Heavy-tailed GARCH Models: Pricing and Risk Management Applications in Power Markets

Shi-Jie Deng
Industrial and Systems Engineering
Georgia Institute of Technology

(Joint works with Jiang, Peng, et al.)
Agenda

• Background and motivations

• Quantile-based GARCH Models
  – Parameter inference
  – Applications
    • Electricity derivatives pricing
    • Risk management measures

• Semi-parametric Estimation of Confidence Intervals of Conditional Quantiles of GARCH Models
  – Normal approximation and data-tilting
  – Applications

• Conclusion
Background and Motivations

• Rapid developments of power markets starting in the mid-1990s
  – Power Exchange/ISO in California.
  – PJM, NY, NE power pools.

• Market setbacks since 2000
  – Fallen or financially distressed power merchants
  – Dropping liquidity in power exchange/OTC markets

• The nature of incompleteness in energy (power) markets
  – Almost non-storable underlying
  – Limited physical supply and inelastic demand
  – Tremendous price and quantity volatility (e.g., price spikes)
  – Limited ability in hedging quantity risks
Background and Motivations (con’t)

• Redesign of power markets
  – FERC Standard Market Design
  – Development of futures/contract markets

• Risk management needs
  – Independent power producers hedge their production.
  – Power marketers quantify, monitor and control trading risks in wholesale and retail markets.

• Trading, Asset valuation, and project selection and financing
  – Pricing and risk management tools to support trading
  – Evaluation of potential investment opportunities in power generation
  – Support for project financing
Implied Volatility of Call (Sept.) Options at Cinergy
(Inferred from Black-Scholes formula based on broker quotes for calls and forwards.)

Resource stack includes:
Hydro units;
Nuclear plants;
Coal units;
Natural gas units;
Misc.

Marginal Cost ($/MWh)

IMA Control & Pricing in Communication & Power Networks, 7-13 Mar. 2004
• More empirical observations: PJM log-price

• Monthly average
Literature on Energy Price Modeling and GARCH Modeling

• Energy commodity spot price models.
  – Schwartz (J.of Fin 1997)
  – Miltersen and Schwartz (JFQA 1998)
  – Hilliard and Reis (JFQA 1998)

• Electricity spot price models and electricity derivatives.
  – Kaminski (Risk Book 1997)
  – Barz and Johnson (1998)
  – Deng (PSERC 1998)
  – Mount and Ethier (PSERC 1998)
  – Deng, Sun and Meliopoulos (PSERC 2003)

• ARCH/GARCH modeling and option pricing.
  – Engle (Econometrica 1982), Bollerslev (J. Econ. 1986)
  – Nelson (Econometrica 1991)
  – Hall and Yao (Econometrica, to appear)
  – Duan (Math Fin. 1995), Heston and Nandi (RFS 2002)
Quantile-based GARCH Models  
(Deng and Jiang, 2003)

- Exhibit heavy, flexible and asymmetric tail behaviors
- Enable maximum likelihood estimation (MLE)/quasi-MLE combined with quantile-based estimation
- Have explicit conditional quantile functions
- Allow efficient computation in pricing and risk management applications  
  - Easy and fast simulation
Quantile GARCH Models

- Quantile GARCH(1,1)
  \[ X_t = \sigma_t \cdot \varepsilon_t \]
  \[ \sigma_t^2 = c + b \cdot X_{t-1}^2 + a \cdot \sigma_{t-1}^2 \]
  where
  \[ \varepsilon_t \sim \text{Class I or Class II} \]

- Extension: GARCH type AR(1) process
  \[ X_t = X_{t-1} + \kappa(\theta - X_{t-1}) + \sigma_t \cdot \varepsilon_t \]
  \[ \sigma_t^2 = c + b \cdot X_{t-1}^2 + a \cdot \sigma_{t-1}^2 \]
  where
  \[ \varepsilon_t \sim \text{Class I or Class II} \]
Quantile Class-I and II Distributions
Quantile Function and Quantile Modelling

• Definition: Suppose that $F(x)$ is a probability distribution function, then the quantile function of $F$ is the generalized inverse

$$F^{-1}(x) = \inf\{y : F(y) \geq x\}$$

• Quantile modelling: Directly specify the quantile function of the distribution in which you are interested.

