Convex risk measures

and

robust optimization problems

Based on

- joint work with Alexander Schied
- results of A. Schied (2003)
- results of Anne Gundel (2004)
- joint work with Irina Penner

<u>complete</u> financial market:

unique equivalent martingale measure $P^* \approx P$

- perfect replication / hedge
- no preferences! (no microeconomic theory, ...)

<u>incomplete</u> financial market:

$\#$ sources of uncertainty
$\gg$ $\#$ financial instruments

<u>many</u> martingale measures $P^* \approx P$

- superreplication
  (preference free / extreme;
  stay on the safe side)
- partial/efficient hedging:
  accept some "<u>downside risk</u>"
- preferences come back!

"<u>utility based</u>" approaches: typically

- high tech (mathematically)
- standard (economically):

Quantifying

Financial Risk

&

Preferences

in the face of

Risk and Uncertainty

"expected utility"
and beyond

(e.g. Kahneman, Smith Nobel 2002)

"<u>Risk</u>"

of a financial position

$$X : \Omega \longrightarrow \mathbb{R}^1$$

Ω = set of scenarios

$X(\omega)$ = discounted net worth at end of trading period

?

- long history (D. Bernoulli, ...)
- "<u>hot" topic</u> (→ risk managemen
  Supervising)

What do we really want?
(Artzner, Delbaen, Eber, Heath)

Focus on

- downside risk
- monetary aspect

$\mathcal{X}$ = space of financial positions
$\subseteq \mathbb{R}^{\Omega}$

$X(\omega)$ = discounted net worth in scenario $\omega \in \Omega$

$\mathcal{A} \subseteq \mathcal{X}$ "acceptable positions"

$$X \in \mathcal{A},\ Y \geq X \Rightarrow Y \in \mathcal{A}$$

$$\Downarrow$$

$$\rho(X) := \inf \{ m \mid X + m \in \mathcal{A} \}$$

= minimal amount which should be added (and invested in a risk free manner) to make the position acceptable

= capital requirement

$$\Downarrow$$

i) <u>monotone:</u>

$$X(\omega) \geq Y(\omega) \quad \Longrightarrow \quad \rho(X) \leq \rho(Y)$$

ii) <u>translation invariant:</u>

$$\rho(X + m) = \rho(X) - m$$

or: $m(1+r)$

<u>"monetary risk measure"</u>

<u>"coherent"</u> (A.D.E.H):

$$\rho(X+Y) \leq \rho(X) + \rho(Y)$$

? $\rho(\lambda X) = \lambda \rho(X) \qquad \forall \lambda > 0$

Two standard examples of "acceptability", given a probabilistic model

P = probability measure on $(\Omega, \mathcal{F})$

① $\mathcal{A} = \{ X \in L^0 \mid P[X<0] \leq \alpha \}$

tolerance $\alpha \in (0,1)$

$\Rightarrow$

$$\rho(X) = \inf \{ m \mid P[X+m<0] \leq \alpha \}$$

$$= - q_X^+(\alpha)$$

$$=: \mathrm{VaR}_\alpha(X)$$

"Value at Risk" at level α

- monetary, pos. homogeneous
- in general not subadditive / convex
- $\mathrm{VaR}_\alpha(X) = E[-X] + \Phi^{-1}(1-\alpha)\,\sigma(X)$

(2) "Acceptable" if
<u>Sharpe ratio</u> is good enough:

$$\mathcal{A} = \{ X \in L^2 \,/\, E[X] \geq c\,\sigma(X) \} \qquad c > 0$$

$$\Rightarrow \quad \rho(X) = E[-X] + c\,\sigma(X)$$

- convex, pos. homogeneous
- in general not monetary (translation invariant, but not monotone)

Both coincide, and thus are coherent, on any Gaussian subspace!

