Revenue Management for a Telecommunications Network

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Outline

Dynamic Pricing: Model Fluid Approximation Diffusion Approximation	Joint work with Rami Atar
Admission Control: Model Fluid Approximation Diffusion Approximation	Joint work with Qiong Wang

The Service Provider's Problem

A service provider (SP) has a communication network of fixed capacity and 2 types of customers:

Immediate demand (dial-up) customers –

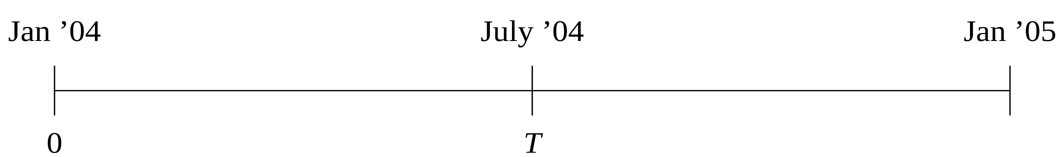
Poisson arrival processes with fixed rates

i.i.d. holding times with means ∼ minutes-hours

fixed prices

Advanced reservation customers –

Customers reserve in advance for connections over a long fixed time interval (6 months – 1 yr) at a price set in a contract



The SP's problem is to (dynamically) set the price of advanced reservation bandwidth to maximize

E[advanced reservation revenue + immediate demand revenue]

The Model

An SP has a network:

- L links ($1 \leq L < \infty$)
- Link l has C_l circuits, $1 \leq l \leq L$
- J call types (routes)
- Type j calls need (for entire duration of call) A_{lj} circuits on link l, $1 \leq l \leq L$, $1 \leq j \leq J$

The SP sells capacity to advanced reservation customers over time horizon $[0, T]$.

Advanced reservation demand arrives as independent Poisson processes,

$$\lambda_j(\boldsymbol{p}) = \text{advanced reservation demand rate for route } j \text{ with prices } \boldsymbol{p},$$
$$\boldsymbol{p} = (p_1, \ldots, p_J); \quad p_j = \text{price for route } j$$

Assume that $\boldsymbol{\lambda}(\boldsymbol{p})$ has a unique inverse $\boldsymbol{p}(\boldsymbol{\lambda})$.

Revenue rate: $\quad r(\boldsymbol{\lambda}) = \boldsymbol{\lambda} \cdot \boldsymbol{p}(\boldsymbol{\lambda})$

(r is assumed to be twice continuously differentiable and concave, with $\lim_{\boldsymbol{\lambda} \to 0} r(\boldsymbol{\lambda}) = 0$)

[Similar to model in Gallego and van Ryzin (1997)]

At time T the remaining circuits are available for immediate demand customers.

Terminal Reward for Service Provider

$g(\boldsymbol{C}^I)$ = immediate demand value (per unit of time) of network with C_l^I circuits on link l, $1 \leq l \leq L$

Immediate demand call parameters:

$$\begin{aligned} \lambda_j^I &= \text{arrival rate of route } j \text{ immediate demand calls} \\ \tau_j^I &= \text{mean holding time of route } j \text{ immediate demand calls} \\ p_j^I &= \text{price for route } j \text{ immediate demand calls} \\ a_j &= \lambda_j^I \tau_j^I : \text{route } j \text{ immediate demand `offered load'} \end{aligned}$$

Immediate demand calls arriving to find insufficient resources are blocked and lost.

Let $B_j(\boldsymbol{C}^I, \boldsymbol{a})$ denote the blocking probability of route j.

Then

$$g(\boldsymbol{C}^I) = \sum_{j=1}^{J} \lambda_j^I \tau_j^I p_j^I (1 - B_j(\boldsymbol{C}^I, \boldsymbol{a}))$$

The Dynamic Pricing Problem

At each time $t \in [0, T]$, the SP chooses the advanced reservation arrival rate $\boldsymbol{\lambda}(t)$. Let

$X_l(t) = \#$ of circuits on link l remaining at time t, $0 \le t \le T$

Then

$$X_l(t) = C_l - \sum_{j=1}^{J} A_{lj} \pi_j \left(\int_0^t \lambda_j(s) ds \right)$$

where $\pi_1, \ldots, \pi_J$ are independent Poisson processes with rate 1.

