An Efficient Mechanism for Allocation of a Divisible Good

with its application to network resource allocation

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1 – Introduction

Network Resource Allocation Problem

- Divisible goods
- Strategic buyers with different valuation functions
- Seek a mechanism to promote social efficiency

How to characterize “good” use of the network?

- **Efficiency:** What is the aggregate value of the allocation compared to the maximum possible?
- **Fairness:** How is the network value distributed among buyers?

Market with Divisible Goods — Single link case, N buyers, $N \geq 2$. Total amount of capacity C is infinitely divisible. Buyer i has strictly concave, strictly increasing and continuously differentiable valuation function $U_i(x_i)$ on $[0, C]$. ($U_i'(0) = \infty$ is ok).

SYSTEM Problem:

$$\begin{aligned} \text{maximize} \quad & \sum_i U_i(x_i) \\ \text{subject to} \quad & \sum_i x_i \leq C \\ & x_i \geq 0, i = 1, ..., N. \end{aligned}$$

Solution to SYSTEM Problem:

$$\begin{cases} U_i'(x_i) = \lambda, & \text{if } x_i > 0 \\ U_i'(0) \leq \lambda, & \text{if } x_i = 0 \end{cases}$$

$\sum_i x_i = C$ and $\lambda \geq 0$

Both λ and $\mathbf{x}$ are unique.

Definition 1 *An allocation is* **efficient** *if it's the solution to the system problem. The* **efficiency** *of an allocation is the ratio of aggregate value it achieves to the maximum possible.*

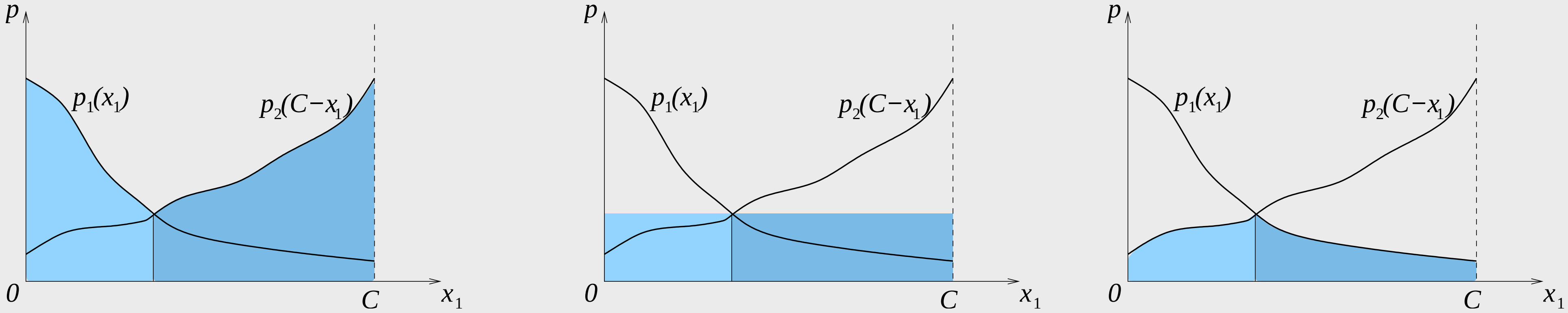
Several Models

- **Kelly** Buyers are price takers. The efficient allocation can be achieved if seller select market clearing price.
- **Johari and Tsitsiklis** Buyers are strategic and they each submit a bid to the seller. Seller allocates the good proportionally according to buyers' bids and payments equal to bids. The worst case efficiency is determined to be 75%.
- **Sanghavi and Hajek** Buyers are strategic and they each submit a bid to the seller. A nonuniform price mechanism with payments equal to bids makes the worst case efficiency about 87%.

Inefficiency of Nash Equilibrium Point (NEP)

- Can the efficiency be up to 100% if all the buyers are strategic?
- Mechanism Design Idea (similar to VCG philosophy) — Let the buyer's payment compensate for the other buyers' loss due to his competition.

