Optimal Risk Transfer
under Dynamic Risk Measures

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Agenda

- New financial instruments and motivation of this study
- A “toy model” : the exponential utility framework
- Some results on risk measures : basic properties and new developments
- Optimal risk transfer problem
- Inf-convolution and BSDEs
Introduction and motivation

Development of new financial products

A new type of assets:

- Recent introduction of a new type of financial contracts with a non-financial underlying risk: “cat-bonds”, weather derivatives...
- Illiquid instruments, with an underlying asset which is not traded on financial markets.
- Underlying risk possibly related to a financial market risk.

Question

How to design derivatives, and price them in such a way that transaction may occur in an illiquid market?
**Interplay between finance and insurance:**

⇒ Use of the knowledge of financial risk management to the management of other kinds of risk.
⇒ Use of the insurance technology to design structured products.

From now on, we will use the insurance risk transfer approach, and vocabulary.

**Main question**

What is the "**optimal**" transfer of a non-tradable risk between different agents having access to other possible investments?

⇒ The notion of "optimality" requires a choice criterion. We consider first exponential utility and then convex risk measures.
Related works

- Convex risk measure: Foellmer and Schied, Artzner, Delbaen, Heath
- Indifference pricing: Musiela, Zariphopoulou, Davis, Rouge-NEK, Becherer, DGRSSS (six authors)
- Calibration and stress testing: Geman, Madan, Avellaneda.
An exponential utility “toy model”

Uncertainty is modelled via a standard probability space \((\Omega, \mathcal{F}, \mathbb{P})\). \(T\) is the time horizon. All flows are capitalized till \(T\) by using the capitalization factor \(\beta_{0,T}\).

Transaction involving two agents

- **Agent A**: at time \(T\), agent A is exposed towards a non-tradable risk \(\Theta\) for an amount \(X(\Theta, \omega)\) in the market scenario \(\omega\). It calls on an investor, agent B, to reduce its exposure by the sale of a structured contract \(F(\Theta, \omega)\).

- **Agent B**: it pays a premium \(\pi\) at time 0 and receives in exchange the structure \(F\) at time \(T\). Its initial wealth is denoted by \(x\).

Utility criterion

Both agents are risk-averse. Their respective attitude towards risk is modelled by an exponential utility function \(U(z) = -\gamma \exp(-\frac{1}{\gamma}z), z \in \mathbb{R}\) (with risk tolerance coefficient \(\gamma_A\), resp. \(\gamma_B\)).
Relationship between both agents

Both agents do not have the same goals

⋆ Agent A looks for an **optimal reduction of its exposure**, that is for the structure \((F, \pi)\) as to maximize its expected utility:

\[
\max_{F \in \mathcal{X}, \pi} \mathbb{E}_P(U_A(X - F + \pi \beta_{0,T}))
\]

⋆ Agent B wants to **improve its expected utility** by doing the \(F\)-transaction, in the following sense

\[
\mathbb{E}(U_B(F + (x - \pi) \beta_{0,T})) \geq \mathbb{E}(U_B(x \beta_{0,T}))
\]

Using the properties of the exponential utility, the program becomes

\[
\min_{F \in \mathcal{X}, \pi} \gamma_A \mathbb{E}\left(\exp\left[-\frac{1}{\gamma_A}(X - F + \pi \beta_{0,T})\right]\right) \text{ given } \mathbb{E}\left(\exp\left[-\frac{1}{\gamma_B}(F - \pi \beta_{0,T})\right]\right) \leq 1
\]


Pricing Rule and Optimal structure

⇒ Binding the constraint leads immediately to the **optimal pricing rule**

\[
\pi^*(F)\beta_{0,T} = -\gamma_B \ln \mathbb{E}(\exp(-\frac{1}{\gamma_B}F)) = -e^{\gamma_B}(F)
\]

⇒ Using Lagrangian multiplier, the optimal structure is minimizing for

\[
\mathbb{E}(\gamma_A \exp[-\frac{1}{\gamma_A}(X-F+\pi\beta_{0,T})] + \lambda \exp[-\frac{1}{\gamma_B}(F-\pi\beta_{0,T})])
\]

