A new property of Cholesky factorization of matrices arising in interior point methods.

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Overview

Numerical linear algebra motivation.

Interior point methods motivation.

The result for Cholesky factorization in LP and QP.

“Extensions” for SOCP (and SDP).

Cholesky Factorization

Given a matrix $M \in \mathbf{R}^{m \times m}$ its CF: $M = L \Lambda L^T$

$$L = \begin{bmatrix} 1 & & \\ \times & 1 & \\ \times & \times & 1 \end{bmatrix}, \; \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{bmatrix}$$

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The error of back/forward-solving depends on $||L||$.

Forward error of CF depends on $||L||$.

If M is nearly singular, symmetric pivoting can be used to reduce $||L||$.

Example 1

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1+\epsilon^2 & 1+\epsilon \\ 1 & 1+\epsilon & 3 \end{bmatrix}$$

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$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1/\epsilon & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & & \\ & \epsilon^2 & \\ & & 1 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1/\epsilon \\ 0 & 0 & 1 \end{bmatrix}}_{L^T}$$

Forward Error

Assume that $\epsilon^2 \sim$ machine precision

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$$M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1+\epsilon^2 & 1+\epsilon \\ 1 & 1+\epsilon & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & & \\ & 2\epsilon^2 & \\ & & \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{1}{2\epsilon} \\ 0 & \epsilon & 2 \end{bmatrix}$$

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Assume that $\epsilon^2 \sim$ machine precision

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1+\epsilon^2 & 1+\epsilon \\ 1 & 1+\epsilon & 3 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & & \\ & 2\epsilon^2 & \\ & & 1\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{1}{2\epsilon} \\ 0 & 0 & 1 \end{bmatrix}$$

Example 2.

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1+\epsilon^2 & 1+\epsilon^2 \\ 1 & 1+\epsilon^2 & 3 \end{bmatrix}$$

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$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & & \\ & \epsilon^2 & \\ & & 2-\epsilon^2 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{L^T}$$

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$$\begin{bmatrix} 1 & & \\ & (1+\alpha)\epsilon^2 & \\ & & \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{1+\gamma}{1+\alpha} \\ 0 & (1+\beta)\epsilon^2 & 2 + \delta\epsilon^2 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & & \\ & (1+\alpha)\epsilon^2 & \\ & & 2+{\color{red}\Theta(\epsilon^2)} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{1+\gamma}{1+\alpha} \\ 0 & 0 & 1 \end{bmatrix}$$

Linear Programming

$$
\begin{aligned}
XZe &= \mu e,\\
Ax &= b,\\
A^{T}y + z &= c,\\
x \geq 0,\ z &\geq 0.
\end{aligned}
$$

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$$D = XZ^{-1} = \begin{bmatrix} \frac{x_1}{z_1} & & \\ & \ddots & \\ & & \frac{x_n}{z_n} \end{bmatrix}$$

Linear Programming

Near the central path

$$x_i \to 0 \;\Rightarrow\; x_i = \Theta(\mu),\; z_i = \Theta(1)$$

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$$D = \begin{bmatrix} \Theta(\frac{1}{\mu}) & & & & & \\ & \ddots & & & & \\ & & \Theta(\frac{1}{\mu}) & & & \\ & & & \Theta(\mu) & & \\ & & & & \ddots & \\ & & & & & \Theta(\mu) \end{bmatrix}$$

Linear Programming

NOT near the central path

$$x_i \to 0 \;\Rightarrow\; x_i \neq \Theta(\mu),\; z_i = \Theta(1)$$

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LP Theorem

Theorem 0 *Given a fixed matrix* $A \in \mathbf{R}^{m \times n}$

$$\text{sup}\{||L|| : L\Lambda L^T = ADA^T, \forall \text{ positive diagonal } D\} < \infty$$

Sketch of the proof

$$
\begin{bmatrix}
\times & \times & \times & \times & \times & \times \\
 & \times & \times & \times & \times & \times \\
 & & \times & \times & \times & \times \\
 & & & \color{red}{\times} & \color{blue}{\times} & \color{blue}{\times} \\
 & & & \color{blue}{\times} & \times & \times \\
 & & & \color{blue}{\times} & \times & \times
\end{bmatrix}
$$

