IMA
University of Minnesota

Tools for Modeling and Data Analysis in Finance/Asset Pricing

Derivatives in Commodity Markets

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March 31, 2004
Figure 1: Crude Price Evolution
Figure 2: Gold Price Evolution
Figure 3: Cotton Price Evolution
Figure 4: Soy Beans Price Evolution
Figure 5: Electricity Price Evolution
Figure 6: SP 500
Figure 7: Microsoft
Commodity Prices
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Similar to stock prices: random walk.
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Different from stock prices: more stationary.
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Forwards $\rightarrow$ Over-The-Counter

Futures $\rightarrow$ Exchange Traded
Arbitrage

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Example: Gold Futures

Suppose that we want to invest in gold for a year, we can do two things:
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Example: Gold Futures

Suppose that we want to invest in gold for a year, we can do two things:
1) We can borrow money and use it to buy gold.
2) We can buy a 1–year forward contract in gold.

If the price of an ounce is currently $410, how much should the 1–year forward price be?
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Arbitrage (Cont.)

Suppose that

\[ F(0, T) > e^{r_410}. \]

Then we can follow the following strategy:
Arbitrage (Cont.)

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1) Short the forward contract.
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2) Borrow $410, buy an ounce of gold.
Arbitrage (Cont.)

Suppose that

\[ F(0, T) > e^r \cdot 410. \]

Then we can follow the following strategy:

1) Short the forward contract.

2) Borrow $410, buy an ounce of gold.

3) In a year we repay our loan, deliver the gold and make \( F(0, T) - e^r \cdot 410 > 0 \).
Arbitrage (Cont.)

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\[ F(0, T) < e^r 410 \]

we do the opposite:
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4) We take delivery of an ounce of gold, pay \( F(0, T) \) and collect \( e^r 410 \).

In this case we will make, free of risk, \( e^r 410 - F(0, T) > 0 \)
By arbitrage, the forward price of a financial asset $S_t$ is given by

$$F(0, T) = S_0 e^{(r-q)T}$$

where $T$ is the expiration of the contract, $r$ is the (continuously compounded) interest rate and $q$ is the dividend yield.
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However, since they provide different cash flow profile some care should be taken when trading one vs the other (tail-hedging).
Market Participants
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Hedgers: Consumers and producers.
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Speculators.
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Market Makers.
Market Participants

Hedgers: Consumers and producers.

Speculators.

Market Makers.
Forward Curves
Forward Curves

In most commodity markets: variety of expirations.

Figure 8: Crude Oil Forward Curve
Figure 9: Crude Oil Forward Curve

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<th>Contract Table</th>
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<td>Pricing Date: 3/26/04</td>
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Typeset by FoilTEX
Figure 10: Gold Forward Curve
Figure 11: Heating Oil Forward Curve
Figure 12: Natural Gas Forward Curve

<table>
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<tr>
<th>Contract</th>
<th>Last</th>
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</table>
Forward Curves

Two different structures: Contango and Backwardation.

Gold $\rightarrow$ Contango

Crude $\rightarrow$ Backwardation
Forward Curves

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Heating Oil $\rightarrow$ Mixture
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How could a curve be in backwardation?
Convenience Yield

Example:

A food manufacturer for whom corn is an essential input.
Convenience Yield

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A food manufacturer for whom corn is an essential input.

Always keep an inventory of corn.
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There are two explanations for this:

• Theory of storage (Kaldor, Working)

• ”Aggegative approach”. (Working aggregated stocks from different locations and grades and plotted them vs spreads in Chicago)
Figure 13: Working Curve. US Stocks vs CL6-CL1
Figure 14: CL1 vs CL6-CL1
Seasonality

Examples: Corn, Natural Gas.
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Figure 15: Natural Gas Forward Curve
Seasonality (Cont.)

Natural Gas: As opposed to corn, it has constant supply but seasonal demand.
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Complex seasonal patterns: by season, by day of the week, by hour.
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Complex seasonal patterns: by season, by day of the week, by hour.

There isn’t anything the market can do to prepare for the high season (peak hours), so the shape of the curve could be very extreme.
Swaps

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If the interest rate curve is flat at 1% then

\[
\frac{x}{(1 + .01)} + \frac{x}{(1 + .01)^2} = \frac{34}{(1 + .01)} + \frac{31}{(1 + .01)^2}
\]  

(1)
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If the spread is wide it pays to refine.
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So, producers can hedge by mimicking this process using derivatives markets.
Spreads (Cont.)