For a non-anticipating $\boldsymbol{\Lambda} = \{\boldsymbol{\lambda}(s), 0 \le s \le T\}$, we let

$$V(\boldsymbol{\Lambda}) = E\left[\int_0^T r(\boldsymbol{\lambda}(s)) ds + g(\boldsymbol{X}(T)) \right]$$

Dynamic pricing problem: Choose a (non-anticipating) $\boldsymbol{\Lambda}$ to maximize $V(\boldsymbol{\Lambda})$ subject to $\boldsymbol{X}(T) \ge 0$.

A Numerical Example

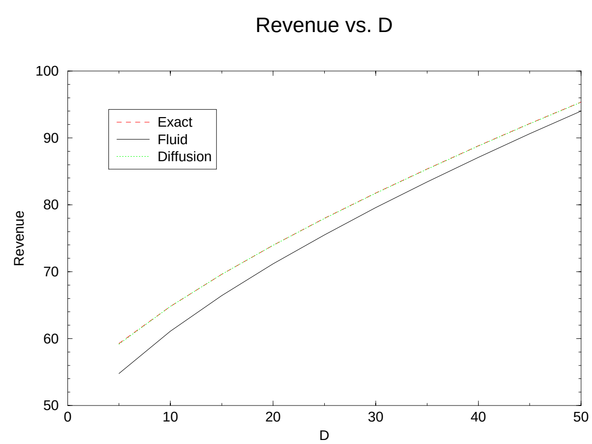
Consider an SP with $L = J = 1$, $C_1 = 100$, $T = 1$, $\lambda^I \tau^I = 50$, $p^I = 1$, and

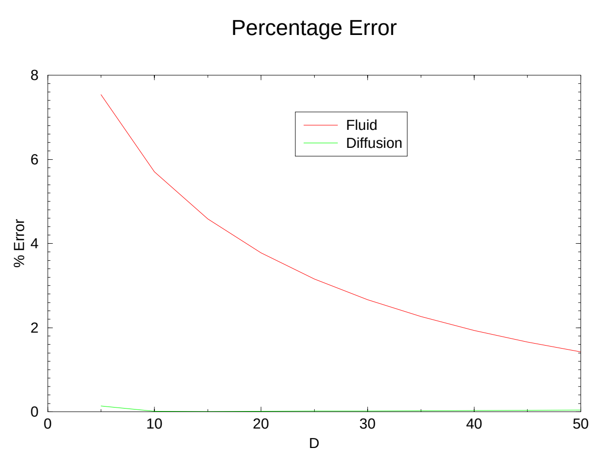
$$\lambda(p) = Dp^{-\epsilon}, \quad p > 0,$$

where $\epsilon > 1$. Then

$$\begin{aligned} p(\lambda) &= D^{1/\epsilon}\lambda^{-1/\epsilon} \\ \text{and} \quad r(\lambda) &= D^{1/\epsilon}\lambda^{1-1/\epsilon} \end{aligned}$$

Set $\epsilon = 1.5$ and vary D from 5 to 50:





The 'Asymptotically Optimal' Controls

Fluid: $\lambda(s) = 50$, $0 \le s \le 1$

Diffusion: Let $b = 0.01$

$$\lambda(s) = \lambda(s, X(s)) = \left[50 + \frac{x^* + \frac{1}{10}[X(s) - 100 + 50s]}{10(b + 1 - s)}\right]^+$$

where x^* is the unique solution to ...

The Fluid Limit

Consider a sequence of systems indexed by n, with $C_l(n) \to \infty$ as $n \to \infty$: $C_l(n) = \lfloor n\ \bar{C}_l \rfloor$ with $0 < \bar{C}_l < \infty$.

Scale the demand function:

$$\lambda_j^{(n)}(\boldsymbol{p}) = n\lambda_j(\boldsymbol{p})\ ,$$

so that $p_j^{(n)}(n\boldsymbol{\lambda}) = p_j(\boldsymbol{\lambda})$ and $r^{(n)}(n\boldsymbol{\lambda}) = nr(\boldsymbol{\lambda})$.