Sketch of the proof

$$
\begin{bmatrix}
\times & \times & \times & \times & \times & \times \\
 & \times & \times & \times & \times & \times \\
 & & \times & \times & \times & \times \\
 & & & {\color{red} m_{kk}} & {\color{blue}\times} & {\color{blue}\times} \\
 & & & {\color{blue} m_{ik}} & \times & \times \\
 & & & {\color{blue}\times} & \times & \times
\end{bmatrix}
$$

Sketch of the proof

$$
\begin{bmatrix}
\times & \times & \times & \times & \times & \times \\
 & \times & \times & \times & \times & \times \\
 & & \times & \times & \times & \times \\
 & & & m_{kk} & \times & \times \\
 & & & m_{ik} & \times & \times \\
 & & & \times & \times & \times
\end{bmatrix}
$$

$$
l_{ik} = m_{kk}/m_{ik} \qquad m_{kk} = r^T A_1 D_1 A_1^T r + r^T A_2 D_2 A_2^T r
$$

Sketch of the proof

$$\begin{bmatrix} \times & \times & \times & \times & \times & \times \\ & \times & \times & \times & \times & \times \\ & & \times & \times & \times & \times \\ & & & m_{kk} & \times & \times \\ & & & m_{ik} & \times & \times \\ & & & \times & \times & \times \end{bmatrix}$$

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$$l_{ik} = m_{kk}/m_{ik} \qquad m_{kk} = r^T A_1 D_1 A_1^T r + r^T A_2 D_2 A_2^T r$$

$$m_{ik} = \rho^T A_1 D_1 A_1^T r + \rho^T A_2 D_2 A_2^T r$$

Sketch of the proof

$$
\begin{bmatrix}
\times & \times & \times & \times & \times & \times \\
 & \times & \times & \times & \times & \times \\
 & & \times & \times & \times & \times \\
 & & & m_{kk} & \times & \times \\
 & & & m_{ik} & \times & \times \\
 & & & \times & \times & \times
\end{bmatrix}
$$

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Quadratic Programming

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Theorem 1 *Given fixed matrices* $A \in \mathbf{R}^{m \times n}$ *and* $Q \in \mathbf{R}^{n \times n}$ *p.s.d.*

$$\sup\{||L|| : L\Lambda L^T = A(Q+D)^{-1}A^T, \forall \text{ positive diagonal } D\} < \infty$$

Second Order Cone Programming

$$\min (c^1)^T x^1 + (c^2)^T x^2 + \ldots + (c^N)^T x^N$$

$$A^1 x^1 + A^2 x^2 + \ldots + A^n x^n = b$$

$$x^1 \in \mathcal{K}_1,\ x^2 \in \mathcal{K}_2,\ \ldots,\ x^N \in \mathcal{K}_N$$

$$x^i = (x_0^i, \bar{x}^i) \in \mathcal{K}_i \subset \mathbf{R}^{n_i} :\ \mathbf{x_0^i} \geq ||\bar{\mathbf{x}}^i||$$

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$$\max b^T y$$

$$y^T A^i + z^i = c^i, \quad \forall i = 1, N$$

$$z^1 \in \mathcal{K}_1,\ z^2 \in \mathcal{K}_2,\ \ldots,\ z^N \in \mathcal{K}_N$$

Second Order Cone Programming

$$
\begin{aligned}
A^1x^1 + A^2x^2 + \ldots + A^nx^n &= b \\
y^\top A^i + z^i &= c^i, \quad \forall i = 1, N \\
\mathrm{Arr}(x)\mathrm{Arr}(z)e_0 &= 0
\end{aligned}
$$

$$
\mathrm{Arr}(x) = \begin{bmatrix} x_0 & x_1 & \ldots & x_n \\ x_1 & x_0 & & \\ \vdots & & \ddots & \\ x_n & & & x_0 \end{bmatrix}
$$

Normal equation, primal method

$$AFA^T, \quad F = \begin{bmatrix} F_1 & & & \\ & F_2 & & \\ & & \ddots & \\ & & & F_N \end{bmatrix},$$

