_k \\ & \qquad \forall (\Delta_k : \|\Delta_k\| \leq 1 \ \& \ \Delta_k = \delta_k I_{d_k},\ k \in \mathcal{I}_\mathrm{s}) \\ (b) & \mathcal{A}_0(x) \succeq \rho \sum_{k=1}^{K} X_k \end{array}$$

For $k \in \mathcal{I}_\mathrm{s}$, (a) is clearly equivalent to

$$X_k \succeq \pm [L_k^T R_k(x) + R_k^T(x) L_k],$$

while for $k \notin \mathcal{I}_\mathrm{s}$, (a) is equivalent to

$$\exists \lambda_k : \left[\begin{array}{cc} X_k - \lambda_k L_k^T L_k & R_k^T(x) \\ R_k(x) & \lambda_k I_{d_k} \end{array} \right] \succeq 0$$

With these observations, our sufficient condition becomes exactly (A[ρ]); thus, (A[ρ]) is an approximation of (R[ρ]).

B. **In order to bound the tightness factor of the approximation, assume that a given x cannot be extended to a feasible solution of (A$[\rho]$); we should prove that then x is infeasible for (R$[\vartheta(\max\limits_{k\in\mathcal{I}_s} d_k)]$).**

Our assumption is equivalent to the fact that the optimal value in the semidefinite program

$$\min_{\tau,\{X_k\},\{\lambda_k\}}\left\{\tau : \begin{array}{c} X_k \succeq \pm A_k,\ k\in\mathcal{I}_s \\ \left[\begin{array}{cc} X_k-\lambda_k L_k^T L_k & R_k^T \\ R_k & \lambda_k I_{d_k}\end{array}\right]\succeq 0,\ k\not\in S \\ \tau I + A \succeq \rho\sum\limits_k X_k \end{array}\right\}$$
$$[A=\mathcal{A}_0(x), R_k=R_k(x), A_k=L_k^T R_k + R_k^T L_k]$$

is positive. Since the problem is strictly feasible, this means that the dual problem admits a feasible solution with positive value of the objective. This reduces to

$$\exists Z\succeq 0:$$
$$\rho\left[\sum_{k\in\mathcal{I}_s}\|\lambda(Z^{1/2}A_kZ^{1/2})\|_1 + 2\sum_{k\not\in\mathcal{I}_s}\|L_kZ^{1/2}\|_2\|R_kZ^{1/2}\|_2\right] > \mathrm{Tr}(Z^{1/2}AZ^{1/2})$$

where $\lambda(B)$ is the vector of eigenvalues of $B\in\mathbf{S}^m$ and $\|B\|_2=\sqrt{\mathrm{Tr}(BB^T)}$.

$$\exists Z \succeq 0:$$
$$\rho\left[\sum_{k\in\mathcal{I}_s}\|\lambda(Z^{1/2}A_kZ^{1/2})\|_1 + 2\sum_{k\not\in\mathcal{I}_s}\|L_kZ^{1/2}\|_2\|R_kZ^{1/2}\|_2\right] > \mathrm{Tr}(Z^{1/2}AZ^{1/2}) \tag{$*$}$$

<u>Lemma</u>: Let $\xi\sim\mathcal{N}(0,I_m)$. Then

(i) For a matrix $B\in\mathbf{S}^m$, one has

$$\mathbf{E}\left\{|\xi^TB\xi|\right\} \geq \vartheta^{-1}(\lfloor\mathrm{Rank}(B)/2\rfloor)\|\lambda(B)\|_1$$

(ii) For matrices $P,Q\in\mathbf{R}^{d\times m}$,

$$\mathbf{E}\left\{\|P\xi\|_2\|Q\xi\|_2\right\} \geq \vartheta^{-1}(1)\|P\|_2\|Q\|_2.$$