Scale immediate demand arrival rates ($\boldsymbol{\tau}^I, \boldsymbol{p}^I$ stay fixed):

$$\lambda_j^I(n) = n\bar{\lambda}_j^I.$$

Let $$\bar{X}_l^{(n)}(t) = n^{-1}X_l^{(n)}(t).$$

(So $\bar{X}_l^{(n)}(0) \to \bar{C}_l$.) Recall that

$$X_l^{(n)}(t) = C_l(n) - \sum_{j=1}^{J} A_{lj}\pi_j\left(\int_0^t \lambda_j(s)ds\right)\ .$$

If, as $n \to \infty$, $n^{-1}\boldsymbol{\lambda}^{(n)}(s) \to \bar{\boldsymbol{\lambda}}(s)$ a.s. uniformly on $[0, T]$, then

$$\bar{X}_l^{(n)}(t) \overset{\text{a.s.}}{\to} \bar{X}_l(t) \equiv \bar{C}_l - \sum_{j=1}^{J} A_{lj}\int_0^t \bar{\lambda}_j(s)ds, \quad 1 \le l \le L,$$

and

$$n^{-1}\int_0^T r^{(n)}(\boldsymbol{\lambda}^{(n)}(s))ds \overset{\text{a.s.}}{\to} \int_0^T r(\bar{\boldsymbol{\lambda}}(s))ds\,.$$

Fluid Limit of Terminal Reward

By assumption, $a_j(n) = n\bar{\lambda}_j^I \tau_j^I$, so that $\bar{a}_j(n) = n^{-1} a_j(n)$ $= \bar{\lambda}_j^I \tau_j^I \equiv \bar{a}_j$.

Suppose that $\bar{\boldsymbol{X}}^{(n)}(T) \stackrel{\text{a.s.}}{\to} \bar{\boldsymbol{X}}(T)$.

Let

$$\rho_l = \frac{\sum_{j=1}^{J} A_{lj}\bar{a}_j}{\bar{X}_l(T)}, \quad 1 \le l \le L\,.$$

From Kelly (1986),

$$B_j(\boldsymbol{X}^{(n)}(T), \boldsymbol{a}(n)) \to \hat{B}_j \quad \text{as} \quad n \to \infty\,,$$

where $\hat{B}_j = \ldots$

If $\rho_l \le 1$, $1 \le l \le L$, then $\hat{B}_j = 0$, $1 \le j \le J$.

Fluid Optimization Problem

Fluid Optimization Problem (FOP):

$$\text{choose} \quad \{\bar{\boldsymbol{\lambda}}(s), 0 \le s \le T\}$$

$$\text{such that} \quad \bar{\boldsymbol{\lambda}}(s) \ge 0, \quad 0 \le s \le T$$

$$\sum_{j=1}^{J} A_{lj} \int_0^T \bar{\lambda}_j(s) ds \le \bar{C}_l, \; 1 \le l \le L$$

$$\text{to maximize} \quad \int_0^T r(\bar{\boldsymbol{\lambda}}(s)) ds + \bar{g}\left(\bar{\boldsymbol{C}} - A \int_0^T \bar{\boldsymbol{\lambda}}(s) ds\right)$$

2 key observations:

1) The 2nd term depends on $\{\bar{\boldsymbol{\lambda}}(s), 0 \le s \le T\}$ only through $\left\{\int_0^T \bar{\lambda}_j(s) ds, \; 1 \le j \le J\right\}$.

2) Due to the concavity of $r(\cdot)$, given that $\int_0^T \bar{\lambda}_j(s) ds = y_j$, $1 \le j \le J$, the 1st term is maximized by choosing $\bar{\lambda}_j(s) = y_j/T$, $0 \le s \le T$, $1 \le j \le J$.

