Max-Plus Decomposition of Supermartingale
Application to Portfolio Insurance

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IMA Workshop April 2004
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Motivations

Insurance Portfolio

- The problem was first motivated by portfolio insurance which is designed to give the investor the ability to limit downside risk while allowing some participation in upside markets.
- The aim of the portfolio manager is to keep portfolio value from falling below a minimum wealth, commonly termed the floor at any time.
- Classical optimization problems generally try to maximize an expected utility criterion, related to individual preferences, through concave increasing utility function.

Related works

Martingale optimization w.r. to convex order

- For power utility functions, the problem may be transformed into an optimization program for martingales with the same initial value subjected to the constraint to dominate a floor $X$.
- The optimality has to hold for any utility functions.
- This last point is related to the convex order.

Max-Plus decomposition of supermartingale

- The solution is given through the decomposition of the Snell envelope $Z$ (American option) of the floor in terms of an adapted increasing process $\Lambda_t$ and martingale $M$ such that $M_t = \sup(Z_t, \Lambda_t)$.
- $\Lambda_t = \sup_{0 \leq u \leq t} L_u$, $L_u \in [-\infty, +\infty]$.
- Study strongly related to the work of H.Foellmer and P.Bank.

Outline of the presentation

These different steps in the reverse order.

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Assumptions

- Fix some horizon date \( T \) (finite or not).
- The uncertainty is modelled by a some filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})\), satisfying the usual conditions of right-continuity and augmentation by \( \mathbb{P} \)-negligible sets.
- All processes \( X \) that we consider are adapted, right continuous with left limits \((\text{CadLag})\), and of class \((D)\),
  
  the family of random variables \((X(S), S \leq T, \ S \in T)\) where \( T \) is 
  the family of stopping times less than \( T \) is uniformly integrable, or equivalently dominated by an u.i.martingale.
- If the process \( X \) has not accessible jumps, it is said to be quasi-left continuous. It is equivalent to said that \( \mathbb{E}(X_{S_n}) \to \mathbb{E}(X_S) \) if the sequence of stopping times \( S_n \to S \).
Supermartingale Decompositions

Let $Z$ be a supermartingale satisfying the previous assumptions

- **Doob-Meyer decomposition**
  There exist a previsible non-decreasing process $A_t$ ($A_0 = 0$) and a u.i. martingale $N^{DM}$, such that $Z_t = N_t^{DM} - A_t$ or equivalently
  \[ N_t^{DM} = Z_t + A_t. \]
  If $Z$ is quasi-left continuous, $A$ is continuous.

- **Multiplicative Decomposition of positive surmartingale**
  There exist a previsible non-decreasing process $B_t$ ($B_0 = 1$) and a u.i. martingale $N^{multi}$, such that $N_t^{multi} = Z_t \times B_t$

- **Max-Plus decomposition**
  There exist a non-decreasing adapted process $\Lambda_t$ ($\Lambda_{-0} = -\infty$) and a u.i. martingale $M$, such that $M_t = Z_t \vee \Lambda_t$.
  - In all these decompositions, the martingale is unique, with the additional assumption in the Max-Plus representation that $\Lambda$ only increases if $M = Z$. $\Lambda$ may be maximally chosed.
Max-plus algebra

**Definition**: The “exotic” algebraic structure $\mathbb{R}_{\text{max}}$

- The symbol $\mathbb{R}_{\text{max}}$ denotes the set $\mathbb{R} \cup \{-\infty\}$ with $\max$ and $+$ as the two binary operations $\oplus$ and $\otimes$, respectively.
- We call this structure the **max-plus algebra**. Sometimes this is also called an ordered group.
- We remark that the natural order on $\mathbb{R}_{\text{max}}$ may be defined using the $\oplus$ operation $a \leq b$ if $a \oplus b = b$.

**Theorem**: The algebraic structure $\mathbb{R}_{\text{max}}$ is an idempotent commutative **semifield**.

- the operation $\oplus$ is associative, commutative and has a zero element $\epsilon = -\infty$. As $a \otimes a = a$, $\mathbb{R}_{\text{max}}$ is idempotent.
- the operation $\otimes$ defines a commutative group on $\mathbb{R}_{\text{max}} - \{-\infty\}$, it is distributive with respect to $\oplus$ and its identity element $e = 0$ satisfies $\epsilon \otimes e = e \otimes e = \epsilon$. 

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Why Max-Plus ?

