Optimal Allocation of a Divisible Good to Strategic Buyers

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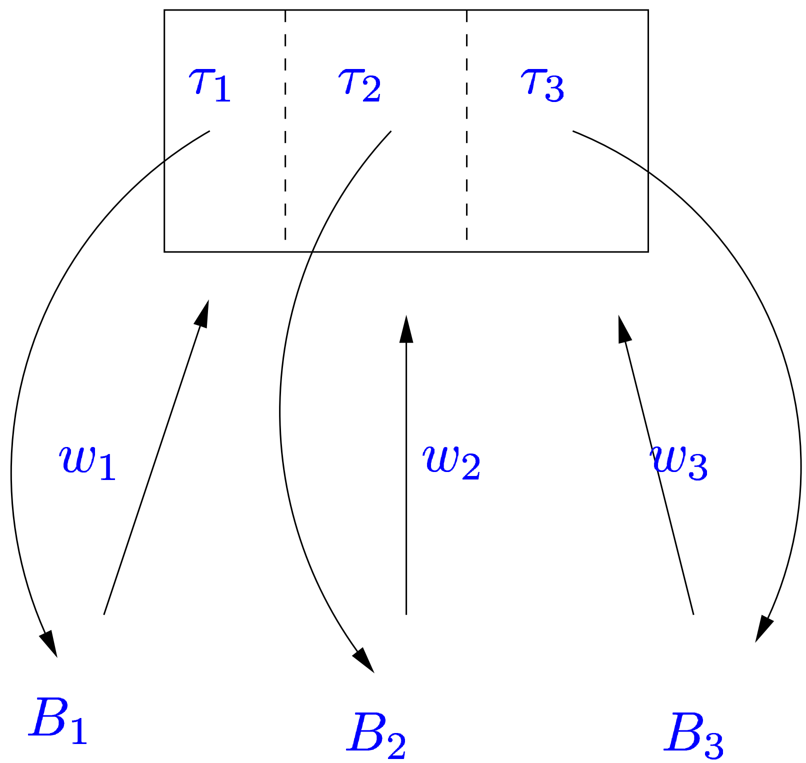
The Framework



The Allocation Rule τ: Buyer B_i pays w_i and recives $\tau_i(\mathbf{w})$.

τ is known to the buyers before they submit *bids*, which we assume are the payments w_i they will make.

Buyers have value functions U_i for the amounts they receive.

Thus, if allocations made according to τ, the *profit* of buyer i if the pament vector is $\mathbf{w}$ is

$$P_i(w_i, \mathbf{w}_{-i}) \;=\; U_i(\tau_i(\mathbf{w})) - w_i$$

Buyers want to make as large a profit as possible.

Nash Equilibrium: A vector of payments $\tilde{\mathbf{w}}$ s.t. any individual deviation leads to lower profit for the deviant:

$$P_i(\tilde{w}_i, \tilde{\mathbf{w}}_{-i}) \;\geq\; P_i(w_i, \tilde{\mathbf{w}}_{-i}) \;\; \forall\; i,\, w_i$$

There can be several Nash equilibria.

The Social Goal

Total (or *social*) value created by an allocation is $\sum_i U_i(\tau_i(\mathbf{w}))$.

Mechanism designer wants to maximize this, but buyers want to maximize their profit P_i.

So, task of mechanism designer: design an allocation rule τ to minimize social value lost due to buyer behavior.

In particular: minimize value lost if buyers payments are at Nash Equilibrium, *in the worst case*.

τ should depend *only* on payments $\mathbf{w}$

Best Value $= \sum_i U_i(x_1^*)$

The Worst Case

The *fractional Efficiency* at a Nash equilibrium is

$$\frac{\sum_i U_i(\tau_i(\tilde{\mathbf{w}})}{\sum_i U_i(x_i^*)}$$

at the worst Nash equilibrium is

$$\inf_{\tilde{\mathbf{w}} \in \mathcal{N}} \frac{\sum_i U_i(\tau_i(\tilde{\mathbf{w}})}{\sum_i U_i(x_i^*)}$$

for the worst possible buyer value functions

$$\inf_{\{U_i\}} \inf_{\tilde{\mathbf{w}} \in \mathcal{N}} \frac{\sum_i U_i(\tau_i(\tilde{\mathbf{w}}))}{\sum_i U_i(x_i^*)}$$

= worst case efficiency if allocation rule τ is used.