Here $\vartheta(\cdot)$ is as required in the MCT. Setting

$$\mu = \max_{k\in\mathcal{I}_s} d_k,$$

so that

$$k\in\mathcal{I}_s \Rightarrow \mathrm{Rank}(Z^{1/2}A_kZ^{1/2}) \leq 2\mu,$$

we get from Lemma combined with $(*)$ the inequality

$$\begin{array}{l}\rho\vartheta(\mu)\mathbf{E}\{\sum_{k\in\mathcal{I}_s}|\xi^TZ^{1/2}A_kZ^{1/2}\xi) \\ \qquad +2\sum_{k\not\in\mathcal{I}_s}\|L_kZ^{1/2}\xi\|_2\|R_kZ^{1/2}\xi\|_2\} > \mathrm{Tr}(Z^{1/2}AZ^{1/2}) \\ \qquad\qquad = \mathbf{E}\left\{\xi^TZ^{1/2}AZ^{1/2}\xi\right\}\end{array}$$

$$\begin{array}{l}\rho\vartheta(\mu)\mathbf{E}\{\sum\limits_{k\in\mathcal{I}_{\rm s}}|\xi^TZ^{1/2}A_kZ^{1/2}\xi| \\ \qquad +2\sum\limits_{k\not\in\mathcal{I}_{\rm s}}\|L_kZ^{1/2}\xi\|_2\|R_kZ^{1/2}\xi\|_2\} > {\rm Tr}(Z^{1/2}AZ^{1/2}) \\ \qquad\qquad\qquad\qquad\qquad\qquad = \mathbf{E}\left\{\xi^TZ^{1/2}AZ^{1/2}\xi\right\}\end{array}$$

It follows that there exists a realization ζ of the random vector $Z^{1/2}\xi$ such that

$$\rho\vartheta(\mu)\left\{\sum_{k\in\mathcal{I}_{\rm s}}|\zeta^TA_k\zeta|+2\sum_{k\not\in\mathcal{I}_{\rm s}}\|L_k\zeta\|_2\|R_k\zeta\|_2\right\}>\zeta^TA\zeta \quad (*)$$

• **For $k\in\mathcal{I}_{\rm s}$, we can choose $\Delta_k=\pm I_{d_k}$ in such a way that**

$$|\zeta^TA_k\zeta|\equiv|\zeta^T[L_k^TR_k+R_k^TL_k]\zeta|=\zeta^T[L_k^T\Delta_kR_k+R_k^T\Delta_k^TL_k]\zeta$$

• **For $k\not\in\mathcal{I}_{\rm s}$, we can choose $\Delta_k\in\mathbf{R}^{d_k\times m}$, $\|\Delta_k\|=1$, in such a way that**

$$2\|L_k\zeta\|_2\|R_k\zeta\|_2 = \zeta^T\left[L_k^T\Delta_kR_k+R_k^T\Delta_k^TL_k\right]\zeta$$

Recalling that $A=\mathcal{A}_0(x)$, $R_k=R_k(x)$, $(*)$ reads

$$\zeta^T\left[\mathcal{A}_0(x)-\rho\vartheta(\mu)\sum_k[L_k^T\Delta_kR_k(x)+R_k^T(x)\Delta_k^TL_k]\right]\zeta<0,$$

while by construction $\|\Delta_k\|\leq 1$ and $\Delta_k=\delta_kI_{d_k}$ for $k\in\mathcal{I}_{\rm s}$. Thus,

$$x\not\in\mathcal{X}[\vartheta(\mu)\rho]. \qquad \blacksquare$$

♣ **Application, I: Maximizing convex quadratic form over the unit cube.**

We have seen that if $G \succ 0$, then

$$\omega(G) \equiv \max_{\|\eta\|_\infty \leq 1} \eta^T G \eta = \frac{1}{\rho^*},$$
$$\rho^* = \max\left\{\rho : G^{-1} \succeq \rho A \;\forall (A = A^T : |A_{ij}| \leq 1)\right\}.$$