Comparison $R_{\text{max}}$ and $R$

$\Rightarrow$ $\oplus$ is idempotent in place of invertible for $+$
$\Rightarrow$ there are no zero divisors in $R_{\text{max}}$ $(a \oplus b = -\infty \Rightarrow a = -\infty \text{ or } b = -\infty)$
$\Rightarrow$ Algebraic computations are efficient

A new old object

This idempotent semiring has been reinvented many times these late fifties

$\star$ discret event system theory
$\star$ graph theory (path algebra)
$\star$ performance evaluation of manufacturing systems, Markov decision theory
$\star$ Hamilton-Jacobi theory (McEneaney, Fleming...)
$\star$ Asymptotic analysis, large deviations...(Quadrat, Akian,...

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Uniqueness in the Max-Plus decomposition

Let $Z$ be a supermartingale satisfying the previous assumptions, and suppose

1. there exist two non-decreasing adapted process $\Lambda^1_t$ and $\Lambda^2_t$ ($\Lambda^i_{-0} = -\infty$) and two u.i. martingale $M^1$ and $M^2$ such that $M^i_T = \Lambda^i_T$ and $M^i_0 = Z_0$
2. $\Lambda^i$ only increases at times when the martingale $M^i$ hits the supermartingale $Z$
3. $(M^i, \Lambda^i)$ are two (max-+) decompositions of $Z$ ($\oplus = \vee = \max$)

$$M^1_t = Z_t \oplus \Lambda^1_t, \quad M^2_t = Z_t \oplus \Lambda^2_t.$$ 

Then, $M^1$ and $M^2$ are indistinguishable martingales. There exists a maximal increasing process $\Lambda$.

**Remark** Let us observe that the equality $M_t = Z_t \oplus \Lambda^1_t = Z_t \oplus \Lambda^2_t$ does not imply that $\Lambda^1_t = \Lambda^2_t$, due to the non uniqueness of the linear equation $a \oplus x = b$ in the max-plus algebra. In particular, $\Lambda^1_t \oplus \Lambda^2_t$ is also solution.

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Sketch of the proof when $Z$ and $\Lambda$ are bounded by below

Recall the assumption $\int_0^T |M_s^i - Z_s| d\Lambda_s^i = 0$. Then, for any regular convex function ($C^2$ with linear growth) $g$, $g(0) = 0$.

$g(M^1_T - M^2_T) \leq g'(M^1_T - M^2_T)(M^1_T - M^2_T) = g'(\Lambda^1_T - \Lambda^2_T)(M^1_T - M^2_T)$

$\mathbb{E}(g(M^1_T - M^2_T)) \leq$

$\mathbb{E}(g'(\Lambda^1_0 - \Lambda^2_0)(M^1_T - M^2_T)) + \mathbb{E}((M^1_T - M^2_T) \int_0^T g''(\Lambda^1_t - \Lambda^2_t)(d\Lambda^1_t - \Lambda^2_t) )$

$= \mathbb{E} \left( \int_0^T (M^1_t - M^2_t)g''(\Lambda^1_t - \Lambda^2_t)(d\Lambda^1_t - \Lambda^2_t) \right)$

$= \mathbb{E} \left( \int_0^T (Z_t - M^2_t)g''(\Lambda^1_t - \Lambda^2_t)d\Lambda^1_t - \int_0^T (M^1_t - Z_t)g''(\Lambda^1_t - \Lambda^2_t)d\Lambda^2_t \right) \leq 0$

by the flat condition and the convexity of $g$.

In particular, $\mathbb{E}(g(M^1_T - M^2_T)) = 0$ for $g(x) = x^+$
Remark Let $Z = N - A$ the Doob Meyer decomposition of $Z$. By Itô’s formula for $Z \lor \Lambda$, and the martingale property (in the continuous case)

$$dM_t = 1_{\{Z_t > \Lambda_t\}}(dN_t^{DM} - dA_t) + 1_{\{Z_t \leq \Lambda_t\}}d\Lambda_t + \frac{1}{2}dL_t^{loc}$$

$$1_{\{Z_t = \Lambda_t\}}d\Lambda_t + \frac{1}{2}dL_t^{loc} = 0, \quad 1_{\{Z_t > \Lambda_t\}}dA_t = 0.$$  

The flat-off condition on $\Lambda$ is in fact a necessary condition.
Existence via A Convex family of supermartingales

Quite similar to the construction given in papers of NEK-Bank, NEK-Foellmer.