The Problem

How "good" can this worst case be made:

$$\sup_{\tau} \inf_{\{U_i\}} \inf_{\tilde{\mathbf{w}} \in \mathcal{N}} \frac{\sum_i U_i(\tau_i(\tilde{\mathbf{w}}))}{\sum_i U_i(x_i^*)}$$

and which is the mechanism τ^* that achieves the sup above.

Which τ^* is the best in the worst case ?

Assumptions:

1. Concave increasing U_i.
2. Allocation rule is symmetric, concave increasing in buyer's own payment.

Our Results(1)

For two buyers, the highest worst case ratio is $\frac{7}{8}$, i.e.

$$\sup_{\tau} \inf_{U_1, U_2} \inf_{\tilde{\mathbf{w}} \in \mathcal{N}} \frac{U_1(\tau_1(\tilde{\mathbf{w}})) + U_2(\tau_2(\tilde{\mathbf{w}}))}{\sum_i U_i(x_i^*)} = \frac{7}{8}$$

and the rule that achieves it is:

$$\tau_l^*(\mathbf{w}) = \frac{w_l}{2w_h} \text{ and } \tau_h^*(\mathbf{w}) = 1 - \frac{w_l}{2w_h}$$

Our Results(2)

For $n \geq 3$ buyers, highest worst case ratio not known. However, for the machanism given by

$$\tau_i^*(\mathbf{w}) \;=\; \frac{w_i}{w_{max}} \int_0^1 \prod_{j \neq i} \left(1 - s\frac{w_j}{w_{max}}\right) ds$$

This has worst case ratio given by

$$\min_{\mathbf{v} \in [0,1]^{n-1}} \left(1 + \sum_{i=1}^{n-1} v_i\right) \int_0^1 \prod_{i=1}^{n-1} (1 - sv_i)\, ds - \left(\sum_{i=1}^{n-1} v_i\right) \prod_{i=1}^{n-1} (1 - v_i)$$

numerical evaluation suggests value is 0.8735.

Related Work

Johari and Tsitsiklis showed that the worst case ratio (as defined) for the proportional allocation rule is $\frac{3}{4}$.

Gopalkrishnan and Hajek, Basar and Maheswaran: study the proportional allocation rule (e.g. unique equilibria etc.)

Kelly: if buyers are not strategic but *price taking*, possible to achieve full efficiency (and there is a “decentralized” way to get there)

Roughgarden and Tardos: Worst case loss in total throughput when data is routed selfishly on networks.

Reduction to Linear Utilities

Proposition:[a] For every allocation rule τ, value functions $\{U_i\}$ and Nash equilibrium $\tilde{\mathbf{w}} \in \mathcal{N}(\tau, \{U_i\})$ there exist linear value functions $\hat{U}_i(x_i) = \alpha_i x_i$ such that $\tilde{\mathbf{w}} \in \mathcal{N}(\tau, \{\hat{U}_i\}) \triangleq \mathcal{N}(\tau, \alpha)$ and

$$\frac{\sum_i U_i(\tau_i(\tilde{\mathbf{w}}))}{\sum_i U_i(x_i^*)} \geq \frac{\sum_i \alpha_i \tau_i(\tilde{\mathbf{w}})}{\max_i \ \alpha_i}$$

$\Rightarrow$ There are linear value functions with Nash equilibria as bad as that of any other value functions.

[a]Using method in Johar i-Tsitsiklis

So, can rewrite

$$\sup_{\tau} \inf_{\{U_i\}} \inf_{\tilde{\mathbf{w}} \in \mathcal{N}} \frac{\sum_i U_i(\tau_i(\tilde{\mathbf{w}}))}{\sum_i U_i(x_i^*)} = \sup_{\tau} \inf_{\alpha} \inf_{\tilde{\mathbf{w}} \in \mathcal{N}} \frac{\sum_i \alpha_i \tau_i(\tilde{\mathbf{w}})}{\max_i \alpha_i}$$

Intuition:
If optimizing for the linear case, efficiency maximized by giving maximum possible to the buyer with highest slope, who will also be the highest bidder.