It is obtained as:

\[
F^* = \frac{\gamma_B}{\gamma_A + \gamma_B} X \quad \mathbb{P} \text{ a.s.} \quad + \text{constant}
\]

Put \(\gamma_C = \gamma_A + \gamma_B\). Then

\[
e^{\gamma_C}(X) = e^{\gamma_A}(X - (F^* - \pi^*_B\beta_{0,T})) = \inf_F \left(e^{\gamma_A}(X - F) + e^{\gamma_B}(F)\right) = E_{AB}(X)
\]

**Remark**: The optimal structure is Pareto optimal.

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Risk measures : basic properties

Let $(\Omega, \mathcal{F})$ be a standard measurable space and $\mathcal{X}$ the linear space of bounded functions (including constant functions).

**Definition :** The functional $\rho$ is said to be a **convex risk measure** in the sense of Föllmer and Schied (2002) if for any $X$ and $Y$ in $\mathcal{X}$

- **Convexity ;** $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda \rho(X) + (1 - \lambda)\rho(Y)$
- **Decreasing** monotonicity ; $\rho(X) \geq \rho(Y)$ if $X \leq Y$
- **Translation invariance :** $\forall m \in \mathbb{R}, \rho(X + m) = \rho(X) - m$.

The following set plays a key role

- The **acceptance set** associated with $\rho$ is defined as :
  
  $\mathcal{A}_\rho = \{\Psi \in \mathcal{X}, \rho(\Psi) \leq 0\}$

- $\rho$ may be defined from the acceptance set by
  
  $\rho(X) = \inf\{m, m + X \in \mathcal{A}_\rho\}$
Penalty function

The dual formulation of this convex functional is a key point for our study.

**Theorem : (FS)** There exists a **penalty function** \( \alpha \) taking values in \( \mathbb{R} \cup \{+\infty\} \) s.t.

\[
\forall \Psi \in \mathcal{X}, \quad \rho(\Psi) = \sup_{Q \in \mathcal{M}_{1,f}} \{\mathbb{E}_{Q}(-\Psi) - \alpha(Q)\}
\]

\[
\forall Q \in \mathcal{M}_{1,f}, \quad \alpha(Q) = \sup_{\Psi \in \mathcal{X}} \{\mathbb{E}_{Q}(-\Psi) - \rho(\Psi)\} = \sup_{\Psi \in \mathcal{A}_\rho} \{\mathbb{E}_{Q}(-\Psi)\}
\]

where \( \mathcal{M}_{1,f} \) is the set of all additive measures on \( (\Omega, \mathcal{F}) \), dual set of \( \mathcal{X} \).

**Example : Entropic risk measure**

\[
e_{\gamma}(X) = \gamma \mathbb{E}(\exp(-\frac{1}{\gamma}X)) = \sup_{Q \in \mathcal{M}_1} (\mathbb{E}_{Q}(-X) - \gamma h(Q|P))
\]

where \( h \) is the **entropic function**

\[
h(Q|P) = \begin{cases} 
\mathbb{E}_{P}(\frac{dQ}{dP} \ln \frac{dQ}{dP}) & \text{if } Q \ll P \\
+\infty & \text{otherwise}
\end{cases}
\]
Risk measure generated by a convex subset $\mathcal{H}$

Let $\mathcal{H}$ be a convex set such that $\inf \{ m \in \mathbb{R}, \exists \xi_T \in \mathcal{H} \text{ s.t. } m \geq \xi_T \} > -\infty$.

- A convex risk measure $\nu^\mathcal{H}$ is defined by:
  \[ \forall \Psi \in \mathcal{X} \quad \nu^\mathcal{H}(\Psi) = \inf \{ m \in \mathbb{R}, \exists \xi_T \in \mathcal{H} \text{ s.t. } m + \Psi \geq \xi_T \} \]

- The associated penalty function is
  \[ \forall Q \in \mathcal{M}_{1,f}, \quad \kappa^\mathcal{H}(Q) = \sup_{\Psi \in \mathcal{H}} \{ \mathbb{E}_Q(-\Psi) \} \]

- Moreover, $\mathcal{H}$ is a cone, $\nu^\mathcal{H}$ is a coherent risk measure in the sense of Artzner and alii, and the penalty function is
  \[ l^\mathcal{H}(Q) = 0 \quad \text{if } Q \in \mathcal{M}_\mathcal{H} \quad ; \quad +\infty \quad \text{otherwise} \]
  where $\mathcal{M}_\mathcal{H}$ is the set of all additive measures such that $\forall \xi \in \mathcal{H}, \mathbb{E}_Q(\xi) \geq 0$. 