Z is to be assumed left quasi-continuous

The supermartingale convex family

- Introduce the Snell envelope of the convex family of processes \((Z_t \vee m)_{t \geq 0}\) indexed by a real parameter \(m\) and defined by

\[
Z_t(m) = \operatorname{esssup}_{\tau \in T_{t,T}} \mathbb{E} (Z_\tau \vee m | \mathcal{F}_t),
\]

- Given Snell envelope properties, \(Z_t(m)\) is the smallest Cadlag supermartingale dominating \(Z \vee m\), and \(T_t(m)\) is the smallest of the optimal stopping times

\[
T_t(m) = \inf \{s \in [t, T], Z_s(m) = Z_s \vee m\}.
\]

- Each martingale dominating \(Z_t \vee m\) necessarily dominates \(Z_t(m)\).

- If \((Z_t)_{t \geq 0}\) is a martingale, then \((Z_t \vee m)_{t \geq 0}\) is a sub-martingale and

\[
Z_t(m) = \mathbb{E} (Z_T \vee m | \mathcal{F}_t).
\]
Convex Analysis

Properties

1. For every $t \in [0, T]$, $m \mapsto Z_t(m)$ is convex and non-decreasing.

2. $m \mapsto Z_t(m) - m$ is non-negative, convex and non-increasing, and 
   $(Z_t(m) - m)_{t \geq 0}$ is the Snell envelope of $(Z_t - m)^+_{t \geq 0}$

3. The family of optimal stopping times $T_t(m)$ is a non-decreasing, 
   left-continuous 
   
   \[
   Z_t(m) = \mathbb{E} \left( Z_{T_t(m)} \lor m \mid \mathcal{F}_t \right). 
   \]

Main theorem

By the envelop theorem, $m \mapsto Z_t(m)$ has left-hand derivatives given by

\[
\frac{\partial^{-}}{\partial m} Z_t(m) = \mathbb{E} \left( \mathbf{1}_{\{m > Z_{T_t(m)}\}} \mid \mathcal{F}_t \right) = \mathbb{E} \left( \mathbf{1}_{\{T_t(m) = T\} \cap \{m > Z_T\}} \mid \mathcal{F}_t \right). 
\]

since $m > Z_{T_t(m)} \Leftrightarrow T_t(m) = T$ and $m > X_T$. 

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Partial representation result

**Theorem:**
Define for \( \alpha \in (t, T] \), the left-continuous inverse of \( T_t(\cdot) \) w.r.t. \( m \) by

\[
L_t^*(\alpha) := \sup\{m; T_t(m) < \alpha\} \iff \{L_t^*(\alpha) \leq m\} = \{T_t(m) \geq \alpha\}
\]

Put \( \Lambda_{t,T} = L_t^*(T) \vee Z_T \), then

\[
Z_t(m) = \mathbb{E} (\Lambda_{t,T} \vee m|\mathcal{F}_t), \quad Z_t = Z_t(-\infty) = \mathbb{E} (\Lambda_{t,T}|\mathcal{F}_t)
\]

**Proof:** That is because for a.e. \( m \)

\[
\frac{\partial}{\partial m} Z_t(m) = \mathbb{E} \left( 1_{L_{t,T}^* \leq m} | \mathcal{F}_t \right).
\]

Since \( \lim_{m \to +\infty} Z_t(m) - m = 0 \).

\[
\Rightarrow Z_t(m) - m = \int_m^{+\infty} -\frac{\partial}{\partial \alpha} (Z_t(\alpha) - \alpha) \, d\alpha = \int_m^{+\infty} 1_{L_{t,T}^* \geq \alpha} \, d\alpha
\]
Max-plus density of $\Lambda_{t,T}$

**Max-plus density**

Let $L_t$ be $L_t := \sup\{m, Z_t(m) = Z_t\}$ the right-point of the closed interval 
\{m \in R|Z_t(m) = Z_t\}, with the convention $L_T := Z_T$.