Beyond: not coherent!

extension
(Heath, Fritelli - Gianin, F. - Schied):

<u>convex</u> risk measure:

- monetary
- $\rho(\lambda X + (1-\lambda)Y)$
  $\leq \lambda \rho(X) + (1-\lambda)\rho(Y)$

<u>general structure</u>?
(ADEH, Delbaen, ...
F. - Schied, Fritelli, ...)

general remark (Fritelli):

ρ = <u>convex</u> functional on $\mathcal{X}$

if $\sigma(\mathcal{X}, \mathcal{X}')$ - <u>lower semicontinuous</u>

then

$$\boxed{\rho = \rho^{**}}$$

where

$$\rho^*(L) := \sup_{X \in \mathcal{X}} (L(X) - \rho(X))$$

Fenchel-Legendre transform (conjugate convex function)

i.e.

$$\boxed{\rho(X) = \sup_{L \in \mathcal{X}'} (L(X) - \rho^*(L))}$$

+ <u>monetary</u> $\Rightarrow$ $L \leq 0$, $L(-1) = 1$ if $\rho^*(L) < \infty$

Illustration:

(1) $\mathcal{X}$ = all bounded measurable functions on $(\Omega, \mathcal{F})$

Banach space with $\|\cdot\|$

$$\boxed{\rho(X) = \max_{Q \in \mathcal{M}_{1,f}} \left(E_Q[-X] - \alpha_{min}(Q) \right)}$$

$$\alpha_{min}(Q) := \sup_{X \in \mathcal{A}_\rho} E_Q[-X]$$

$$= \rho^*(-E_Q)$$

$\mathcal{M}_{1,f}$ = finitely additive probability measures on $(\Omega, \mathcal{F})$

Indeed:

$\{\rho \leq c\}$ strongly closed, convex
hence $\sigma(\mathcal{X}, \mathcal{X}^*)$ - closed

coherent:

$$\boxed{\rho(X) = \max_{Q \in \mathcal{Q}_{max}} E_Q[-X]}$$

$$= \{Q \mid \alpha_{min}(Q) = 0\}$$

truly probabilistic representation?

(2) $\mathcal{X} \supseteq C_b(\Omega)$

polish

Ω <u>compact</u>:

$$\rho(X) \underset{\text{Riesz}}{=} \max_{Q \in \mathcal{M}_1(\Omega)} \left(E_Q[-X] - \alpha_{\min}(Q) \right)$$

more generally (via Prohorov):

ρ <u>"tight"</u>: $\exists\, K_1 \subseteq K_2 \subseteq \dots$ compact:

$$\rho(\lambda I_{K_n}) \to \rho(\lambda) \quad \forall \lambda \geq 1$$

$\Rightarrow$

$$\boxed{\rho(X) = \sup_{Q \in \mathcal{M}_1(\Omega)} \left(E_Q[-X] - \alpha(Q) \right)}$$

---

does not necessarily extend to bounded measurable functions

(correction of F.-Schied)

with reference measure

P on $(\Omega, \mathfrak{F})$:

③ $\mathcal{X} = L^\infty(\Omega, \mathfrak{F}, P)$

ρ convex risk measure on $\mathcal{X}$

($i.e.$ $X = Y$ P-a.s. $\Rightarrow \rho(X) = \rho(Y)$)

F. Delbaen (F.-Schied, Frittelli)

ρ lower-semicontinuous in $\sigma(L^\infty, L^1)$

$\Longleftrightarrow$ $\mathcal{A}_\rho$ $\sigma(L^\infty, L^1)$-closed

$\Longleftrightarrow$ "Fatou property":

$X_n \to X$ a.s., $\sup \|X_n\|_\infty < \infty$

$\Rightarrow \underline{\lim}_n \rho(X_n) \geq \rho(X)$

Fenchel-
$\Longleftrightarrow$
Legendre

$$\boxed{\rho(X) = \sup_{Q \ll P} \left(E_Q[-X] - \alpha_{min}(Q) \right)}$$

coherent:

$\alpha_{min}(Q) \in \{0, \infty\}$

$\Longrightarrow$

$$\rho(X) = \sup_{Q \in \mathcal{Q}(\rho)} E_Q[-X]$$

Examples

① Financial market model

$$\mathcal{A} := \{ X \in L^\infty \mid X + \int \eta \, dY \geq 0 \ P\text{-a.s.}$$

for some admissible trading strategy $\eta \in \mathcal{S}$

$\mathcal{S}$ convex

$\Rightarrow$

$$\rho \text{ convex}, \quad \alpha_{min}(Q) = \sup_{\eta \in \mathcal{S}} E_Q[\int \eta \, dY]$$

② accept some risk:

$\rho_0, \mathcal{A}_0$

$$\tilde{\mathcal{A}} := \{ X \in L^\infty \mid X + \int \eta \, dY \geq A_0 \}$$

for some $\eta \in \mathcal{S}$, some $A_0 \in \mathcal{A}_0$

$\Rightarrow$ $\tilde{\rho}$ convex,

$$\tilde{\alpha}_{min}(Q) = \alpha_{min}(Q) + \alpha_0(Q)$$

(3) $\mathcal{A} = \{ X \mid \underbrace{E_P[\ell(-X)]}_{\text{"shortfall risk"}} \le x_0 \}$

convex
not a cone

$\Rightarrow$ ρ <u>convex risk measure</u>

$$\alpha(Q) = \frac{1}{\lambda}\left(x_0 + E\left[\ell^*\left(\lambda \frac{dQ}{dP}\right)\right]\right)$$

e.g. for $\ell(x) = e^{\alpha x}$:

$$\alpha(Q) = \frac{1}{\alpha}\left(\underbrace{H(Q \mid P)}_{\text{relative entropy}} + \log x_0 \right)$$

$$= \begin{cases} \int \log \frac{dQ}{dP}\, dQ & Q \ll P \\ +\infty & \text{else} \end{cases}$$

"<u>entropic" risk measure</u>:

$$\rho(X) = \frac{1}{\alpha} \log E\left[e^{-\alpha X}\right]$$

(normalized to $\rho(0) = 0$)

(4) $\mathcal{Q} = \{ Q \ll P \mid \frac{dQ}{dP} \leq \frac{1}{\lambda} \}$

$\Rightarrow$

$$\rho(X) = \frac{1}{\lambda} \int_0^\lambda VaR_\alpha(X)\, d\alpha$$

$=$ "average value at risk"
(tail value at risk, ...)

$$=: AVaR_\lambda(X)$$ coherent

Theorem (coherent case: Kusuoka; convex case: F.-Kunze, Frittelli, Dana)

ρ convex risk measure:

distribution invariant
($law(X) = law(Y) \Rightarrow \rho(X) = \rho(Y)$)

$$\Longleftrightarrow \quad \rho(X) = \sup_{\mu \in \mathcal{M}_1[0,1]} \left(\int_0^1 AVaR_\lambda(X)\, \mu(d\lambda) - \alpha(\mu) \right)$$

Proof:

$$E_Q[-X] - \alpha_{min}(Q)$$

$$\underset{\varphi := \frac{dQ}{dP}}{=} E[(-X)\varphi] - \alpha_{min}(Q)$$

$$\underset{\substack{\text{Hardy-Littlewood} \\ \text{(monotone} \\ \text{rearrangements)}}}{\leq} \int_0^1 q_{-X}(t)\, q_\varphi(t)\, dt - \alpha_{min}(Q)$$

$$\underset{\text{atomless}}{=} \sup_{\hat{X} \sim X} E[-\hat{X}\varphi] - \alpha_{min}(Q)$$

$$\leq \sup_{\hat{X} \sim X} \rho(\hat{X}) \underset{\text{law-invariant}}{=} \rho(X)$$