The FOP can thus be transformed into a J variable optimization problem:

$$\text{choose} \quad \boldsymbol{y} \in \mathbb{R}_+^J$$

$$\text{such that} \quad \sum_{j=1}^{J} A_{lj} y_j \le \bar{C}_l, \quad 1 \le l \le L$$

$$\text{to maximize} \quad Tr\left(\frac{\boldsymbol{y}}{T}\right) + \bar{g}(\bar{\boldsymbol{C}} - A\boldsymbol{y}) \equiv f(\boldsymbol{y})$$

Fluid Optimization Problem: Solution

$$\left.\begin{array}{l} \text{choose} \quad \boldsymbol{y} \in \mathbb{R}_+^J \\ \text{such that} \quad \displaystyle\sum_{j=1}^{J} A_{lj} y_j \le \bar{C}_l, \ 1 \le l \le L \\ \text{to maximize} \quad Tr\left(\frac{\boldsymbol{y}}{T}\right) + \bar{g}(\bar{\boldsymbol{C}} - A\boldsymbol{y}) \equiv f(\boldsymbol{y}) \end{array}\right\} \qquad \text{(FOP)}$$

Consider the following related problem:

$$\left.\begin{array}{l} \text{maximize} \quad f(\boldsymbol{y}) \\ \text{subject to} \quad \displaystyle\sum_{j=1}^{J} A_{lj} y_j \le \bar{C}_l - \sum_{j=1}^{J} A_{lj} \bar{a}_j, \ 1 \le l \le L \end{array}\right\} \qquad \text{(FOP2)}$$

Claim: If, at the solution to (FOP2), on every link the marginal value of bandwidth to immediate demand is at least as great as the marginal value of bandwidth to advanced reservation, the solution to (FOP2) is also the solution to (FOP).

Marginal value of immediate demand bandwidth on link l

$$= -\sum_{j=1}^{J} \frac{\partial \hat{B}_j}{\partial \bar{C}_l^I} p_j^I \bar{a}_j = \frac{\sum_{j=1}^{J} A_{lj} p_j^I \bar{a}_j}{\sum_{j=1}^{J} A_{lj}^2 \bar{a}_j}$$

The Diffusion Limit

Assume that the Fluid Optimization Problem leads to a critical solution: $\rho_l = 1$ for at least 1 $l \in \{1, \ldots, L\}$.

Let $\boldsymbol{y}^*$ denote the solution to FOP.

Let

$$\begin{aligned} \boldsymbol{\lambda}^0 &= \frac{\boldsymbol{y}^*}{T} \quad \text{and} \\ \lambda_j^{(n)}(s) &= n\lambda_j^0 + \sqrt{n}\, u_j^{(n)}(s)\,. \end{aligned}$$

Then

$$r^{(n)}(\boldsymbol{\lambda}^{(n)}(s)) = n\boldsymbol{p}^0 \cdot \boldsymbol{\lambda}^0 + \sqrt{n}\,\boldsymbol{u}^{(n)}(s) \cdot \nabla r(\boldsymbol{\lambda}^0) + o(\sqrt{n})\,,$$

where $\boldsymbol{p}^0 = \boldsymbol{p}(\boldsymbol{\lambda}^0)$. For $\boldsymbol{x} \in \mathbb{R}^J$, let

$$\hat{r}^{(n)}(\boldsymbol{x}) = n^{-1/2}[r^{(n)}(n\boldsymbol{\lambda}^0 + \sqrt{n}\,\boldsymbol{x}) - n\boldsymbol{p}^0 \cdot \boldsymbol{\lambda}^0]$$

Then, as $n \to \infty$,

$$\hat{r}^{(n)}(\boldsymbol{x}) \to \hat{r}(\boldsymbol{x}) = x \cdot \nabla r(\boldsymbol{\lambda}^0)\,.$$

The Controlled Diffusion Process

For $u^{(n)}(\cdot)$ 'well enough behaved', as $n \to \infty$,

$$n^{-1/2}\left[\pi_j\left(\int_0^t \lambda_j^{(n)}(s)ds\right) - n\lambda_j^0 t\right] \stackrel{\text{a.s.}}{\to} W_j(t) + \int_0^t u_j(s)ds,$$

where $\{W_j(t), t \geq 0\}$, $1 \leq j \leq J$ are independent Wiener processes with zero drift and variances λ_j^0.