We have

$$\rho^* = \max\{\rho :\; G^{-1} + \rho \sum_{i \leq j} [L_{ij}^T \Delta_{ij} R_{ij} + R_{ij}^T \Delta_{ij}^T L_{ij}] \succeq 0$$
$$\forall (\Delta_{ij} \in \mathbf{R} : |\Delta_{ij}| \leq 1)\}$$
$$L_{ij} = \begin{cases} e_i^T, & i \neq j \\ \frac{1}{\sqrt{2}} e_i, & i = j \end{cases}, \quad R_{ij} = \begin{cases} e_j^T, & i \neq j \\ \frac{1}{\sqrt{2}} e_j, & i = j \end{cases}$$

By the MCT, the efficiently computable quantity

$$\widehat{\rho} = \max_{\rho, \{X_{ij}\}, \{\lambda_{ij}\}} \left\{ \rho : \begin{array}{l} \begin{bmatrix} X_{ij} - \lambda_{ij} L_{ij}^T L_{ij} & R_{ij}^T \\ R_{ij} & \lambda_{ij} \end{bmatrix} \succeq 0,\; i \leq j \\ G^{-1} \succeq \rho \sum_{i \leq j} X_{ij} \end{array} \right\}$$

is a lower bound, tight within the factor $\vartheta(1) = \frac{\pi}{2}$, on ρ^*, so that $\frac{1}{\widehat{\rho}}$ is an efficiently computable upper bound on $\omega(G)$.

On a closest inspection, the efficiently computable upper bound

$$\frac{1}{\widehat{\rho}} = \frac{1}{\max\limits_{\rho, X_{ij} \lambda_{ij}} \left\{ \rho : \begin{array}{l} \begin{bmatrix} X_{ij} - \lambda_{ij} L_{ij}^T L_{ij} & R_{ij}^T \\ R_{ij} & \lambda_{ij} \end{bmatrix} \succeq 0,\ i \leq j \\ G^{-1} \succeq \rho \sum\limits_{i \leq j} X_{ij} \end{array} \right\}}$$

on $\omega(G) = \max\limits_{\eta : \|\eta\| \leq 1} \eta^T G \eta$ **turns out to be exactly the standard semidefinite relaxation bound**

$$\begin{array}{rcl} \widehat{\omega}(G) & = & \min\limits_{\lambda} \left\{ \sum\limits_{i} \lambda_i : \mathrm{Diag}\{\lambda\} \succeq G \right\} \\ & \equiv & \max\limits_{X} \{ \mathrm{Tr}(GX) : X \succeq 0, X_{ii} = 1 \} \end{array},$$

and we arrive at $\frac{\pi}{2}$**-Theorem of Nesterov (1996):**

For $G \succeq 0$, $\widehat{\omega}(G)$ *is a tight, within the factor* $\frac{\pi}{2}$, *upper bound on* $\omega(G)$**.**

which originally was proved via the random hyperplane technique of Goemans and Williamson.

♣ **Application, II: Lyapunov Stability Analysis under interval uncertainty**

The possibility to certify the stability of uncertain time-varying dynamical system

$$\dot{z}(t) = A(t)z(t) \qquad [A(t) \in \mathcal{U} \forall t]$$

by a Lyapunov stability certificate is equivalent to solvability of the semi-infinite LMI

$$X \succeq I, \quad A^T X + XA \preceq -I \quad \forall A \in \mathcal{U} \qquad \textbf{(L)}$$

♠ **Assume that uncertainty comes from structured norm-bounded perturbations:**

$$\mathcal{U} = \left\{ A = A_* + \rho \sum_{k=1}^{K} P_k^T \Delta_k^T Q_k : \begin{array}{c} \Delta_k \in \mathbf{R}^{d_k \times d_k} \\ \|\Delta_k\| \leq 1,\, k \leq K \\ \Delta_k = \delta_k I_{d_k},\, k \in \mathcal{I}_{\rm s} \end{array} \right\}$$

and that we are interested to compute the *Lyapunov Stability Radius* ρ^* – **the supremum of those** ρ **for which (L) is solvable.**