The left-inverse of $T_t(m)$ is also given by

$$L_t^*(\alpha) = \sup_{t \leq s \leq \alpha} L_s = \bigoplus_t L_s \quad \Lambda_{t,T} = \sup_{t \leq s \leq T} L_s = \bigoplus_t L_s$$

**Max-plus decomposition**

Since $\Lambda_{0,t} \bigoplus \Lambda_{t,T} = \Lambda_{0,T}$, we obtain the $Z$ max-plus decomposition via the increasing process $\Lambda_{0,t}$.

$$Z_t = \mathbb{E}(\Lambda_{t,T}|\mathcal{F}_t) = \mathbb{E}(\sup_{t \leq s \leq T} L_s|\mathcal{F}_t), \quad M_t^\boxplus = \mathbb{E}(\Lambda_{0,T}|\mathcal{F}_t) = Z_t \lor \Lambda_{0,t}$$

**Decomposition of the Snell Envelope of process $X$**

$L_t := \sup\{m, Z_t(m) = Z_t\}$ and $\Lambda_{0,t}$ only increases on $Z = M^\boxplus$
Example

Martingale case

$L_t := \sup\{m, \mathbb{E}(M_T \vee m|\mathcal{F}_t) = M_t = \mathbb{E}(M_T|\mathcal{F}_t)\}$

So, $M_T \vee m = M_T$, $\mathcal{F}_t - \mathbb{P}$, a.s., and $L_t$ is the conditionnal ess inf of $M_T$. In this case $L_t$ is an non decreasing process.

Monotone case

Suppose $Z$ to be a decreasing process. The $L_t = Z_0$. 
Stochastic order and Max-plus decomposition
Stochastic orders

Definition:
Let $X_1$ and $X_2$ be two random variables. Then we say that $X_1$ is \textbf{less variable} than $X_2$ in the \textbf{convex stochastic order}, and we write $X_1 \preceq_{cx} X_2$ if for all \textbf{convex} functions $g$ (if that makes sense)

\[ \mathbb{E}[g(X_1)] \leq \mathbb{E}[g(X_2)] \]

- If the inequality holds only for all \textbf{decreasing} convex functions, then $X_1$ is said to be smaller than $X_2$ in the \textbf{decreasing convex order} (denoted by $X_1 \preceq_{dcx} X_2$).

\[ \Rightarrow \quad X_1 \preceq_{cx} X_2 \Rightarrow \mathbb{E}(X_1) = \mathbb{E}(X_2) \] provided the expectations exist and $\text{var}(X_1) \leq \text{var}(X_2)$, whenever $\text{var}(X_2)$ is finite.

\[ \Rightarrow \quad X_1 \preceq_{dcx} X_2 \Rightarrow \mathbb{E}(X_1) \geq \mathbb{E}(X_2) \] provided the expectations exist.
Formulation of the martingale optimization problem

X-Envelope de Snell

In that follows, the previous results are applied to the Snell envelope $Z^X$ of $X$.

The optimization problem

Set $\mathcal{M}(x) = \left\{ (M_t)_{t \geq 0} \text{ u.i.martingale} | M_0 = x \text{ and } M_t \geq X_t \ \forall t \in [0, T] \right\}$

- We aim at finding a martingale $(M^*_t)$ in $\mathcal{M}(x)$ such that for all martingales $(M_t)$ in $\mathcal{M}(x)$

$$M^*_T \leq_{cx} M_T$$

- The initial value of any martingale dominating $X$ must be at least equal to the one of the Snell envelope $Z^X_0 = \sup_{\tau \in \mathcal{T}_0,T} \mathbb{E}[X_\tau]$, 

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Necessary and Sufficient condition of optimality

Theorem

• Let us consider a martingale \((M_t)_{t \geq 0}^*\) in \(\mathcal{M}(Z_0^X)\), satisfying the terminal condition \(M_T^* = K_T\), where \((K_t)_{t \geq 0}\) is an adapted increasing process, which only increases when the martingale hits the floor \(X\).

⇒ Then \((M_t^*)_{t \geq 0}\) is the “smallest martingale” in \(\mathcal{M}(Z_0^X)\) dominating the floor, with respect to the convex stochastic ordering.

⇒ Suppose \(x = Z_0^X\). The martingale \(M^\oplus\) of the max-plus \(Z\)-decomposition is an optimale solution.

• In particular, \(M_T^\oplus\) is less variable than \(M_T^DM\) where \(M^DM \in \mathcal{M}(Z_0^X)\) is the martingale of the Doob’s decomposition of \(Z\).