Classical way in dividing a good



(a) Aggressive discriminatory (b) Uniform price (c) Clark/VCG

Comments on Clark/VCG mechanism

- It is an efficient mechanism.
- Buyers' bids are functions, so strategy space is infinite dimensional.
- The revenue of seller depends on the buyers' valuations, and can be very small.

2 – Efficient Mechanism

Define the Game

GAME: $N > 2$ buyers compete for finite amount C of divisible goods. Each buyer submits a bid to the seller. Buyers' bids are within the strategy space $\mathcal{B} = \{\boldsymbol{b} : b_i \geq 0, \forall i\}$. Seller allocates capacity and charges buyer according to $\boldsymbol{b}$ under some rule, which is denoted as $x_i(b_i, \boldsymbol{b}_{-i})$ and $m_i(b_i, \boldsymbol{b}_{-i})$ respectively. Hence, buyer i has payoff $\Pi_i(b_i, \boldsymbol{b}_{-i}) = U(x_i) - m_i$. A Nash equilibrium point (NEP) can be defined as a bid vector $\boldsymbol{b}$ such that for all i:

$$\Pi_i(b_i; \boldsymbol{b}_{-i}) \geq \Pi_i(\bar{b}_i; \boldsymbol{b}_{-i}), \ \forall\, \bar{b}_i \geq 0 \tag{1}$$

Notations

- ☞ $\boldsymbol{b} = (b_1, b_2, ..., b_N)$ and $\boldsymbol{b}_{-i} = (b_1, ..., b_{i-1}, b_{i+1}, ..., b_N)$
- ☞ B and B_{-i} are functionals of $\boldsymbol{b}$ given by $B(\boldsymbol{b}) = \sum_j b_j$ and $B_{-i}(\boldsymbol{b}) = \sum_{j \neq i} b_j$.

Our Mechanism

Proportional Allocation Rule: $$x_i = \begin{cases} \frac{b_i}{B}C, & \text{if } b_i \neq 0 \\ 0, & \text{if } b_i = 0 \end{cases}, \forall\, i$$

Payment Rule: $$m_i = B_{-i}[\varphi(B) - \varphi(B_{-i})]$$

Some Observations

- $\varphi(u)$ is a *scale function* which indicates the severity of the market competition.
- Buyer's bid b_i can be seen as i-th buyer's thirstiness for the good.
- In some sense, m_i indicates the loss of value to other buyers due to the allocation to the i-th buyer.

Payment for several choices of φ (for $N = 2$)

☞ $\varphi(u) = u \Rightarrow \begin{cases} m_1 = b_2[(b_1 + b_2) - b_2] = b_1 b_2 \\ m_2 = b_1[(b_1 + b_2) - b_1] = b_1 b_2 \end{cases}$

☞ $\varphi(u) = \log u \Rightarrow \begin{cases} m_1 = b_2[\log(b_1 + b_2) - \log b_2] = b_2 \log\left(1 + \frac{b_1}{b_2}\right) \\ m_2 = b_1[\log(b_1 + b_2) - \log b_1] = b_1 \log\left(1 + \frac{b_2}{b_1}\right) \end{cases}$

☞ $\varphi(u) = -u^{-1+\epsilon}, 0 < \epsilon < 1 \Rightarrow \begin{cases} m_1 = b_2[b_2^{-1+\epsilon} - (b_1 + b_2)^{-1+\epsilon}] \\ m_2 = b_1[b_1^{-1+\epsilon} - (b_1 + b_2)^{-1+\epsilon}] \end{cases}$

Definition 2 *Define a scale function φ on $[0, +\infty)$ to be* good *if*

1. *it is strictly increasing and continuously differentiable.*
2. $u^2\varphi'(u) : [0, +\infty) \rightarrow [0, +\infty)$ *is a strictly increasing onto map.*