• The advantages of quantile modelling:
  - Fast sampling
  - Easy Q-Q plots
  - Easy probability calculation
  - Explicit Quantile function
Two classes of new distributions

The first Class $Q_I(\alpha, \beta, \delta, \mu)$ is defined by the following quantile function:

$$q(y; \alpha, \beta, \delta, \mu) = \delta \alpha \left\{ \log \frac{y^\beta}{1 - y^\beta} \right\}^{(\frac{1}{\alpha})} + \mu$$

where $\delta, \alpha, \beta \in \mathbb{R}_+, \mu \in \mathbb{R}$, and the superscript ’$(\alpha)$’ for $\alpha > 0$ represents the operation below.

$$x(\alpha) = \begin{cases} 
  x^\alpha & \text{if } x > 0 \\
  0 & \text{if } x = 0 \\
  -(-x)^\alpha & \text{if } x < 0 
\end{cases}$$

- $\mu$: location
- $\delta$: scaling parameter
- $\alpha$: tail thickness
- $\beta$: tail balancing factor
Properties of Class I Distributions

• Explicit form of probability distribution function
  \[ q^{-1}(x; \alpha, \beta, \delta, 0) = \left\{ \frac{1}{1 + e^{-\frac{1}{\delta}x(\alpha)}} \right\}^{\frac{1}{\beta}} \]

• Explicit form of probability density function
  \[ p(x; \alpha, \beta, \delta, 0) = \frac{1}{\delta \beta} \cdot \frac{x(\alpha)}{x} \cdot e^{-\frac{1}{\delta}x(\alpha)} \cdot \frac{1}{(1 + e^{-\frac{1}{\delta}x(\alpha)})^{1 + \frac{1}{\beta}}}, \]

• Potentially different tail behaviors at two sides
  
  - Right side: as \( x \to +\infty \) the right tail is about \( C x^{\alpha-1} e^{-\frac{1}{\delta}x^{\alpha}} \).
  
  - Left side: as \( x \to -\infty \) the left tail is about \( C (-x)^{\alpha-1} e^{-\frac{1}{\delta \beta}(-x)^{\alpha}} \).
The Second Class: \( Q_{II}(\alpha_-, \alpha_+, \delta_-, \delta_+, \mu) \)

The quantile function of the second Class is

\[
q(y; \alpha_-, \alpha_+, \delta_-, \delta_+, \mu) = -\frac{1}{\alpha_-} (\log \frac{1}{y})^{\frac{1}{\alpha_-}} + \frac{1}{\alpha_+} (\log \frac{1}{1-y})^{\frac{1}{\alpha_+}} + \mu
\]

where \( \alpha_-, \alpha_+, \delta_-, \delta_+ \in \mathbb{R}_+, \mu \in \mathbb{R} \). We are mostly interested in the cases \( \alpha_- \leq 1, \alpha_+ \leq 1 \).

- \( \mu \): location

- \( \delta_+ / \delta_- \): scaling parameters at the right / left hand side.

- \( \alpha_+ / \alpha_- \): tail thickness parameters at the right / left hand side.