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$$AFA^T, \quad F = \begin{bmatrix} F_1 & & & \\ & F_2 & & \\ & & \ddots & \\ & & & F_N \end{bmatrix},$$

$$F_i = \begin{bmatrix} (x_0^i)^2 + \|\overline{x}^i\|^2 & 2x_0^i(\overline{x}^i)^T \\ 2x_0^i\overline{x}^i & \gamma(x^i)^2 * I + 2 * \overline{x}^i(\overline{x}^i)^T \end{bmatrix},$$

$$\gamma^2(x^i) = (x_0^i)^2 - \|\overline{x}^i\|^2 \geq 0$$

Eigenvalue/vector decomposition of F

$$F = QDQ^T,$$

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$$Q = \begin{bmatrix} 1/\sqrt{2} & 0 & & 0 & 1/\sqrt{2} \\ \bar{x}/\sqrt{2}\|\bar{x}\| & q^2 & \cdots & q^{n_i - 1} & -\bar{x}/\sqrt{2}\|\bar{x}\| \end{bmatrix},$$

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$q^2, \dots, q^{n_i-1}$ are such that Q is an orthogonal matrix.

Three types of cones

By complementarity

$$(x^*)^T z^* = 0, \quad \gamma(x^*)\gamma(z^*) = 0$$

Under strict complemetarity, there are three types of cones:

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I. $x^* = 0$, $\gamma(z^*) > 0$

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Under strict complemetarity, there are three types of cones:

I. $x^* = 0$, $\gamma(z^*) > 0$

II. $\gamma(x^*) > 0$, $z^* = 0$

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Under strict complemetarity, there are three types of cones:

I. $x^* = 0$, $\gamma(z^*) > 0$

II. $\gamma(x^*) > 0$, $z^* = 0$

III. $\gamma(x^*) = 0$, $x^* \neq 0$, $\gamma(z^*) = 0$, $z^* \neq 0$

On the central path

$$\mathrm{Arr}(x)\mathrm{Arr}(z)e_0 = \mu e_0, \quad \gamma(x)\gamma(z) = \Theta(\mu)$$

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II. $\gamma(x(\mu)) = \Theta(1)$, $z(\mu) = \Theta(\mu)$, $\gamma(z(\mu)) = \Theta(\mu)$

III. $\gamma(x(\mu)) = \sqrt{\mu}$, $x(\mu) = x^* + \Theta(\mu)$,
$\gamma(z(\mu)) = \sqrt{\mu}$, $z(\mu) = z^* + \Theta(\mu)$

Three types of cones

$$D = \begin{bmatrix} (x_0 + ||\bar{x}||)^2 & & & & \\ & \gamma^2(x) & & & \\ & & \ddots & & \\ & & & \gamma^2(x) & \\ & & & & (x_0 - ||\bar{x}||)^2 \end{bmatrix},$$

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I : $x(\mu) \to 0$ $D = \begin{bmatrix} \Theta(\mu^2) & & & & \\ & \Theta(\mu^2) & & & \\ & & \ddots & & \\ & & & \Theta(\mu^2) & \\ & & & & \Theta(\mu^2) \end{bmatrix}$

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II : $\gamma(x(\mu)) \geq \gamma^* > 0 \; D = \begin{bmatrix} \Theta(1) & & & & \\ & \Theta(1) & & & \\ & & \ddots & & \\ & & & \Theta(1) & \\ & & & & \Theta(1) \end{bmatrix}$

Three types of cones

$$D = \begin{bmatrix} (x_0 + \|\bar{x}\|)^2 & & & & \\ & \gamma^2(x) & & & \\ & & \ddots & & \\ & & & \gamma^2(x) & \\ & & & & (x_0 - \|\bar{x}\|)^2 \end{bmatrix},$$

$$\text{III} : \gamma(x(\mu)) \to 0, \; D = \begin{bmatrix} \Theta(1) & & & & \\ & \Theta(\mu) & & & \\ & & \ddots & & \\ & & & \Theta(\mu) & \\ & & & & \Theta(\mu^2) \end{bmatrix}$$

Bad Example

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}, \; x = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 3\mu \\ \sqrt{\mu} \\ 0 \end{pmatrix}$$

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$$\gamma^2(x) = \Theta(\mu), \; Ax = \begin{pmatrix} \Theta(\sqrt{\mu}) \\ \Theta(\sqrt{\mu}) \\ \Theta(1) \end{pmatrix}$$