Let

$$\hat{X}_l^{(n)}(t) = n^{-1/2}[X_l^{(n)}(t) - n\bar{X}_l(t)].$$

(So $\hat{\boldsymbol{X}}^{(n)}(0) \to 0$.) Then, as $n \to \infty$,

$$\hat{\boldsymbol{X}}^{(n)}(t) \stackrel{\text{a.s.}}{\to} \hat{\boldsymbol{X}}(t), \quad 0 \leq t \leq T,$$

where

$$\hat{X}_l(t) = -\sum_{j=1}^{J} A_{lj}\left[\int_0^t u_j(s)ds + W_j(t)\right].$$

The Diffusion Scale Terminal Reward, $L = J = 1$

Jagerman (1974) showed that, if

$$M_n = na + \sqrt{n}\beta + o(\sqrt{n}), \quad -\infty < \beta < \infty\,,$$

then

$$\sqrt{n}B(M_n, na) \to \frac{1}{\sqrt{a}} h\left(\frac{-\beta}{\sqrt{a}}\right)\,,$$

where

$$h(x) = \frac{\phi(x)}{1 - \Phi(x)}, \quad -\infty < x < \infty\,,$$

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad \text{and} \quad \Phi(x) = \int_{-\infty}^{x} \phi(z)dz\,.$$

Thus

$$\sqrt{n}B(X_n(T), \lambda_n^I \tau^I) \to \frac{1}{\sqrt{\bar{\lambda}^I \tau^I}} h\left(\frac{-\hat{X}(T)}{\sqrt{\bar{\lambda}^I \tau^I}}\right)\,.$$

If we let

$$\hat{g}_n(x) = n^{-1/2}[g_n(\lambda_n^I \tau^I + \sqrt{n}\,x) - n\bar{\lambda}^I \tau^I p^I]\,,$$

then

$$\hat{g}_n(\hat{X}_n(T)) \to \hat{g}(\hat{X}(T)) \equiv -p^I \sqrt{\bar{\lambda}^I \tau^I}\, h\left(\frac{-\hat{X}(T)}{\sqrt{\bar{\lambda}^I \tau^I}}\right)$$

(Related results for network due to Hunt and Kelly (1989).)

The Diffusion Control Problem $(L = J = 1)$

Let

$$\hat{V}(\boldsymbol{u}) = E\left[r'(\lambda^0) \int_0^T u(s)ds \right. \\ \left. - p^I \sqrt{\bar{\lambda}^I \tau^I} h\left(\frac{\int_0^T u(s)ds + \sqrt{\lambda^0} W(T)}{\sqrt{\bar{\lambda}^I \tau^I}} \right) \right]$$

The Diffusion Control Problem (DCP) is:

choose (non-anticipating) $\{u(s),\ 0 \leq s \leq T\}$

to maximize $\hat{V}(\boldsymbol{u})$.

Let

$$d(x) = r'(\lambda^0)x - p^I \sqrt{\bar{\lambda}^I \tau^I} h\left(\frac{x}{\sqrt{\bar{\lambda}^I \tau^I}} \right) .$$

Then d is concave, and

$$d'(x) = 0 \Leftrightarrow h'\left(\frac{x}{\sqrt{\bar{\lambda}^I \tau^I}} \right) = \frac{r'(\lambda^0)}{p^I}$$

Let x^* denote unique solution: global maximum of d

$$\hat{V}(\boldsymbol{u}) = E\left[d\left(\int_0^T u(s)ds + \sqrt{\lambda^0} W(T) \right) \right] \leq d(x^*) .$$

A Linear Quadratic Gaussian Control Problem

Let

$$Y(t) = x^* - \int_0^T u(s)ds - \sqrt{\lambda^0}W(t)\,,$$

and let $b > 0$ be fixed.

The 'smoothed' DCP is:

$$\text{choose}\ \ \{u(s),\ 0 \le s \le T\}$$

$$\text{to minimize } E\left[Y^2(T) + b\int_0^T u^2(s)ds\right]\ .$$

This is an LQG problem:

$$\text{optimal control:}\ \ u(s) = u^*(s, Y(s))\,,$$

$$u^*(s,y) = \frac{y}{b+T-s}$$

$$\text{value}:\ \ \mathcal{L}^*(b) = \min_{\boldsymbol{u}} E\left[Y^2(T) + b\int_0^T u^2(s)ds\right]$$

$$\mathcal{L}^*(b) = \frac{(x^*)^2 b}{b+T} + \lambda^0 b\ ln\left(1 + \frac{T}{b}\right)$$

$$\to 0 \quad \text{as} \quad b \downarrow 0$$

For any $\epsilon > 0$, can find $b(\epsilon)$ such that

$$\hat{V}(\boldsymbol{u}_b^*) \ge d(x^*) - \epsilon$$

Admission Control: Model

SP has network as before (L links, C_l circuits on link l, J call type (routes), A_{lj} circuits and on link l by call of type j).