Assumption 1 $U_i'(0) = +\infty$ *for at least two values of i.*

Proposition 1 *Suppose φ is good and Assumption 1 holds, then there is a unique NEP* **b** *in* GAME*. Furthermore,* **b** *is the solution to*

$$\left\{ \begin{array}{lcll} U_i'(x_i) & = & \frac{B^2\varphi'(B)}{C} & \text{if } b_i > 0 \\ U_i'(0) & \leq & \frac{B^2\varphi'(B)}{C} & \text{if } b_i = 0 \end{array} \right. , \forall\, i \tag{2}$$

$$\text{where } x_i = \frac{b_i}{B} C$$

The allocation **x** *achieved under* **b** *is efficient.*

Remark 1 *Suppose i is a buyer and* **b** *is a bid vector such that* $\mathbf{b}_{-i}$ *satisfies the sufficient condition for an NEP (for b_i fixed). Then if b_i is increased slightly, the first order increase in the payment of i is equal to the first order decrease in the sum of values of the other buyers.*

Proof. Suppose buyer i changes his bid from b_i to $b_i + \delta$. Then the change of j-th buyer's value is

$$\begin{aligned} \Delta_{U_j} &= \frac{\partial[U_j(x_j(b_i, \mathbf{b}_{-i}))]}{\partial b_i}\delta = -U_j'(x_j)\frac{b_j}{B^2}C\delta \\ &= -b_j\varphi'(B)\delta \end{aligned}$$

So

$$\Delta_{m_i} = \frac{\partial[m_i(b_i, \mathbf{b}_{-i})]}{\partial b_i}\delta = B_{-i}\varphi'(B)\delta = -\sum_{j \neq i}\Delta_{U_j}$$

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3 – Buyers' Payment and Seller's Revenue

Go back to the examples we have mentioned before, we study the efficient NEP and corresponding seller's revenue.

1. $\varphi(u) = u$
   The equilibrium condition is $U_i'(x_i) = \frac{B^2}{C} = \lambda$
   implies that $B = b_1 + b_2 = \sqrt{\lambda C}$
   The revenue $R = 2b_1 b_2 \leq 2(\frac{b_1+b_2}{2})^2 = \frac{\lambda C}{2}$

2. $\varphi(u) = \log u$
   The equilibrium condition is $U_i'(x_i) = \frac{B}{C} = \lambda$
   implies that $B = b_1 + b_2 = \lambda C$
   The revenue $R = b_1 \log\left(1 + \frac{b_2}{b_1}\right) + b_2 \log\left(1 + \frac{b_1}{b_2}\right) \leq (\log 2)\lambda C \doteq 0.69\lambda C$

The different choice of the scale function can take the different revenue to the seller.

Proposition 2 *Assuming the conditions of Proposition 1, at the NEP the buyers' payment price is always less than* λ.

Proof. The assumption $u^2\varphi'(u)$ is strictly increasing over $u \geq 0$ implies (in fact, equivalent to) the condition $u\varphi(u)$ is strictly convex. So $b\varphi(b) - a\varphi(a) < [u\varphi(u)]'|_{u=b}(b-a)$, for $0 < a < b$, then

$$\begin{aligned}
p_i &= \frac{m_i}{x_i} = \frac{B_{-i}[\varphi(B) - \varphi(B_{-i})]}{x_i} \\
&= \frac{(B_{-i}+b_i)\varphi(B_{-i}+b_i) - B_{-i}\varphi(B_{-i}) - b_i\varphi(B_{-i}+b_i)}{C\frac{b_i}{B}} \\
&< \frac{b_i[(B_{-i}+b_i)\varphi'(B_{-i}+b_i) + \varphi(B_{-i}+b_i)] - b_i\varphi(B_{-i}+b_i)}{C\frac{b_i}{B}} \\
&= \frac{b_i B\varphi'(B)}{C\frac{b_i}{B}} = \frac{B^2\varphi'(B)}{C} = \lambda
\end{aligned}$$

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Corollary 1 *Let R be the revenue of the seller. Assuming the conditions of Proposition 1, the supremum of R with respect to the choice of φ is $\sup R = \lambda C$*

Proof. From Proposition 2, $p_i < \lambda$ for all i implies $R < \lambda C$.