• If \(x \geq Z_0^X\), the same results holds, in terms of increasing process \(\Lambda_{0,T} \vee x\).
Proof

- Let $(M_t)_{t \geq 0}$ be an arbitrary element of $\mathcal{M}(x)$ and $g$ be a real convex function, for which $\mathbb{E}[g(M_T)]$ and $\mathbb{E}[g(M_T^*)]$ are well defined.

\[ g \text{ convex } \Rightarrow g(M_T) - g(M_T^*) \geq g'(M_T^*)(M_T - M_T^*) \Rightarrow \]
\[ \mathbb{E}[g(M_T)] - \mathbb{E}[g(M_T^*)] \geq \mathbb{E}[g'(M_T^*)(M_T - M_T^*)]. \]

- Remark that $(g'(K_t), t \geq 0)$ is a nondecreasing process and that $g'(K_T) = g'(K_0) + \int_0^T dg'(K_t)$.

\[ \Rightarrow \quad \mathbb{E}[g'(M_T^*)(M_T - M_T^*)] = \mathbb{E}[g'(K_T^*)(M_T - M_T^*)] \]
\[ = \mathbb{E}[g'(K_0)(M_T - M_T^*)] + \mathbb{E}\left[ \int_0^T (M_T - M_T^*) \ dg'(K_t) \right]. \]

\[ \Rightarrow \quad \mathbb{E}[g'(K_0)(M_T - M_T^*)] = g'(K_0)(M_0 - M_0^*) = 0 \]

\[ \Rightarrow \quad \mathbb{E}(\int_0^T (M_T - M_T^*) \ dg'(K_t)) = \mathbb{E}(\int_0^T (M_t - M_t^*) \ dg'(K_t)) \]
\[ = \mathbb{E}\left[ \int_0^T (M_t - X_t) \ dg'(K_t) \right] \geq 0 \]
Portfolio Insurance with American Constraint
Portfolio management with guarantee

We are concerned with the portfolio problem where the goal of the manager is

*to exceed the performance of a given benchmark process at any time during the life of the fund.*

**Example:** A fund guarantees

1. a part of an index performance, $I_t$
2. at any time the investor could receive 90% of his initial investment, without capitalization, that is

$$V_t \geq G_t = \sup(\alpha V_0, \beta \frac{I_t}{I_0})$$

Similar problems appear when the manager is submitted to legal constraints.

The guarantee may be called the floor.

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Investment funds strategies

In practice, the first step in the management of investment fund or pension fund is to define a **Strategic allocation**

According to the **investor’s risk aversion**, the manager decides the proportion of Indexs, securities, coupon bonds, in a well-diversified portfolio with present value $S_t$.

In mathematical framework $S_t$ is the **optimal portfolio** for a non constrained problem associated with given utility function

**Tactic Allocation**

or How to manage the **strategic portfolio**(underlying) in such a way that they are

- high performance of the fund (optimality?)
- no large **losses**

This last condition may be required by the **regulator** as a legal constraint

As we would see, by doing that **the manager has an optimal behaviour**.

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Links with the classical optimization problem

Let us introduce a financial market, where interest rate and risk premium are given through the state price density $H_t$, in such way that for any self-financing portfolio $V_t$, $H_t V_t$ is a local martingale.

- Consider the following “non-constrained” problem:

$$\max_{V_t} \mathbb{E}\{u(V_T), (V_t H_t)_{t \geq 0} \text{ martingale and } V_0 = x\}$$

⇒ $(V^*_t x)_{t \geq 0}$ is optimal iff $\mathbb{E}[u'(V^*_T x)(V_T x - V^*_T x)] = 0$, for any self-financing portfolio $(V^*_t x)_{t \geq 0}$, where $x$ stands for the initial capital.

⇒ In a complete market $u'(V^*_T x) = \lambda H_T$, where $\lambda$ is a Lagrange multiplier, and $H$ the state prices process.

⇒ For **CRRA utility function** : $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$, for all $x \in R^+$, with $\gamma \in ]0, 1[$, $V^*_T x = x S_T$, where $S_T = V^*_T 1$ denotes the optimal portfolio with initial capital $S_0 = 1$. 

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Change of numeraire and martingale problem

- **Constrained** decision problem: $\max_{V_t} \mathbb{E}(u(V_T))$, subject to $(V_t)_{t \geq 0}$ self-financing portfolio, $V_t \geq G_t \ \forall t \in [0, T]$ and $V_0 = x$.