♠ **With structured norm-bounded uncertainty, the semi-infinite LMI Lyapunov LMI reads**

$$
\begin{array}{l}
\overbrace{[-I - A_*^T X - X A_*]}^{\mathcal{A}_0(X)} \\
\qquad +\rho \sum\limits_{k=1}^{K} [Q_k^T \Delta_k (P_k X) + (P_k X)^T \Delta_k^T Q_k] \succeq 0 \\
\qquad \forall \left(\{\Delta_k\} : \begin{array}{c} \Delta_k \in \mathbf{R}^{d_k \times d_k} \\ \|\Delta_k\| \leq 1,\ k \leq K \\ \Delta_k = \delta_k I_{d_k},\ k \in \mathcal{I}_{\rm s} \end{array} \right)
\end{array} \qquad (*)
$$

The MCT implies a sufficient condition for solvability of $(*)$ **and thus – an efficiently computable lower bound** $\widehat{\rho}$ **on** ρ^***:**

$$
\widehat{\rho} = \sup_{\rho, X, \{X_k\}, \{\lambda_k\}} \left\{ \rho : \begin{array}{l} X \succeq I \\ X_k \succeq \pm [Q_k^T P_k X + X P_k^T Q_k],\ k \in \mathcal{I}_{\rm s} \\ \left[\begin{array}{cc} X_k - \lambda_k Q_k^T Q_k & X P_k^T \\ P_k X & \lambda I_{d_k} \end{array} \right] \succeq 0,\ k \not\in \mathcal{I}_{\rm s} \\ \mathcal{A}_0(X) \succeq \rho \sum\limits_k X_k \end{array} \right\}
$$

The bound is tight within the factor $\vartheta \left(\max\limits_{k \in \mathcal{I}_{\rm s}} d_k \right)$**:**

$$
\widehat{\rho} \leq \rho^* \leq \vartheta \left(\max_{k \in \mathcal{I}_{\rm s}} d_k \right) \widehat{\rho}.
$$

For example, in the case of interval uncertainty $d_k = 1,\ k = 1, ..., K$**, the factor becomes** $\frac{\pi}{2}$**.**

♣ **Many important properties of a linear time-invariant dynamical system**

$$\begin{array}{rcl} \dot{z}(t) & = & Az(t) + Bu(t) \\ y(t) & = & Cz(t) + Du(t) \end{array} \qquad \textbf{(S)}$$

are *LMI-representable* — **(S) possesses the property iff certain LMI**

$$\boldsymbol{\Sigma}_{\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]}(X) \succeq 0$$

associated with (S) is solvable:

♠ **Open-loop stability:**

$$\lim_{t\to\infty} u(t) = 0 \Rightarrow \lim_{t\to\infty} z(t) = 0$$
$$\Updownarrow$$
$$\exists X : X \succeq I,\ A^T X + XA \preceq -I$$

♠ **Stabilizability via state feedback:**

$$\exists K : \left\{ \begin{array}{rcl} \dot{z} & = & Az + Bu \\ u & = & Kz \end{array} \right. \Rightarrow \lim_{t\to\infty} z(t) = 0$$
$$\Updownarrow$$
$$\exists X = [Y, Z] : Y \succeq I,\ YA^T + AY + ZB^T + BZ^T \preceq -I$$

♠ **Positive realness:**

$$z(0) = 0 \Rightarrow \int_0^T u^T(t)y(t)dt \geq 0 \quad \forall T \geq 0$$

$$\Updownarrow$$

$$\exists X : X \succ 0, \begin{bmatrix} A^T X + XA & XB - C^T \\ B^T X - C & -D^T - D \end{bmatrix} \preceq 0$$

♠ **Real boundedness:**

$$z(0) = 0 \Rightarrow \int_0^T y^T(t)y(t)dt \leq \int_0^T u^T(t)u(t)dt \quad \forall T \geq 0$$

$$\Updownarrow$$

$$\exists X : X \succ 0, \begin{bmatrix} A^T X + XA & XB & C^T \\ B^T X & -I & D^T \\ C & D & -I \end{bmatrix} \preceq 0$$