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Main result

**Theorem:** Let $\rho_1$ and $\rho_2$ be two convex risk measures with respective penalty functions $\alpha_1$ and $\alpha_2$. Let $\rho_{1,2} = \rho_1 \square \rho_2$ be the inf-convolution of $\rho_1$ and $\rho_2$ defined by

$$\Psi \rightarrow \rho_{1,2}(\Psi) \overset{def}{=} \inf_H [\rho_1(\Psi - H) + \rho_2(H)]$$

and assume that $\rho_{1,2}(0) > -\infty$. Then $\rho_{1,2}$ is a finite convex risk measure. The associated penalty function is given by

$$\forall Q \in M_{1,f} \quad \alpha_{1,2}(Q) = \alpha_1(Q) + \alpha_2(Q)$$

The related acceptance set $A_{\rho_{1,2}}$ is the ”pseudo-closure” of $A_{\rho_1} + A_{\rho_2}$.

**Remark:** Note also that

$$\rho_{1,2}(\Psi) = \inf \{ \rho_1(\Psi - H), \quad H \in A_{\rho_2} \}$$
**\( \mathcal{H} \)-reduced risk measure**

Let \( \mathcal{H} \) be a cone of \( \mathcal{X} \) and \( \rho \) be a convex risk measure with penalty function \( \alpha \) such that \( \inf \{ \rho(-H), H \in \mathcal{H} \} > -\infty \).

The inf-convolution \( \rho^\mathcal{H} \) of \( \rho \) and \( \nu^\mathcal{H} \) is the \( \mathcal{H} \)-reduced \( \rho \) defined as

\[
\rho^\mathcal{H}(\Psi) \overset{\text{def}}{=} \rho \square^\mathcal{H} \nu^\mathcal{H}(\Psi) = \inf \{ \rho(\Psi - H), H \in \mathcal{H} \} = \sup_{Q \in \mathcal{M}^\mathcal{H}} \{ \mathbb{E}_Q(-\Psi) - \alpha(Q) \}
\]

is a convex risk measure with penalty function

\[
\alpha^\mathcal{H}(Q) = \alpha(Q) \quad \text{if} \quad \mathbb{E}_Q(H) \geq 0, \forall H \in \mathcal{H} \; ; \; +\infty \quad \text{otherwise}
\]

**Comments :**

★ A typical example of a cone is the set \( \mathcal{V}_T \) of the gain processes associated with financial investments (most liquid part of the market).

\( \rho^\mathcal{V}_T \overset{\text{def}}{=} \rho^m \) is the **market modified risk measure**.

★ The financial market plays there the same role as an intermediate agent having a risk measure \( \nu^\mathcal{V}_T \).
Dilatation and semi-group property

Let $\rho$ be a convex risk measure with penalty function $\alpha$.

**Definition:** The associated dilated risk measure $\rho_\gamma$ is defined by

$$\forall \Psi \in \mathcal{X} \quad \rho_\gamma(\Psi) \overset{\text{def}}{=} \gamma \rho\left(\frac{1}{\gamma} \Psi\right) \quad \text{with} \quad \alpha_{\rho_\gamma}(Q) = \gamma \alpha(Q)$$

where $\gamma > 0$ is the risk tolerance coefficient.

**Semi-group property w.r.to inf convolution**

If $\rho$ and $\rho'$ are two risk measures

$$\rho_\gamma \Box \rho_\gamma' = \rho_{\gamma + \gamma'}, \quad \rho_\gamma \Box \rho'_\gamma = (\rho \Box \rho')_\gamma$$

**Typical example:** entropic risk measure.
Properties

1. $\rho_{\gamma} \boxtimes \rho_{\gamma'}(X) = \inf_F \{\rho_{\gamma}(X - F) + \rho_{\gamma'}(F)\} = \rho_{\gamma}(X - \frac{\gamma'}{\gamma + \gamma'} X) + \rho_{\gamma'}(\frac{\gamma'}{\gamma + \gamma'} X)$

   $\frac{\gamma'}{\gamma + \gamma'} X$ is optimal, as in the entropic case.