$\Rightarrow$ Let $M_t^S = H_tS_t$ be the S-martingale and define $Q^S$ the risk-neutral probability w.r.to $S$

$$\frac{dQ^S}{d\mathbb{P}} = \frac{M_T^S}{M_0^S} = H_TS_T.$$ 

- In the case of a **CRRA utility function**, since $S$ is the optimal portfolio, $(S_T)^{-\gamma} = \lambda H_T$

$$\mathbb{E}(u(V_T)) = \frac{1}{1 - \gamma} \mathbb{E}(S_T^{1-\gamma}(\frac{V_T}{S_T})^{1-\gamma}) = \frac{1}{1 - \gamma} \lambda \mathbb{E}(H_T S_T (\frac{V_T}{S_T})^{1-\gamma})$$

$$= \lambda \mathbb{E}_{Q^S}(u(V_T^S)).$$

- **Under $Q^S$,** $(V_t^S = \frac{V_t}{S_t})_{t \geq 0}$ is a martingale and the problem becomes:

$$\max_{V_t^S} \left\{ \mathbb{E}_{Q^S}(u(V_T^S)), \text{subject to } V_t^S \geq \frac{G_t}{S_t} = X_t \text{ and } V_0^S \text{ given} \right\}.$$
Characterization of the optimal solution

Suppose \( x = Z_0 \)

The solution is given by the martingale of the max-plus decomposition of the Snell envelope \( Z^X \) of \( X \) in the \( Q_S \)-market s.t.

\[
Z^X_t(m) = \text{esssup}_{\tau \in T_t, T} \mathbb{E}^{Q_S} [(Z^X_\tau \vee m) | \mathcal{F}_t] = \text{esssup}_{\tau \in T_t, T} \mathbb{E}^{Q_S} [(X_\tau \vee m) | \mathcal{F}_t] .
\]

\( L^S_t = \sup\{m, Z^X_t(m) = Z^X_t = X_t\} \) is the boundary in strike of the \( Q_S \)-American Call option with pay-off \( (X_t - m)^+ \).

Then, the optimal \( Q_S \)-martingale is

\[
M^{S,*}_t = \mathbb{E}^{Q_S} [\Lambda^S_{0,T} | \mathcal{F}_t] = \mathbb{E}^{Q_S} [\Lambda^S_{0,t} \vee \Lambda^S_{t,T} | \mathcal{F}_t] ,
\]

if \( \Lambda^S_{0,t} = \sup_{0 < u \leq t} L^S_u \)
American Option in BS Framework

We assume the underlying to be a geometrical Brownian motion. The American Put with strike $K$ is a function, $U^a(t,x)$ satisfying the variational inequality

$$\partial_t U^a + \frac{1}{2}\sigma^2 x^2 \partial^2_{xx} U^a + rx \partial_x U^a - rU^a = 0$$

on the continuation region \{ $x > b(t)$ \} = \{ $U^a(t,x) > K - x$ \} ($b(t)$ is the exercise boundary) with the smooth-fit condition at the boundary $\partial_x U^a(t,b(t)) = -1$.

Then, for $S_t^x = xS_t$, $U^a(t,xS_t) = \text{esssup}_{U \geq t} \mathbb{E}(e^{-r(U-t)}(K - xS_U)^+) | \mathcal{F}_t)$

By change of numeraire, under the probability $\mathbb{Q}_{S_t}$

$$\frac{U^a(t,xS_t)}{S_t} = C^a(t,KS_t^{-1},x) = \text{esssup}_{U \geq t} \mathbb{E}_{Q_S}(KS_U^{-1} - x)^+ | \mathcal{F}_t)$$
Closed form of the American Call option

Using previous notation

\[ L^S_t = \sup\{x, C^a(t, KS_t^{-1}, x) = (KS_t^{-1} - x)^+\} = b(t)/S_t \]
\[ \Lambda^S_{t,s} = \sup_{t \leq u \leq s} \frac{b(u)}{S_u} \]

- The strategy \( M^{S,*}_t = \Lambda^S_{0,t} + C^a(t, KS_t^{-1}, \Lambda^S_{0,t}) \geq KS_t^{-1} \) is the optimal \( Q_S \) self-financing martingale strategy dominating \( X_t = KS_t^{-1} \) with terminal value

\[ M^{S,*}_T = (K/S_T) \lor \sup_{0 \leq u \leq T} \left( \frac{b(u)}{S_u} \right) \]

- The price of the American Call option is given by

\[ C^a(t, KS_t^{-1}, x) = \mathbb{E}_{Q_S}(KS_T^{-1} - x \lor \Lambda^S_{t,T})^+ | \mathcal{F}_t) \]