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\[
S_t = E(t) - HG(t)
\]

where \( H \) is the heat rate.
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Buy a futures contract in NY, sell one in London.
Hedging

Two main risks when using derivatives to hedge:
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A farmer growing corn does not know the amount of bushels to hedge until harvest.
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If $F$ is the current fwd price, $S$ the final price, $Q$ the amount produced and $H$ the amount hedged, the hedged revenue is

$$R = SQ + (H(S - F))$$
We choose $H$ by minimizing the variance of that quantity:

$$H = -\frac{\rho_{SQ, S} \sigma_{SQ}}{\sigma_S}$$
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**Stack Hedge:** Hedge the exposure with a single contract.

**Stack-and-roll:** Roll the hedge into the next contract when the one we are using expires.
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In general people like to used short-dated contracts because of liquidity
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Given that there is enormous basis risk in a deal like this: what is the best strategy to follow?
Options

Suppose that in a risk neutral world:

\[ \frac{dS}{S} = (r - q)dt + \sigma dW \]

What is the stochastic differential equation for \( F \) (forward contract)?
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The futures price can therefore be treated like a stock paying a dividend yield of \( r \)
Options

European Calls and Puts on Futures can then be valued using Black-Scholes.
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\[ c = e^{-rT} \left( F_0 N(d_1) - K N(d_2) \right) \]
\[ p = e^{-rT} \left( K N(-d_2) - F_0 N(-d_1) \right) \]

where

\[ d_1 = \frac{\ln(F_0/K) + \sigma^2 T/2}{\sigma \sqrt{T}} \]
\[ d_2 = d_1 - \sigma \sqrt{T} \]
Options (Cont.)

Two ways to get to Black-Scholes’ formulae:
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Therefore we can form a portfolio that cancels \( dW \).

That portfolio should not return more money than a bank account.

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Girsanov’s Theorem:

\[ dY(t) = a(t, \omega)dt + dW(t) \] 
\[ (2) \]

Set

\[ M_t = e^{-\int_0^t a(s, \omega)dW(s)} - \frac{1}{2} \int_0^t a^2(s, \omega)ds \]

and assume that \( E(e^{\frac{1}{2} \int_0^T a^2(s, \omega)ds}) < \infty \)

Define a new measure by \( dQ(\omega) = M_T(\omega)dP(\omega) \)

Then \( Y(t) \) is a Brownian Motion wrt \( Q(t \leq T) \).
Options (Cont.)

Risk Neutral Valuation
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Risk Neutral Valuation

The value of a call should be:

\[ C(S_0, K, T, r) = e^{-rT} \hat{E}(\max(S_T - K, 0)) \]  

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Idea of proof:
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- Find a replicating portfolio that equals the payoff at the end, its initial value is 3
Asian Options

Payoffs:

\[ C(S) = \max\left(\sum_{i=1}^{n} \frac{S(i)}{n} - K, 0\right) \]

\[ P(S) = \max(K - \sum_{i=1}^{n} \frac{S(i)}{n}, 0) \]
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Edgeworth expansions (Jarrow-Rudd):

Suppose that \( F(s) \) is the true distribution for the average.
\( A(s) \) an approximating distribution.

\[
f(s) = a(s) + \frac{(\kappa_2(F) - \kappa_2(A))}{2!} \frac{d^2a(s)}{ds^2} - \frac{(\kappa_3(F) - \kappa_3(A))}{3!} \frac{d^3a(s)}{ds^3} \\
+ \frac{(\kappa_4(F) - \kappa_4(A)) + 3(\kappa_2(F) - \kappa_2(A))^2}{4!} \frac{d^4a(s)}{ds^4} + \epsilon(s)
\]

where
\[ \kappa_1(F) = \mu_1(F), \kappa_2(F) = \mu_2(F), \]

\[ \kappa_3(F) = \mu_3(F) \text{ and } \kappa_4(F) = \mu_4(F) - 3\mu_2^2(F) \]

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\[ C(S_1(T), S_2(T)) = \max(S_1(T) - S_2(T) - K, 0) \]

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Start with the usual Geometric Brownian Motion:

\[ \frac{dS_i}{S_i} = \mu_i dt + \sigma_i dW_i \quad \text{with } i = 1, 2. \quad <dW_1, dW_2> = \rho dt \] (4)
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Same argument as in Black-Scholes produces a PDE that can be solved numerically.
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However, people look for approximations that give closed form solutions to be able to compute greeks.
Figure 16: Distribution of Spread between two assets
\((S_1(0) = 100, S_2(0) = 100, \sigma_1 = 50\%, \sigma_2 = 40\%, \rho = 80\%)\)
Using approximations:

Model the spread directly as a normal variable, make the first two moments of the spread match the ones implied by the dynamics of $S_1$ and $S_2$. 
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Kirk (1995) proposed to write the payoff as

$$C(S_1(T), S_2(T)) = (S_2(T) + K)\max\left(\frac{S_1(T)}{S_2(T) + K} - 1, 0\right)$$

and approximating $\frac{S_1}{(S_2+K)}$. 