Next, consider $\varphi_\epsilon(u) = -u^{-1+\epsilon}$ where $0 < \epsilon < 1$. It is a good scale function. The NEP $\boldsymbol{b}_\epsilon$ satisfies $\frac{B^\epsilon}{1-\epsilon} = \lambda C$. We get $B(\boldsymbol{b}_\epsilon) = [\lambda C(1-\epsilon)]^{\frac{1}{\epsilon}}$. Thus,

$$
\begin{aligned}
R_\epsilon &= \sum_{i=1}^{N} B_{-i}(-B^{-1+\epsilon} + B_{-i}^{-1+\epsilon}) \\
&= B^\epsilon \sum_{i=1}^{N} (1-\alpha_i)[(1-\alpha_i)^{-1+\epsilon} - 1] \quad \text{where } \alpha_i = \frac{b_i}{B} \\
&= [\lambda C(1-\epsilon)] \sum_{i=1}^{N} [(1-\alpha_i)^\epsilon - (1-\alpha_i)] \to \lambda C
\end{aligned}
$$

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To gain insight we ask:

Is the revenue upper bound exactly λC achievable?

Consider $\varphi(u) = -\frac{a}{u}, a > 0$. It is a good scale function since $u^2\varphi'(u) = a$.

☞ If $a = \lambda C$, the efficient NEP exists and $R = \lambda C$

☞ If $a \neq \lambda C$, NO efficient NEP exists.

In this case, the seller specifies the market price. If he has some luck or *a priori* information to know what λ is, then an NEP exists for the GAME and the allocation is efficient.

What to do if Assumption 1 doesn't hold?

Proposition 3 *Suppose φ is good and assumption 1 doesn't hold, then there are infinite many NEPs in* GAME*. One of them satisfies (2) and is thus efficient.*

If there is at most one buyer with $U_i'(0) = +\infty$, we can introduce an ϵ**-modified GAME**. With all the other settings same as GAME, add two strategic virtual buyers indexed by I and II with utility function $U_I(x_I) = \alpha\epsilon \log x_I$ and $U_{II}(x_{II}) = (1-\alpha)\epsilon \log x_{II}$. Let $\tilde{\boldsymbol{b}}$ be the extension of $\boldsymbol{b}$ with $\tilde{\boldsymbol{b}} = (b_1, ..., b_N, b_I, b_{II})$ and $\tilde{\boldsymbol{x}}$ be the extension of $\boldsymbol{x}$ with $\tilde{\boldsymbol{x}} = (x_1, ..., x_N, x_I, x_{II})$. From Proposition 1, there is a unique NEP $\tilde{\boldsymbol{b}}_\epsilon$ in the ϵ*-modified GAME*.

Proposition 4 $\tilde{\mathbf{b}} = \lim_{\epsilon \to 0} \tilde{\mathbf{b}}_\epsilon$ *exists. Let* **b** *be the vector composed of the first N elements of* $\tilde{\mathbf{b}}$*.* **b** *is the efficient NEP of* GAME*.*

4 – Conclusion

☞ The mechanism is easy to implement since the buyer's strategy space is one dimensional and each buyer only needs to report one value to the seller.

☞ The equilibrium of the game under this mechanism yields the efficient allocation.

☞ At the NEP, the buyer's payment is reasonably bounded. The revenue of the seller can approach arbitrarily close to λC, which is the product of the efficient market shadow price and the whole capacity.

☞ Future works? Extend to general networks and implement in decentralized method to reach NEP.