Demand arrives in J independent Poisson processes with rates λ_j, $1 \leq j \leq J$. Each accepted type j call pays p_j. (Calls do not depart.)

The system 'operates' over the interval $[0, T]$.

Admission Control Problem

At each arrival epoch the SP makes an accept/reject decision. Let

$$Y_j(t) = \text{total \# of type } j \text{ calls accepted during } [0, t] \, .$$

We require that Y_j be non-anticipating and that

$$dY_j(t) \le d\pi_j(\lambda_j t) \, .$$

Let

$$V(\mathcal{Y}) = E\left[\sum_{j=1}^{J} p_j Y_j(T)\right] \, ,$$

where $\mathcal{Y} = \{\boldsymbol{Y}(t), 0 \le t \le T\}$

Admission Control Problem: Choose a $\mathcal{Y}$ to maximize $V(\mathcal{Y})$ subject to

$$\sum_{j=1}^{J} Y_j(T) A_{lj} \le C_l, \quad 1 \le l \le L \, .$$

The Fluid Limit for Admission Control

Consider a sequence of systems indexed by n, with $C_l(n) \to \infty$ as $n \to \infty$: $C_l(n) = \lfloor n\ \bar{C}_l \rfloor$ with $0 < \bar{C}_l < \infty$.

Scale the arrival rates:

$$\lambda_j^{(n)} = n\bar{\lambda}_j \ ,$$

Consider a 'thinning' policy, determined by $(q_1, \ldots, q_J)$, with $0 \le q_j \le 1$, $1 \le j \le J$: Accept each type j arrival with probability q_j.

Let

$$\bar{Y}_j^{(n)}(t) = n^{-1} Y_j^{(n)}(t)\,.$$

Then, by the strong law of large numbers,

$$\bar{Y}_j^{(n)}(t) \stackrel{\text{a.s.}}{\to} \bar{Y}_j(t) = q_j \bar{\lambda}_j t$$

Fluid Optimization Problem for Admission Control

Fluid Optimization Problem (FOP):

$$
\begin{aligned}
&\text{choose} \quad && x_1, \ldots, x_J \\
&\text{such that} \quad && 0 \le x_j \le \bar{\lambda}_j \quad 1 \le j \le J\,, \\
&\text{and} \quad && T \sum_{j=1}^{J} A_{lj} x_j \le C_l, \quad 1 \le l \le L \\
&\text{to maximize} \quad && \sum_{j=1}^{J} p_j x_j\,.
\end{aligned}
$$

Denote the optimal solution to this LP by $(x_1^*, \ldots x_J^*)$, and the optimal objective function value by $\bar{V}$.

Fluid optimal policy: If capacity is available, accept each type j arrival with probability $q_j^* = x_j^*/\bar{\lambda}_j$, $1 \le j \le J$. Let V_{q^*} denote the expected total revenue with this policy.

By SLLN, $n^{-1} V_{q^*}^{(n)} \to \bar{V}$ as $n \to \infty$.

The Hindsight Policy

Given the total number of arrivals, $\pi_j(\lambda_j T)$, $1 \le j \le J$, solve the LP:

$$\begin{array}{ll} \text{choose} & y_j, \;\; 1 \le j \le J \\[2ex] \text{such that} & 0 \le y_j \le \pi_j(\lambda_j T), \;\; 1 \le j \le J \\[2ex] \text{and} & \displaystyle\sum_{j=1}^{J} A_{lj} y_j \le C_l, \;\; 1 \le l \le L \\[2ex] \text{to maximize} & \displaystyle\sum_{j=1}^{J} p_j y_j \,. \end{array}$$

Let $V_H^{(n)}$ denote the expected value of the optimal objective function of the above LP for the n^{th} system. Then

$$n^{-1} V_H^{(n)} \to \bar{V} \quad \text{as} \quad n \to \infty \,.$$

The Diffusion Limit for Admission Control

Assume that

$$T \sum_{j=1}^{J} A_{lj} x_j^* = \bar{C}_l$$

for at least 1 $l \in \{1, \ldots, L\}$.