---

**Spread Options (Cont.)**
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Recently Carmona-Durrleman (2003) proposed a formula based on properties of the bivariate normal and convex inequalities.
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Spread Options (Cont.)

A particular case: Option to exchange one asset for another (Margrabe).
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First appearance of the ”forward measure approach” in the literature (very popular in the interest rate world).
Spread Options (Cont.)

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First appearance of the "forward measure approach" in the literature (very popular in the interest rate world).

By risk neutral valuation:

\[ C(0) = e^{-rT}E(\max(S_1(T) - S_2(T), 0)) \]

Payoff can be rewritten:
\[ C(S_1(T), S_2(T)) = \max(S_2(T)(S_1(T)/S_2(T) - 1), 0) \]
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If we are lucky and it is lognormal we will obtain a Black-Scholes-type of formula.

Suppose that

\[ \frac{dS_1}{S_1} = rdt + \sigma_1 dW_1 \quad \text{and} \quad \frac{dS_2}{S_2} = rdt + \sigma_2 dW_2 \]
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\[
S_2(T) = S_2(0)e^{-\frac{1}{2}\sigma_2^2 T + \sigma_2 dW_2(T)}
\]

(6)

so by Girsanov we know that \( d\hat{W}_2 = dW_2(T) - \sigma_2 dt \) is a Brownian Motion under the new measure.

Also, \( d\hat{W}_1 = dW_1(T) - \rho \sigma_2 dt \) is a Brownian Motion.

By Itô

\[
d\left(\frac{S_2}{S_1}\right) = \frac{S_1}{S_2} (\sigma_1^2 - \sigma_1 \sigma_2 \rho) dt + \frac{S_1}{S_2} (\sigma_1 dW_1 - \sigma_2 dW_2)
\]

Under the new measure:
\[ d \left( \frac{S_1}{S_2} \right) = \frac{S_1}{S_2} (\sigma_1 d\hat{W}_1 - \sigma_2 d\hat{W}_2) \]

which can be written as

\[ d \left( \frac{S_1}{S_2} \right) = \frac{S_1}{S_2} \sigma dW \]

and therefore is lognormal with

\[ \sigma = \sqrt{\sigma_1^2 + \sigma^2 + 2 \rho \sigma_1 \sigma_2} \]

giving

\[ C(S_1(0), S_2(0)) = S_1(0) N(d_1) - S_2(0) N(d_2) \]

with
\begin{align*}
d_1 &= \frac{S_1(0)/S_2(0) + \sigma^2 T/2}{\sigma \sqrt{T}} \\
d_2 &= d_1 - \sigma \sqrt{T}
\end{align*}
Timespread Options and Storage

Storage allows us to take advantage of contango markets.
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Renting storage:
market in contango → make money.
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So: renting storage $\sim$ put spread option.
Other Options

Natural Gas: Swing Options.
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$$\max(S - K + \text{fut}[k_0 + 1], \text{fut}[k_0])$$
Other Options (Cont.)

Swaptions
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Swaptions
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For example we can write an option on $x$ from $[1]$
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How do we value that? What is the process followed by \( x \)?
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It is a basket of geometric brownian motions.
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We need implied volatilities for each of the contracts in the swap.
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And options on futures expire a few days before the futures expiration.
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Term Structure Models

Modelling dynamics of commodity prices: Historically later than the same task for interest rates.
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Two types of models:

- spot "+" convenience yield (Black-Karasinski, Gibson-Schwartz)
- forward curve (Following Heath-Jarrow-Morton, Miltersen-Shwatz, Amin-Ng-Pirrong)
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Can we explain the term structure of volatilities?

Remark: Term structure of volatilities in the context of futures means something different from what it means in the context of stocks.
Figure 17: Term Structure of (ATM) Volatilities. Natural Gas.
Term Structure Models. Black-Karasinski

1-factor model. mean reverting in $\log S_t$. 
Term Structure Models. Black-Karasinski

1-factor model. mean reverting in $\log S_t$.

\[ d \log S_t = \alpha (f(t) - \log S_t) dt + \sigma_t dW_t \]  

(7)
Term Structure Models. Black-Karasinski

1-factor model. mean reverting in $\log S_t$.

$$d \log S_t = \alpha(f(t) - \log S_t)dt + \sigma_t dW_t$$  (7)