♣ **In all outlined (and many other) cases, solvability of the** *semi-infinite* **LMI**

$$\mathbf{\Sigma}_{\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]}(X) \succeq 0 \ \forall \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{U} \qquad (*)$$

is a *sufficient* **condition for the property to be possessed by the** *uncertain time-varying system*

$$\begin{array}{rcl} \dot{z}(t) & = & A(t)z(t) + B(t)u(t) \\ y(t) & = & C(t)z(t) + D(t)u(t) \end{array}, \qquad \begin{bmatrix} A(t) & B(t) \\ C(t) & D(t) \end{bmatrix} \in \mathcal{U} \ \forall t$$

Assuming structured norm-bounded uncertainty

$$\mathcal{U} = \left\{ \begin{bmatrix} A_* & B_* \\ C_* & D_* \end{bmatrix} + \rho \sum_k P_k \Delta_k Q_k^T : \begin{array}{c} \Delta_k \in \mathbf{R}^{d_k \times d_k} \\ \|\Delta_k\| \leq 1,\ k \leq K \\ \Delta_k = \delta_k I_{d_k},\ k \in \mathcal{I}_{\rm s} \end{array} \right\}$$

and with "well-structured $\mathbf{\Sigma}$" (as it is the case in all our examples), the MCT yields an $O(1)\sqrt{\max\limits_{k \in \mathcal{I}_{\rm s}} d_k}$-tight computationally tractable approximation to the semi-infinite LMI $(*)$.

♣ **Consider the** *analysis* **version of a semi-infinite LMI with structured norm-bounded uncertainty and scalar perturbation blocks**

$$\rho^* = \max\left\{\rho : A + \rho \sum_{k=1}^{K} \delta_k A_k \succeq 0 \forall (\delta \in \mathbf{R}^K : \|\delta\|_\infty \leq 1)\right\} \qquad \textbf{(P)}$$

When solving this problem, we lose nothing by assuming that $A \succ 0$. In this case, the scaling $A \leftarrow I$, $A_k \leftarrow A^{-1/2} A_k A^{-1/2}$ converts (P) into the problem

$$\frac{1}{\rho^*} = \min\left\{\lambda : \|\sum_k \delta_k A_k\| \leq \lambda \ \forall (\delta : \|\delta\|_\infty \leq 1)\right\}$$

which is the problem of computing the norm of the linear mapping

$$\mathcal{A}(\delta) = \sum_{k=1}^{K} \delta_k A_k : (\mathbf{R}^K, \|\cdot\|_\infty) \to (\mathbf{S}^m, \|\cdot\|).$$

The MCT offers an efficiently computable upper bound, tight within the factor $O(1)\sqrt{\max_k \mathrm{Rank}(A_k)}$, on this norm.

♠ **What about computing the norm of the same mapping regarded as a mapping from $(\mathbf{R}^K, \|\cdot\|_p)$ into $(\mathbf{R}^K, \|\cdot\|_\infty) \to (\mathbf{S}^m, \|\cdot\|)$?**

♠ **How to compute/estimate the norm of a linear mapping**

$$\mathcal{A}(\delta) = \sum_{k=1}^{K} \delta_k A_k : (\mathbf{R}^K, \|\cdot\|_p) \to (\mathbf{S}^m, \|\cdot\|) \quad ?$$

The case of utmost interest here is $p = 2$, where, our question reduces to the following NP-hard problem:

> *Given a K-dimensional ellipsoid in $\mathbf{S}^n$, centered at a positive definite matrix, what is the largest similar ellipsoid with the same center which is contained in the positive semidefinite cone?*

$$\mathcal{A}(\delta) = \sum_{k=1}^{K} \delta_k A_k : \mathbf{R}^K \to \mathbf{S}^m.$$