2. $\rho$ is a coherent risk measure if and only if, for any $\gamma$ strictly positive,
   $\rho_{\gamma} \equiv \rho$. Moreover, $\forall n \geq 1 \quad \rho \boxtimes^n = \rho$

3. Assume $\rho(0) = 0$ Then $\rho_{\gamma}$ is a decreasing function of $\gamma$, with asymptotic behavior
   $\Rightarrow \rho_{\infty} \overset{\text{def}}{=} \lim_{\gamma \to \infty} \rho_{\gamma}$ is a coherent risk measure and

   $\rho_{\infty}(\Psi) = \sup \{\mathbb{E}_Q(-\Psi) | Q \in \mathcal{M}_{1,f}, \alpha(Q) = 0\}$

   $\Rightarrow \rho_0$ is simply the super-pricing rule of $-\Psi$:

   $\rho_0(\Psi) = \sup \{\mathbb{E}_Q(-\Psi) | Q \in \mathcal{M}_{1,f}, \alpha(Q) < \infty\}$
Optimal Risk Transfer: the framework

Transaction involving two agents

★ Agent A: at a future time $T$, agent A is exposed towards a non-tradable risk $\Theta$ for an amount $X(\Theta, \omega)$ in the market scenario $\omega$. It calls on an investor, agent B, to reduce its exposure by the sale of a structured contract $F(\Theta, \omega)$.

★ Agent B: it pays a premium $\pi$ at time 0 and receives in exchange the structure $F$ at time $T$. Its initial wealth is denoted by $x$.

★ Both agents can also invest in financial markets via two cones of bounded terminal gains associated with self-financing investment strategies, $\mathcal{V}_T(A)$ and $\mathcal{V}_T(B)$.

★ Both agents assess their risk exposure using two convex risk measures, $\rho_A$ and $\rho_B$, with respective penalty functions $\alpha_A$ and $\alpha_B$. 

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Optimal transfer : agents goals

Relationship between both agents :

1. Agent A looks for a **hedge of its exposure**. It wants to determine the structure \((F, \pi, \xi_A)\) as to minimize the risk measure of its wealth :

\[
\min_{F,\pi,\xi_A \in \mathcal{V}^{(A)}_T} \rho_A(X - F + \pi + \xi_A) = \min_{F,\pi} \min_{\xi_A \in \mathcal{V}^{(A)}_T} \rho_A(X - F + \pi + \xi_A) = \min_{F,\pi} \rho_m^A(X - F + \pi)
\]

2. Agent B wants to **improve its risk measure** by doing the \(F\)-transaction. Its interest in doing the transaction may be written as :

\[
\min_{\xi_B \in \mathcal{V}^{(B)}_T} \rho_B(F - \pi + x + \xi_B) \leq \min_{\xi_B \in \mathcal{V}^{(B)}_T} \rho_B(x + \xi_B)
\]

or equivalently, using the cash invariance translation property and the market modified risk measure \(\rho^m_B\) :

\[
\rho^m_B(F - \pi) \leq \rho^m_B(0)
\]
Optimal pricing rule and residual risk measure

★ The optimal pricing rule is simply obtained by binding the constraint of Agent B and using the cash invariance property:

\[ \pi^*(F) = \rho_B^m(0) - \rho_B^m(F) \]

★ The program \( \min_F \rho_A^m(X - F + \pi) \) subject to \( \rho_B^m(F - \pi) \leq \rho_B^m(0) \) may be reduced to (to within the constant \( \rho_B^m(0) \))

\[
R_{AB}^m(X) \overset{\text{def}}{=} \min_F (\rho_A^m(X - F) + \rho_B^m(F)) = \rho_A \Box \nu_T^{(A)} \Box \rho_B \Box \nu_T^{(B)}(X)
\]

⇒ Using the previous results, \( R_{AB}^m \) is a convex risk measure with penalty function \( \alpha_{AB}^m \) defined for any \( Q \in \mathcal{M}_{1,f} \) by:

\[
\alpha_{AB}^m(Q) = \alpha_A^m(Q) + \alpha_B^m(Q) = \alpha_A(Q) + \alpha_B(Q) \quad \text{if} \ Q \in \mathcal{M}^{(A)} \cap \mathcal{M}^{(B)} ; +\infty \quad \text{otherwise}
\]
Characterization of the optimal structure

The program to be solved is

\[
\min_F (\rho^m_A (X - F) + \rho^m_B (F))
\]

The case of dilated risk measures

Both agents have dilated risk measures, \( \rho_{\gamma_A} \) and \( \rho_{\gamma_B} \).