Solving, we see that $\log(S_T)$ is normally distributed with

$$\text{mean} = e^{-\alpha(T-t)} \log S_t + \alpha e^{-\alpha T} \int_t^T e^{\alpha s} f(s)ds$$
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standard deviation $= \sigma e^{-\alpha T} \sqrt{\frac{e^{2\alpha T} - e^{2\alpha t}}{2\alpha}}$
We can write the processes for $S_t$ and for $F(t, T)$ in terms of the initial forward curve:

\begin{align*}
S_t &= F(0, t)e^{-\frac{\sigma^2}{4\alpha}(1-e^{-2\alpha t})}e^{\alpha t}\sqrt{\frac{e^{2\alpha t-1}}{2\alpha}}N(0, 1), \quad (8) \\
F(t, T) &= F(0, T)e^{\frac{\sigma^2}{4\alpha}(e^{-2\alpha T}-e^{-2\alpha(T-t)})}e^{\alpha T}\sqrt{\frac{e^{2\alpha T-1}}{2\alpha}}N(0, 1) \quad (9)
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which can be used to calibrate to the current term structure.
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$$\text{stdev}(\log(F(t_2, T)/F(t_1, T))) \sim \sigma e^{-\alpha(T-t_1)}\sqrt{t_2-t_1}$$
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$$\text{stdev}(\log(F(t_2, T)/F(t_1, T))) \sim \sigma e^{-\alpha(T-t_1)}\sqrt{t_2 - t_1}$$

so, instantaneously, the volatility of $F(t, T)$ is $\sigma e^{-\alpha(T-t)}$. 
And the volatility for the whole life:

\[ \sigma_{F(0,T)} = \sigma e^{-\alpha T} \sqrt{\frac{e^{2\alpha T} - 1}{2\alpha}} \frac{1}{\sqrt{T}} \]
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Common to all 1-factor models.

Solution: More factors.
Term Structure Models. Gibson-Schwartz

2-factor model.

\[ dS_t = (r_t - q_t)S_t dt + \sigma S_t dW_t^1 \]  \hspace{1cm} (10)

\[ dq_t = \alpha(\theta - q_t) dt + \gamma dW_t^2 \]  \hspace{1cm} (11)

where \( d < W^1, W^2 >_t = \rho dt. \)

Dynamics of \( S_t = \) Geometric Brownian Motion.

Dynamics of \( q_t = \) Orstein-Uhlenbeck (Mean reverting).
Term Structure Models. Gibson-Schwartz

Carmona-Ludkovski:

Compute implied convenience yield.
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Compute implied convenience yield.

Find that each futures contract seems to carry its own risk.
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Term Structure Models. Heath-Jarrow-Morton

Cortazar-Schwartz (1994) model the whole forward curve simultaneously

\[
\frac{dF(t, T)}{F(t, T)} = \sum_{i=1}^{n} \sigma_i(t, T) dW_i(t) \tag{12}
\]

So,

\[
F(t, T) = F(0, T) e^{-\frac{1}{2} \int_0^t \sum_{i=1}^{n} \sigma_i^2(s, T) ds + \int_0^t \sum_{i=1}^{n} \sigma_i(s, T) dW_i(s)}
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The current forward curve is entered as starting point.

Setting \( T = t \) we obtain the process for \( S(t) \):

\[
S(t) = F(0, t)e^{-\frac{1}{2} \int_0^t \sum_{i=1}^{n} \sigma_i^2(s, t)ds + \int_0^t \sum_{i=1}^{n} \sigma_i(s, t)dW_i(s)}
\]
Term Structure Models. Heath-Jarrow-Morton (Cont.)

The HJM model for interest rates:
Term Structure Models. Heath-Jarrow-Morton (Cont.)

The HJM model for interest rates:

\( P(t, T) \) a zero coupon bond. We define the zero rate as \( R(0, T) \) so that

\[
P(t, T) = e^{-R(0,T)T}.
\]

Also, we define the forward rate between \( T_1 \) and \( T_2 \) observed at time \( t \) as \( f(t, T_1, T_2) \) so that

\[
e^{R(0,T_2)T_2} = e^{R(0,T_1)T_1} e^{f(0,T_1,T_2)(T_2-T_1)}
\]

So it is the rate that we can lock in today for money borrowed between times \( T_1 \) and \( T_2 \). Then
\[ f(0, T_1, T_2) = \frac{\ln P(0, T_1) - \ln P(0, T_2)}{(T_2 - T_1)} \]

Letting \( T_2 \) approach \( T_1 \) we obtain

\[ f(0, T) = -\frac{\partial \ln P(0, T)}{\partial T} \]