♣ **By the standard arguments, the function**

$$\phi(\alpha) \equiv \ln(\|\mathcal{A}\|_{\frac{1}{\alpha}}) = \ln\left(\max\left\{\|\mathcal{A}(\delta)\| : \|\delta\|_{\frac{1}{\alpha}} \leq 1\right\}\right)$$

is

- **convex and nonincreasing in** $\alpha \in [0, 1]$
- **Lipschitz continuous, with constant** $\ln(K)$**.**

By elementary arguments, it follows that

$$\left\{\begin{array}{lcl} \phi(\alpha) & \leq & (1-\alpha)\phi(0) + \alpha\phi(1) \\ \phi(\alpha) & \geq & (1-\alpha)\phi(0) + \alpha\phi(1) - a(1-\alpha)\ln(K) \end{array}\right.$$

$$\Updownarrow$$

$$K^{-\frac{p-1}{p^2}} \|\mathcal{A}\|_\infty^{1-1/p} \|\mathcal{A}\|_1^{1/p} \leq \|\mathcal{A}\|_p \leq \|\mathcal{A}\|_\infty^{1-1/p} \|\mathcal{A}\|_1^{1/p}$$

The quantity $\|\mathcal{A}\|_1$ **is easily computable:**

$$\|\mathcal{A}\|_1 = \max_k \|A_k\|,$$

while the MCT provides us with an upper bound

$$\gamma_\infty(\mathcal{A}) = \min_{\gamma, \{X_k\}} \left\{\gamma : \begin{array}{c} X_k \succeq \pm A_k \\ \gamma I \succeq X_1 + \ldots + X_K \end{array}\right\}$$

on $\|\mathcal{A}\|_\infty$**, and this bound is tight within the factor** $O(1)\sqrt{\max\limits_k \mathrm{Rank}(A_k)}$**.**

$$\mathcal{A}(\delta) = \sum_{k=1}^{K} \delta_k A_k : \mathbf{R}^K \to \mathbf{S}^m.$$

♣ **We arrive at the efficiently computable upper bound**

$$\gamma_p(\mathcal{A}) = \left(\max_k \|A_k\|\right)^{1-1/p} \gamma_\infty^{1/p}(\mathcal{A})$$

on

$$\|\mathcal{A}\|_p = \max\left\{\|\sum_{k=1}^{K} \delta_k A_k\| : \|\delta\|_p \leq 1\right\},$$

and this bound is tight within the factor

$$O(1)\left(\max_k \mathrm{Rank}(A_k)\right)^{\frac{1}{2p}} K^{\frac{p-1}{p^2}}.$$

When $p = 2$ **and the ranks of** A_k **are** $O(1)$**, the factor becomes** $O(1)K^{1/4}$ **(cf. the factor** $\sqrt{\min[K, m]}$ **for the only previously known computable upper bound on** $\|\mathcal{A}\|_2$**).**

♣ The MCT admits complex case extension:
Matrix Cube Theorem, Complex Case: **Consider a semi-infinite LMI**

$$\begin{array}{l}\mathcal{A}_0(x)+\sum\limits_{k=1}^{K}[L_k^H\Delta_kR_k(x)+R_k^H(x)\Delta_k^HL_k]\succeq 0\\ \forall\left(\Delta=\{\Delta_k\}: \begin{array}{l}\Delta_k\in\mathbf{C}^{d_k\times d_k},\ k\leq K\\ \|\Delta_k\|\leq 1,\ k\leq K\\ \Delta_k=\delta_kI_{d_k},\ \delta_k\in\mathbf{R},\ k\in\mathcal{I}_{\rm s}^{\mathbf{R}}\\ \Delta_k=\delta_kI_{d_k},\ \delta_k\in\mathbf{C},\ k\in\mathcal{I}_{\rm s}^{\mathbf{C}}\end{array}\right)\end{array}\qquad(\mathbf{R}[\rho])$$

where $L_k, R_k(x)\in\mathbf{C}^{d_k\times m}$**,** $\mathcal{A}_0(x)$ **is Hermitian and** $R_k(x)$**,** $\mathcal{A}_0(x)$ **are affine in** x**.**