Assumption: Allocation rule is *scale free* - for all real $\gamma > 0$ and $\tau_i(\gamma \mathbf{w}) = \tau_i(\mathbf{w})$ for all i and $\mathbf{w}$.

Two Buyers

Theorem 1 (Upper Bound): When two buyers are present,the worst case efficiency at Nash equilibrium for any scale-free mechanism cannot exceed $\frac{7}{8}$, and this is achieved by τ^*.

Proof Sketch:

- Enough to specify what lower buyer l and higher buyer h get.
- For every scale free $\tau(w_l, w_h)$ there is a $\phi(v)$ such that

$$\tau_l(w_l, w_h) = \phi\left(\frac{w_l}{w_h}\right) \quad \text{and} \quad \tau_h(w_l, w_h) = 1 - \phi\left(\frac{w_l}{w_h}\right)$$

  ϕ is concave increasing with $\phi(0) = 0$ and $\phi(1) = \frac{1}{2}$.

- Lowest such ϕ is $\phi(v) = \frac{v}{2}$, which corresponds to τ^*.

n buyers

Buyer i gets

$$\tau_i^*(\mathbf{w}) = \frac{w_i}{w_{max}} \int_{s=0}^{1} \prod_{j \neq i} \left(1 - s\frac{w_j}{w_{max}}\right) ds$$

$$= \frac{w_i}{w_{max}} \sum_{m=0}^{n-1} \frac{(-1)^m}{m+1} \frac{1}{w_{max}^m} \left(\sum_{m \text{ sets of } [n]-i} w_{i_1} \ldots w_{i_m} \right)$$

Eg. for 3 buyers, highest buyer gets

$$1 - \frac{w_1 + w_2}{2w_{max}} + \frac{w_1 w_2}{3w_{max}^2}$$

Efficiency: n Buyers

Theorem 4: The worst case efficiency of a Nash equilibrium for n buyers when τ^* is used is

$$E_n = \min_{\mathbf{v}\in[0,1]^n} \left(1+\sum_{i=1}^{n-1} v_i\right)\int_0^1 \prod_{i=1}^{n-1}(1-sv_i)\,ds - \left(\sum_{i=1}^{n-1} v_i\right)\prod_{i=1}^{n-1}(1-v_i)$$

Cannot (yet ?) evaluate analytically.

Numerical: for all n worst case when all except three of the v_is are 0

worst case with n buyers = worst case with 4 buyers.

Value = 0.8735

Other Properties

Q: Given *any* value functions $\{U_i\}$ does there exist unique Nash equilibrium when τ^* is used ?

A: For two users yes. For n buyers, do not know.

► "Hierarchial property": If there are n buyers and m of them submit 0 payments, the mechanism τ_n^* allocates according to τ_{n-m}^* to the remaining.

? "Strategic Splitting": A buyers cannot gain by joining auction as m buyers who co-operate.

Open Question: Revenue

Mechanism designer may be interested in revenue instead of social good.

Fractional Formulation:

If buyers do not participate in the event of (potential) negative profits, at NE

$$\text{Revenue} \;=\; \sum_i \tilde{w}_i \;\leq\; \sum_i U_i(\tilde{x}_i)$$

so, for τ can define *fraction of value obtained as revenue*

$$\frac{\tilde{w}_i}{\sum_i U_i(\tau_i(\tilde{\mathbf{w}}))}$$

? Should we maximize or minimize ?

Worst case ratio is (probably) 0 for most mechanisms.

Best: For proportional allocation τ_{prop} to n buyers, best:

$$\sup_{\{U_i\}} \sup_{\tilde{\mathbf{w}} \in \mathcal{N}(\tau_{prop}, \{U_i\})} \frac{\sum_i \tilde{w}_i}{\sum_i U_i(\tilde{x}_i)} \leq 1 - \frac{1}{n}$$

$\Rightarrow$ Seller misses out on at least $\frac{1}{n}$ of the value generated by the sale.

(OR) Buyers get at least $\frac{1}{n}$

? What is the best case for τ^* ?

? Which mechanism has the best best case ?

Conclusion + Future Work

1. Found the mechanism that performs the best in the worst case.
2. Final expression needs to be evaluated analytically.
3. Relation between revenue and social value ? (what, if any, is the tradeoff curve)
4. The "network case": multiple inter-related goods (like capacities on links in series/parallel)