1. If the agents have access to the same market, \( \rho^m_A = \rho^m_{\gamma_A} \) and the final risk measure \( R^m_{AB}(X) = \rho^m_{\gamma_A + \gamma_B} \).

   An optimal structure is the given by \( F^* = \frac{\gamma_B}{\gamma_A + \gamma_B} X \)

2. When agents have access to different markets, the final risk measure \( R^m_{AB}(X) \) is the risk measure \( \rho_{\gamma_A + \gamma_B} \) reduced by \( \mathcal{H} = \mathcal{V}_T^A + \mathcal{V}_T^B \)

3. Suppose \( \eta^*_A + \eta^*_B \) is an optimal solution of the final hedging program:

\[
F^* = \frac{\gamma_B}{\gamma_A + \gamma_B} X + \frac{\gamma_B}{\gamma_A + \gamma_B} \eta^*_A - \frac{\gamma_A}{\gamma_A + \gamma_B} \eta^*_B
\]

   is an optimal structure.

4. In a more general framework, we simply obtain a necessary and sufficient condition for the optimality but not an explicit form of \( F^* \).
Comments

One market

★ This result that is optimal to transfer the same ration of the initial risk as in the problem without market is very strong as it does not require any specific underlying modelling either for the non-tradable risk or the financial market.

★ The optimal structure $F^*$ does not depend on the financial market. The impact of the financial market is simply visible through the pricing rule.

★ The underlying logic is non-speculative as the issuer has an interest to sell a structure if and only if it is initially exposed.

Two markets

• Even if the issuer $A$ is not initially exposed, a transaction may take place. It is an opportunity for the agent $B$ to buy “derivatives product” in the $A$-market, which is unaccessible for trading.
Dynamic Risk measures and
Backward Stochastic Differential Equations

Motivation

We want to study risk measures defined by their local specifications and propose a method to characterize the optimal solution of the inf-convolution problem.
BSDEs

We introduce a Brownian filtration $\left( \Omega, \mathcal{F}_t = \sigma(W_s; 0 \leq s \leq t), \mathbb{P} \right)$

$$-dY_t = f(t, Y_t, Z_t)dt - \langle Z_t, dW_t \rangle, \quad Y_T = \Psi$$

1. Given standard Lipschitz assumptions, there exists a unique solution $(Y_t, Z_t)$ with square integrability properties. Moreover a comparison theorem holds.

2. For any bounded $\Psi$, if $f$ has quadratic growth w.r. to $z$, there exists a unique solution $(Y, Z)$ in $L^\infty \otimes H^2 \otimes H^2$ (Kobylanski 2000), Lepeltier-San Martin,(1999)

3. BSDEs and PDE’s : Suppose that $\Psi = g(X_T)$, $f(t, ., y, z) = f(t, X_t, y, z)$ where $X$ is a $R^d$-valued diffusion process with elliptic generator $L$. Then, with some regularity assumptions, the process $Y$ is given by $u(t, X_t)$ where $u$ is viscosity solution of the non linear PDE

$$\left( \partial_t + L \right) u(t, x) + f(t, x, u(t, x), \nabla u(t, x)) = 0, \quad u(T, x) = g(x)$$

The comparison theorem is nothing than the maximum principle for PDEs.