On 'diffusion scale' the fluid optimal policy is *not* asymptotically optimal:

$$\frac{V_H^{(n)} - V_{q^*}^{(n)}}{\sqrt{n}} \to \Delta > 0 \, .$$

Can we do better than the thinning policy?

We introduce a 'trigger point' policy:

- Use fluid optimal policy until trigger point $\tau \leq T$
- At τ, solve new LP and follow a thinning policy with $(\tilde{q}_1, \ldots, \tilde{q}_J)$

Trigger Points

Let

$$\underline{s}_l = \min_{j:A_{lj}x_j^*>0} \{A_{lj}x_j^*\},$$

$$\bar{s}_l = \min_{j:A_{lj}(\bar{\lambda}_j - x_j^*)>0} \{A_{lj}(\bar{\lambda}_j - x_j^*)\},$$

$$b_l(t) = (T-t)\left[\sum_{j=1}^{J} A_{lj}x_j^* - \underline{s}_l\right],$$

and

$$u_l(t) = (T-t)\left[\sum_{j=1}^{J} A_{lj}x_j^* + \bar{s}_l\right].$$

Also, let

$$\underline{\tau}_l = \inf\{t : b_l(t) > X_l(t)\},$$

$$\bar{\tau}_l = \inf\{t : u_l(t) < X_l(t)\},$$

and

$$\tau = \min\left\{\min_{1\le l\le L} \underline{\tau}_l, \quad \min_{1\le l\le L} \bar{\tau}_l\right\}.$$

Let $s_n = \sqrt{n}[T - \tau^{(n)} \wedge T]$.

Claim: $s_n \overset{\text{a.s.}}{\to} s$ as $n \to \infty$.

(There are more trigger points.)

The 2^{nd} LP

At τ solve the LP:

$$\begin{array}{ll} \text{choose} & x_j, \quad 1 \le j \le J \\ \text{such that} & 0 \le x_j \le \bar{\lambda}_j, \quad 1 \le j \le J \\ \text{and} & (T-\tau)\sum_{j=1}^{J} A_{lj} x_j \le X_l(\tau), \quad 1 \le l \le L \\ \text{to maximize} & \sum_{j=1}^{J} p_j x_j \,. \end{array}$$

Denote the optimal solution to this LP by $(\tilde{x}_1, \ldots, \tilde{x}_J)$, and let $\tilde{q}_j = \tilde{x}_j / \bar{\lambda}_j$.

Use a thinning policy with $(\tilde{q}_1, \ldots, \tilde{q}_J)$ during $(\tau, T]$.

Note: There are $O(\sqrt{n})$ arrivals during $(\tau^{(n)}, T]$, so by the SLLN the thinning policy is optimal to $o(\sqrt{n})$. (This is a fluid problem on $\frac{1}{\sqrt{n}}$ time with $\frac{1}{\sqrt{n}}$ capacity.)

An Example: $L = 1,\ J = 2$

Consider a system with 1 link and 2 types, $\bar{C}_1 = 1$, $A_{11} = A_{12} = 1$, $\bar{\lambda}_1 = \bar{\lambda}_2 = 3/4$, $p_1 = 2$, $p_2 = 1$.

The hindsight policy is to accept all $\pi_1(\lambda_1 T)$ type 1 calls, and $C_1 - \pi_1(\lambda_1 T)$ type 2 calls.

The FOP solution is $x_1^* = 3/4$, $x_2^* = 1/4$, so that $q_1^* = 1$, $q_2^* = 1/3$. Also,

$$\begin{aligned} \underline{s} &= 1/4, \quad b(t) = 3/4(T-t)\,, \\ \bar{s} &= 1/2, \quad u(t) = 3/2(T-t)\,. \end{aligned}$$

Trigger type and response:

$\underline{\tau}$: stop admitting type 2

$\bar{\tau}$: admit all type 2