The system of LMIs in Hermitian matrix variables X_k**,** V_k **and real variables** λ_k

$$\begin{array}{l}X_k\succeq\pm\left[L_p^HR_p(x)+R_p^H(x)L_p\right],\ k\in\mathcal{I}_{\rm s}^{\mathbf{R}},\\ \left[\begin{array}{cc}X_k-V_k & L_k^HR_k(x)\\ R_k^H(x)L_k & V_k\end{array}\right]\succeq 0,\ k\in\mathcal{I}_{\rm s}^{\mathbf{C}},\\ \left[\begin{array}{cc}X_k-\lambda_kL_k^HL_k & R_k^H(x)\\ R_k(x) & \lambda_kI_{d_k}\end{array}\right]\succeq 0,\ k\notin\mathcal{I}_{\rm s}^{\mathbf{R}}\cup\mathcal{I}_{\rm s}^{\mathbf{C}}\\ A-\rho\sum\limits_{k=1}^{K}X_k\succeq 0\end{array}\qquad(\mathbf{A}[\rho])$$

is a tight, within the factor $\vartheta_{\mathbf{C}}\left(\max\limits_{k\in\mathcal{I}_{\rm s}^{\mathbf{R}}\cup\mathcal{I}_{\rm s}^{\mathbf{C}}}d_k\right)$**, approximation of (**$\mathbf{R}[\rho]$**). Here** $\vartheta_{\mathbf{C}}(\mu)\leq O(1)\sqrt{\mu}$ **is a universal function such that** $\vartheta_{\mathbf{C}}(1)=\frac{4}{\pi}$**.**

<u>Corollary 1.</u> For a Hermitian $m \times m$ matrix $G \succeq 0$, the Semidefinite Relaxation bound

$$\widehat{\omega}(G) = \min_{\lambda} \left\{ \sum_i \lambda_i : \mathrm{Diag}\{\lambda\} \succeq G \right\}$$

on the quantity

$$\omega(G) = \max_{\eta \in \mathbf{C}^m : \|\eta\|_\infty \leq 1} \eta^H G \eta,$$

and this bound is tight within the factor $\frac{4}{\pi} = 1.27...$**:**

$$\omega(G) \leq \widehat{\omega}(G) \leq \frac{4}{\pi} \omega(G).$$

<u>Corollary 2.</u> Consider a time-varying uncertain dynamical system

$$\dot{z}(t) = A(t)z(t), \quad A(t) \in \mathcal{U}\ \forall t \qquad \textbf{(S)}$$

with complex data. In the case of structured norm-bounded uncertainty

$$\mathcal{U} = \left\{ A_* + \rho \sum_{k=1}^{K} P_k^H \Delta_k Q_k : \begin{array}{l} \Delta_k \in \mathbf{C}^{d_k \times d_k}, k \leq K \\ \|\Delta_k\| \leq 1,\ k \leq K \\ \Delta_k = \delta_k I_{d_k},\ \delta_k \in \mathbf{R},\ k \in \mathcal{I}_{\rm s}^{\mathbf{R}} \\ \Delta_k = \delta_k I_{d_k},\ \delta_k \in \mathbf{C},\ k \in \mathcal{I}_{\rm s}^{\mathbf{C}} \end{array} \right\}$$

the Lyapunov Stability Radius ρ^* of (S):

$$\rho^* = \sup\left\{\rho : \exists X \succeq I : A^H X + X A^H \preceq -I\ \forall A \in \mathcal{U}\right\}$$

admits a tight, within the factor $O(1)\sqrt{\max\limits_{k \in \mathcal{I}_{\rm s}^{\mathbf{R}} \cup \mathcal{I}_{\rm s}^{\mathbf{C}}} d_k}$, efficiently computable lower bound. In the case of $\mathcal{I}_{\rm s}^{\mathbf{C}} = \mathcal{I}_{\rm s}^{\mathbf{R}} = \emptyset$ ("interval uncertainty"), the bound is tight within the factor $\frac{4}{\pi}$.