Remark: \( \alpha_- \) and \( \alpha_+ \) provide the flexibility for this class of distributions to have different tail thickness at the two sides.
Figure 1: Density plot of the first class
Figure 2: Density plot of the second class
Class-I Fit of PJM Daily Price Return

\[ \alpha = 0.99245 \]
\[ \beta = 0.92348 \]
\[ \delta = 0.19064 \]
\[ \mu = -0.027395 \]
Class-II Fit of PJM Daily Load

\[ \alpha_{\text{left}} = 0.91657 \]
\[ \alpha_{\text{right}} = 1.0422 \]
\[ \delta_{\text{left}} = 0.048554 \]
\[ \delta_{\text{right}} = 0.053146 \]
\[ \mu = -0.020806 \]
A Two-step Estimation Scheme

• Step 1: Quasi-MLE for estimating GARCH coefficients
  – Hall and Yao (Econometrica, to appear)

\[
L_\nu(a, b, c) = \sum_{t=\nu}^{n} \left\{ \frac{X_t^2}{\tilde{\sigma}_t^2(a, b, c)} + \log \tilde{\sigma}_t^2(a, b, c) \right\},
\]

• Step 2: Quantile-based estimation for obtaining parameters for the innovation term.
Empirical Estimation

- MLE estimation of GARCH(1,1) coefficients:
  \[ \alpha_0 = 0.0023; \alpha_1 = 0.0587; \beta = 0.9252. \]
- Q-Q Plot of the innovation term.
  - Daily electricity price: PJM Western Hub

\[ \alpha = 0.92191 \]
\[ \beta = 0.77793 \]
\[ \delta = 0.56062 \]
\[ \mu = -0.21724 \]
Empirical Estimation (con’t)

- Q-Q Plot of unconditional marginal distrib.
Application: Value Energy Contracts

- Energy (electricity) derivatives are complex financial instrument
  - Physical characteristics of underlying
  - Path-dependent and American-style (exercisable at any time)

- Examples:
  - Tolling agreements.
    - Independent power producers hedge operational risks.
    - Power merchants implement asset-light operations.
    - Fossil-fueled power producers hedge output risks.
  - Swing contracts
  - Gas storage contracts
Application: Pricing Methodology

• European-style financial contracts
  – Simulation

• American-style path-dependent financial contracts
  – Dynamic programming least-squares approximation:
    Longstaff and Schwartz (2001), Tsitsiklis and Van Roy (2001)
  – Simulation
Interval Estimation for the Conditional Quantile of Fat-tailed GARCH Models
(Chan, Deng, Peng, and Xia, 2003)

• One-step conditional quantile estimation of heavy-tailed GARCH models
• Characterization of confidence intervals for conditional quantiles
Model Specification

- Heavy-tailed GARCH Model

\[ X_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = c + \sum_{i=1}^{p} b_i X_{t-i}^2 + \sum_{j=1}^{q} a_j \sigma_{t-j}^2, \]

- Heavy-tail in innovation

\[ \epsilon_t \sim G(x) \quad \text{(cdf of } \epsilon_t) \]

\[ 1 - G(x) \sim c_1 x^{-\gamma}, \quad G(-x) \sim c_2 x^{-\gamma}, \quad \text{for } x \text{ large} \& \gamma > 2 \]

- 100\(\alpha\)% one step ahead conditional VaR

\[ x_{\alpha,n} = \inf\{x : P(X_{n+1} \leq x|X_{n+1-k}, k \geq 1) \geq \alpha\}. \]
Estimation

• Likelihood function (quasi-MLE)

\[ L_\nu(a, b, c) = \sum_{i=\nu}^{n} \left\{ \frac{X_t^2}{\tilde{\sigma}_t^2(a, b, c)} + \log \tilde{\sigma}_t^2(a, b, c) \right\}, \]

• Tail index

\[ \hat{\gamma} = \left\{ \frac{1}{k} \sum_{i=1}^{k} \log \frac{\hat{\epsilon}_{m,m-i+1}}{\hat{\epsilon}_{m,m-k}} \right\}^{-1}, \]

where \( \hat{\epsilon}_t = X_t/\tilde{\sigma}_t(\hat{a}, \hat{b}, \hat{c}) \) and \( \hat{\epsilon}_{m,1} \leq \cdots \leq \hat{\epsilon}_{m,m} \) denote the order statistics of \( \hat{\epsilon}_\nu, \cdots, \hat{\epsilon}_n \)