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Localization of convex risk measures

We extend the notion of static entropic risk measure to a more localized dynamic one:

**Dynamic Entropic risk measure**:

\[
\gamma, t(X) = \gamma \ln \mathbb{E}_P(\exp(-\frac{1}{\gamma}X)|\mathcal{F}_t), \quad e_{\gamma, T}(X) = -X
\]

The dynamics of the process \( (e_{\gamma, t}(X); t \in [0, T]) \) is given by the BSDE with the quadratic driver \( f(t, z) = \frac{1}{2\gamma} ||z||^2 \):

\[
-d e_{\gamma, t}(X) = \frac{1}{2\gamma} ||Z_t||^2 dt - \langle Z_t, dW_t \rangle \quad e_{\gamma, T}(X) = -X
\]

Given Kobylansky, it is the **only solution** of this BSDE.
Extension to a family of convex risk measures:

The idea is to generalize the notion of static convex risk measure to a more dynamic notion by considering BSDEs:

**Theorem**: Suppose that the regular driver $f(t, z)$ is convex w.r. to $z$.

The solution $(\rho_t(X); t \leq T)$ of the BSDE with terminal condition $-X$.

$$-d\rho_t(X) = f(t, Z_t)dt - \langle Z_t, dW_t \rangle, \quad \rho_T(X) = -X$$

is, for any time $t \leq T$, a convex risk measures.

**Comment**: This convexity result has already been proved in the study of pricing functionals with constraints (for instance, El Karoui-Quenez (1996) or Peng (1997, 2003)).
Dual Representation

Fenchel representation of the driver and penalty function

Let us denote by $\alpha_t(\nu) \in [0, +\infty]$ the Fenchel transform of the convex driver $f$, $(\nu \in \mathbb{R}^d)$

$$f(t, z) = \sup_{\nu} \{ < z, \nu > - \alpha(t, \nu) \} =: \sup_{\nu} f^\nu(t, z)$$

Suppose that the domain $\mathcal{D}_\boxplus = \{ \nu, \quad \alpha(t, \nu) < +\infty \}$ is a bounded subset of $\mathbb{R}^d$ and define the linear BSDE by

$$-d\rho^\nu_t(X) = f^\nu(t, Z_t) dt - \langle Z_t, dW_t \rangle, \quad \rho_T(X) = -X$$

Then, by the comparison theorem

$$f(t, z) = \sup_{\nu} f^\nu(t, z) \quad \Rightarrow \quad \rho(X) = \text{ess sup} \rho^\nu_t(X)$$
**Linear BSDE**

Let us introduce

- the exponential martingale \( H_t^\nu = \exp\left( \int_0^t < \nu_s, dW_s > - \frac{1}{2} \int_0^t |\nu_s|^2 ds \right) \)
- the \( \mathbb{P} \)-equivalent martingale measure \( \mathbb{Q}^\nu \) with density \( H_T^\nu \).
- the \( \mathbb{Q}^\nu \)-Brownian motion \( dW^\nu = dW_t - \nu_t dt \)

The risk-measure

\[
-d\rho^\nu_t(X) = (<Z_t, \nu_t> - \alpha(t, \nu_t))dt - \langle Z_t, dW_t \rangle = -\alpha(t, \nu_t)dt - \langle Z_t, dW^\nu_t \rangle
\]

is given by

\[
\rho^\nu_t(X) = \mathbb{E}_{\mathbb{Q}^\nu}(-X) - \mathbb{E}_{\mathbb{Q}^\nu}\left( \int_t^T \alpha(s, \nu_s)ds|\mathcal{F}_t \right)
\]

**Remark:** The penalty functional \( \alpha(t, \mathbb{Q}^\nu) \) (in the sense of Foellmer and Schied) is given by

\[
\alpha(t, \mathbb{Q}^\nu) = \mathbb{E}_{\mathbb{Q}^\nu}\left( \int_t^T \alpha(s, \nu_s)ds|\mathcal{F}_t \right)
\]
Inf-convolution problem

Notation
Let us assume the both agents risk measures to be given by BSDEs

★ Their dynamic versions are denoted by $\rho_t^A$ and $\rho_t^B$.

● Their respective BSDEs are associated with the convex drivers $f^A(t, z)$ and $f^B(t, z)$.

★ Let $(f^A \Box f^B)(t, z) = \inf_u (f^A(t, z - u) + f^B(t, u))$ be classical inf convolution in $\mathbb{R}^d$ of $f^A$ and $f^B$.