Now, if we propose the bond model:

\[ \frac{dP(t, T)}{P(t, T)} = \sum_{i=1}^{\nu} \sigma_i(t, T) dW_i(t) + r(t) dt \]
\[ f(0, T_1, T_2) = \frac{\ln P(0, T_1) - \ln P(0, T_2)}{(T_2 - T_1)} \]

Letting \( T_2 \) approach \( T_1 \) we obtain

\[ f(0, T) = -\frac{\partial \ln P(0, T)}{\partial T} \]

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By Itô and differentiating wrt \( T \):
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We obtain:

\[
df(t, T) = \sum_{i=1}^{\nu} \frac{\partial \sigma_i(t, T)}{\partial T} dW_i(t) + \left( \sum_{i=1}^{\nu} \frac{\partial \sigma_i(t, T)}{\partial T} \int_t^T \frac{\partial \sigma_i(t, s)}{\partial s} ds \right) dt
\]

which gives the dynamics for the forward rates in a risk neutral world.
In the same spirit as HJM Miltersen-Schwartz (1999) propose to define forward convenience yields. So the model for the curve is obtained from:

\[
    f(t, s) = f(0, s) + \int_0^t \mu_f(u, s) du + \int_0^t \sigma_f(u, s) \cdot dW_u
\]

\[
    \epsilon(t, s) = \epsilon(0, s) + \int_0^t \mu_\epsilon(u, s) du + \int_0^t \sigma_\epsilon(u, s) \cdot dW_u
\]

\[
    S_t = S_0 + \int_0^t S_u \mu_S(u) du + \int_0^t S_u \sigma_S(u) \cdot dW_u
\]
Term Structure Models

Choosing the Volatility Functions
Term Structure Models

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How do we choose the $\sigma$ functions in [12]?
Term Structure Models

Choosing the Volatility Functions

How do we choose the $\sigma$ functions in 12?

Principal Components Analysis.
Term Structure Models

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The same phenomenon was found by Cortazar and Schwartz in copper futures, and it is also true in crude oil, Libor rates, etc.
Term Structure Models

Figure 18: First four Principal Components. Crude Oil.
Term Structure Models

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The correlation structure.
Correlation among different rates is very high.
Possible model:
contiguous rates correlation $= \rho$.
For non-contiguous it decays exponentially.
If $\rho$ is high we recover the level-slope-curvature structure.
Term Structure Models

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Depending on how high is the intercurve correlation we will get ”separation” vectors of different orders.
Figure 19: PCA of crude and heating oil together.
Figure 20: Seasonality in the Eigenvalues (o=heating oil, x=crude)
Figure 21: Term Structure of Volatilities. Crude Forward Curve. Different Smiles on 2/25/04.
Smiles

As usual...
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Challenge: liquidity, data.
Smiles (Cont.)
The Vol Smile and its Implied Tree (Derman & Kani, 1994)
They modify the stock process in order to match option prices.
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Introduce the concept of **local volatility**

Instead of

\[
\frac{dS}{S} = \mu dt + \sigma dZ
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use

\[
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Inverse Problem: Given option prices can we determine \( \sigma \) ?
Smiles (Cont.)

Derman and Kani work in the binomial framework.
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Figure 22: Derman-Kani Tree.

**Figure 3. The Implied Tree**

- Stock price vs. time
Smiles (Cont.)

Stochastic Volatility.

"Black-Scholes Analysis"
Smiles (Cont.)

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"Black-Scholes Analysis"

Assume that \( S \) satisfies (as always)

\[
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and \( \sigma \) is also stochastic following

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The goal is to mimic B-S's line of thought
Main idea in Black-Scholes: The source of risk in an option is the same as the source of risk of the underlying so we can cancel this risk (the market is "complete").
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Form the portfolio

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By doing Itô and killing the BMs:

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\[ \Delta = \frac{\partial V}{\partial S} - \left( \frac{\partial V}{\partial \sigma} \bigg/ \frac{\partial V_1}{\partial \sigma} \right) \frac{\partial V_1}{\partial S} \]
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In the end, by collecting \( V \) terms on the left and \( V_1 \) terms on the right:
\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho \sigma q S \frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial^2 V}{\partial \sigma^2} + rS \frac{\partial V}{\partial S} - rV \\
= \frac{\partial V_1}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + \rho \sigma q S \frac{\partial^2 V_1}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial^2 V_1}{\partial \sigma^2} + rS \frac{\partial V_1}{\partial S} - rV_1
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"Market price of risk"
Real Options

Similarities between options and investment projects:

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The payoff of the call option is then: \( \max(S_{elec} - H S_{natgas}, 0) \)
Figure 23: Scheme of a Refinery
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