• Estimator by method I:

\[ \hat{x}_\alpha^0 = (1 - \alpha)^{-1/\hat{\gamma}} \left( \frac{k}{m} \right)^{1/\hat{\gamma}} \hat{\epsilon}_{m,m-k}, \]

\[ \hat{x}_{\alpha,n} = \tilde{\sigma}_{n+1}(\hat{a}, \hat{b}, \hat{c}) \hat{x}_\alpha^0 \]
Theorem 1

• Suppose regularity conditions hold and

\[ k = k(m) \to \infty, \frac{k}{m} \to 0, \sqrt{k} A(m/k) \to 0, \]

\[ n^{-1} \lambda_n / A(m/k) \to 0, \log \left( \frac{k}{m(1 - \alpha)} \right) / \sqrt{k} \to 0 \]

as \( n \to \infty \). Then

\[ \frac{\hat{\gamma} \sqrt{k}}{\log \left( \frac{k}{m(1 - \alpha)} \right)} \left\{ \frac{\hat{x}_{\alpha,n}}{x_{\alpha,n}} - 1 \right\} \xrightarrow{d} N(0, 1). \]
Confidence Intervals

- **Method I: Normal approximation method.**
  - Based on Theorem 1, a confidence interval with level $\beta$ for $x_{\alpha,n}$ is

  \[
  I^*_\beta = (\hat{x}_{\alpha,n}(1 + \frac{z_\beta}{\sqrt{k}}|\log \frac{k}{m(1 - \alpha)}|)^{-1}, \hat{x}_{\alpha,n}(1 - \frac{z_\beta}{\sqrt{k}}|\log \frac{k}{m(1 - \alpha)}|)^{-1}),
  \]

  with $z_\beta$ satisfies $P(|N(0,1)| \leq z_\beta) = \beta$. 
Data Sets and Their Autocorrelation

- Daily electricity price return and load change in PJM

![Log returns of PJM Real-time LMP](image1)

![Daily adjusted load](image2)
Data Sets and Their Autocorrelation

- 1-Month PJM forward price vs. SP 500

![Log returns of PJM 1-month forward price](image1)

![Log Returns of SP500](image2)
## Comparison with Gaussian GARCH

<table>
<thead>
<tr>
<th></th>
<th>Method I</th>
<th>Conditional Normal</th>
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<tbody>
<tr>
<td><strong>k=30, alpha=0.99</strong></td>
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<tr>
<td>daily load</td>
<td>0.030</td>
<td>0.024</td>
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<tr>
<td>1-m PJM forward</td>
<td>0.009</td>
<td>0.014</td>
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<tr>
<td>3-m PJM forward</td>
<td>0.005</td>
<td>0.005</td>
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<tr>
<td>PJM real time LMP</td>
<td>0.018</td>
<td>0.030</td>
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<tr>
<td>SP500</td>
<td>0.014</td>
<td>0.017</td>
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<tr>
<td><strong>k=60, alpha=0.99</strong></td>
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<tr>
<td>daily load</td>
<td>0.024</td>
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<td><strong>k=100, alpha=0.99</strong></td>
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Confidence Band of CVaR
Conclusion

• Methodology
  – Non-Gaussian fat-tailed GARCH type models based on quantile distributions
  – A two-step procedure for parameter inference: quasi-MLE and quantile-based estimation
  – A one-step semi-nonparametric estimation scheme of high/low quantiles and their confidence intervals

• Applications
  – Modeling financial time series data as well as energy (e.g., electricity price and load) data
  – Financial derivatives/contracts pricing
  – Risk management measures (e.g., CVaR)
Future Work

• More on Quantile-based GARCH models.
  – Parameter inference
  – Multivariate extensions

• Risk-neutralized processes corresponding to the quantile-based GARCH models.

• Efficient simulation and dynamic programming algorithms for asset pricing problems.