Theorem: Suppose $(f^A \Box f^B)(t, z)$ to be a regular driver. Let $(\rho_{t}^{A,B}(X), Z_t)$ the associated BSDE with terminal value $X$. Then

1. For any $F \in \mathcal{X}$, $\rho_t^{A,B}(X) \leq \rho_t^A(X - F) + \rho_t^B(F)$ $\mathbb{P}$ a.s.

2. If there exists an admissible $\hat{Z}_t^B$ such that

$$\forall t \geq 0 \quad f^A \Box f^B(t, Z_t) = f^A(t, Z_t - \hat{Z}_t^B) + f^B(t, \hat{Z}_t^B)$$

then at any time $t$, $\rho_t^{A,B}(X)$ is the inf-convolution of $\rho_t^A$ and $\rho_t^B$

$$\forall t \geq 0 \quad \rho_t^{A,B}(X) = (\rho^A \Box \rho^B)_t(X) \quad \mathbb{P} \text{ a.s}$$
3. Under this assumption, let \( F^* \) the structure defined by the forward equation

\[
F^* = \int_0^T f^B(t, \hat{Z}^B_t) \, dt - \int_0^T \langle \hat{Z}^B_t, dW_t \rangle
\]

Then \( F^* \) is an optimal solution for the inf-convolution problem

\[
(\rho^A \square \rho^B)_t(X) = \inf_{F_t} \{ \rho^A_t(X - F) + \rho^B_t(F) \}
\]

Usual regularity assumptions are made on the different convex drivers. The proof uses intensively the comparison theorem.
Entropic Indifference buyer price

Let us introduce the hedging space $\mathcal{V}_T = \{ \xi = \int_0^T < \phi_s, dW_s + \lambda_s ds, \quad \phi_s \in \mathcal{K}_t \}$. The coherent dynamic risk measure associated is the opposite of the price

$$-d\rho^\mathcal{V}_t(\xi) = -< Z_s, \lambda_s > ds, -< Z_s, dW_s >, \quad \rho^\mathcal{V}_T(\xi) = -\xi$$

if $\forall t, Z_t \in \mathcal{K}_t$, $-\infty$ if not.

The modify entropic risk measure $e^\mathcal{V}_{\gamma,t}(X) = e^{\gamma,t} \square \rho^\mathcal{V}_t(X)$ is the unique solution of the Quadratic BSDE

$$-de^\mathcal{V}_{\gamma,t}(X) = \inf_{\{\phi_t \in \mathcal{K}_t\}} \left( \frac{1}{2\gamma} \| Z_t - \phi_s \|^2 - < \phi_t, \lambda_t > \right) dt - \langle Z_t, dW_t \rangle$$

$$e^\gamma,T(X) = -X$$
Non-speculative logic

We now suppose that the Agent $A$ does not support any risk $(X = 0)$

**Corollary:**

Assume that $f^A(t, 0) = f^B(t, 0) = 0$ and

$$\partial_z f^A(t, 0) = \partial_z f^B(t, 0) = 0$$

1. The inf-convolution $(f^A \square f^B)(t, 0)$ and the associated risk measure $(\rho_A \square \rho_B)(t, 0)$ are **identically null**.

2. Moreover, $F^* \equiv 0$ is an optimal solution for the inf-convolution problem

$$\left(\rho^A \square \rho^B\right)(t, 0) = \inf_F \{\rho^A_t (-F) + \rho^B_t (F)\}$$

3. If both drivers $f^A$ and $f^B$ are **strictly convex**, then $F^* \equiv 0$ is the **unique** optimal solution for the inf-convolution problem.

In this case the logic of the transaction is **non-speculative** since the issuer has an interest to sell a structure if and only if it is initially exposed.
Concluding remarks

* Standard diversification occurs in exchange economies when agents have dilated risk measures.

⇒ The regulator has to impose very different risk measures (or penalty functions) to the different agents in the market to increase the diversification.

* During trade talks preceding the transaction, both agents will reveal some information on their risk measure. This step is crucial, because the structure depends only upon this information.

* Both agents take part in the negotiation process for the transaction. Not only the price is at stake but also the structure (or the amount). Consequently, this increases the probability to reach an agreement.

* In the general framework of convex risk measures, modifications in the investment opportunities of the agent is simply translated by an explicit modification of the risk measure.

* Extensions on localized version of convex risk measures.