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$$\gamma^2(x) = \Theta(\mu), \; Ax = \begin{pmatrix} \Theta(\sqrt{\mu}) \\ \Theta(\sqrt{\mu}) \\ \Theta(1) \end{pmatrix}$$

$$AFA^T = \frac{\gamma^2(x)}{2} A \begin{bmatrix} -1 & & \\ & 1 & \\ & & 1 \end{bmatrix} A^T + Axx^T A^T = \begin{bmatrix} \Theta(\mu) & \Theta(\mu) & \Theta(\sqrt{\mu}) \\ \Theta(\mu) & \Theta(\mu) & \Theta(\sqrt{\mu}) \\ \Theta(\sqrt{\mu}) & \Theta(\sqrt{\mu}) & \Theta(1) \end{bmatrix},$$

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$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}, \; x = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 3\mu \\ \sqrt{\mu} \\ 0 \end{pmatrix}$$

$$\gamma^2(x) = \Theta(\mu), \; Ax = \begin{pmatrix} \Theta(\sqrt{\mu}) \\ \Theta(\sqrt{\mu}) \\ \Theta(1) \end{pmatrix}$$

$$AFA^T = L\Lambda L^T: \quad L = \begin{bmatrix} 1 & & \\ \Theta(1) & 1 & \\ \Theta(1/\sqrt{\mu}) & \Theta(1/\sqrt{\mu}) & 1 \end{bmatrix},$$

Bad Example (cont.)

$$D = \begin{bmatrix} \Theta(1) & & \\ & \Theta(\mu) & \\ & & \Theta(\mu^2) \end{bmatrix},$$

$$Q = \begin{bmatrix} \Theta(1) & \Theta(\mu) & \Theta(1) \\ \Theta(\sqrt{\mu}) & \Theta(1) & \Theta(\sqrt{\mu}) \\ \Theta(1) & \Theta(\sqrt{\mu}) & \Theta(1) \end{bmatrix},$$

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$$AFA^T = AQDQ^TA^T = AQ_1D_1Q_1^TA^T + AQ_\mu D_\mu Q_\mu^TA^T,$$

Bad Example (cont.)

$$D = \begin{bmatrix} \Theta(1) & & \\ & \Theta(\mu) & \\ & & \Theta(\mu^2) \end{bmatrix},$$

$$Q = \begin{bmatrix} \Theta(1) & \Theta(\mu) & \Theta(1) \\ \Theta(\sqrt{\mu}) & \Theta(1) & \Theta(\sqrt{\mu}) \\ \Theta(1) & \Theta(\sqrt{\mu}) & \Theta(1) \end{bmatrix},$$

$$m_{11} = a_1 F (a_1)^T = a_1 Q_1 D_1 Q_1^T (a_1)^T + a_1 Q_\mu D_\mu Q_\mu^T (a_1)^T,$$

Bad Example (cont.)

$$D = \begin{bmatrix} \Theta(1) & & \\ & \Theta(\mu) & \\ & & \Theta(\mu^2) \end{bmatrix},$$

$$Q = \begin{bmatrix} \Theta(1) & \Theta(\mu) & \Theta(1) \\ \Theta(\sqrt{\mu}) & \Theta(1) & \Theta(\sqrt{\mu}) \\ \Theta(1) & \Theta(\sqrt{\mu}) & \Theta(1) \end{bmatrix},$$

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$$D = \Theta(1),\ a_1 Q_1 = \Theta(\sqrt{\mu}), m_{11} = \Theta(\mu)$$

$$m_{31} = a_3 Q_1 D_1 Q_1^T (a_1)^T + \ldots = \Theta(\sqrt{\mu})$$

SOCP Theorem

Theorem 1 *Given a fixed matrix $A \in \mathbf{R}^{m \times n}$, and $\mathcal{F}$ - a set of SOCP primal scaling matrices for $x(\mu)$, with $x(\mu) = x^* + \Theta(\mu)$, and any $\mu > 0$ then*

$$\text{sup}\{||L|| : L \Lambda L^T = AFA^T, \text{ as } \forall F \in \mathcal{F}\} < \infty$$

Idea of the proof

If $x = x^* + \Theta(\mu)$, then $Q_1(\mu) = Q_1^* + \Theta(\mu)$
(in the bad example $Q_1(\mu) = Q^* + \Theta(\sqrt{\mu})$)

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LP: (Dikin-Stewart-Todd-...)

$$\text{sup}\{\|(ADA^T)^{-1}AD\| : \forall \text{diagonal positive } D\} < \infty$$

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SDP: On the central path (Sturm, Zhang)

$$\text{sup}\{\|(AFA^T)^{-1}AF\| : \ \forall \text{diagonal positive } D\} < \infty$$