$\Longrightarrow$ "sandwich"

$$\rho(X) = \sup_{Q \ll P} \left[\int_0^1 q_{-X}(t)\, q_\varphi(t)\, dt - \alpha_{min}(Q) \right]$$

$$q_{-X}(t) = \mathrm{VaR}_{1-t}(X) \qquad \text{a.a. } t$$

$\Longrightarrow$

$$\int_0^1 q_{-X}(t)\, q_\varphi(t)\, dt = \int_0^1 \mathrm{VaR}_t(X)\, \underbrace{q_\varphi^+(1-t)}_{=:\,\nu((t,1])}\, dt$$

$$d\mu := s\,\nu(ds) \qquad \nu((t,1]) = \int_{(t,1]} \frac{1}{s}\,\mu(ds)$$

$$\underset{\text{Fubini}}{=} \int_{(0,1]} \underbrace{\frac{1}{s}\int_0^s \mathrm{VaR}_t(X)\, dt}_{=:\,\mathrm{AVaR}_s(X)}\ \mu(ds)$$

<u>Corollary:</u> $X \succsim Y$ $\forall$ v. Neumann-Morgenstern agents ("second order stochastic dominance")

$\Longleftrightarrow$ $\rho(X) \le \rho(Y)$ $\forall$ ρ convex representation in terms of AVaR:

$$\Longleftrightarrow \quad \rho(X) = \sup_{f \in \mathcal{D}} \left(E_{f(P)}[-X] - \alpha(f) \right)$$

more or less pessimistic distortion of P $\left\{ \begin{array}{l} \text{Choquet integral for capacity} \\ \gamma := f(P) \end{array} \right.$

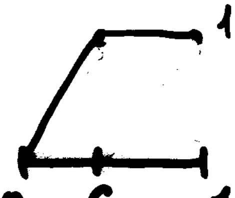
$\mathcal{D}$ = all "concave distortions"

$f : [0,1] \to [0,1]$, $\nearrow$ concave

$f(0) = 0$, $f(1) = 1$

(cf. "Behavioral finance" Yaari, ...)

$$\mu \;\hat{=}\; f'_+(t) = \int_{(t,1]} s^{-1} \mu(ds) = \nu((t,1])$$

$\delta_c \;\hat{=}$ 

$$= \int_0^1 q_Y(\alpha) \underbrace{f'(1-\alpha)}_{\nearrow} d\alpha$$

$$E_f[Y] := \int_0^\infty f(P[Y > x])\, dx$$

~ analogy / connection to
<u>microeconomics</u> beyond
"homo economicus"
à la von Neumann-Morgenstern:

<u>preferences $\succsim$ on $\mathfrak{X}$:</u>

1 <u>von Neumann-Morgenstern:</u>

X: identified with distribution
μ_X of X under P

$\succsim$ on "lotteries" $\xRightarrow{\text{Axioms for } \succsim}$
represented by utility functional

$$U(\mu_X) = \int u(x)\,\mu_X(dx) = E_P[u(X)]$$

"expected <u>utility</u>"

② <u>Savage</u> (Aumann-Anscombe, ...)

no a priori model P

<u>Axioms</u> for $\succsim$ on $\mathcal{X}$ $\Longleftrightarrow$

$\exists$ <u>implicit</u> P, u:

„preferences are represented by "expected utility"

$$E_P[u(X)]$$

- "Paradoxa" (Allais, Ellsberg, ... Kahneman - Tversky, ...)
- Nobel prizes in Economics 2002

"Behavioral Finance"

③ <u>Gilboa-Schmeidler</u>,

Bewley ("Knightian uncertainty"), ..

Shannon, Dana, Châteauneuf, ...
Yaari

$$\inf_{P \in \mathcal{P}} E_P[u(X)]$$

"robust" preferences

<u>robust</u>

<u>utility maximization</u> :

maximize

$$U(H) := \inf_{Q \in \mathcal{Q}} E_Q[u(H)]$$

over

$$\mathcal{H}(x) := \{ H = x + \int_0^T \xi \, dX \mid \xi \text{ admissible} \}$$

$$= \{ H \mid \sup_{P^* \in \mathcal{P}^*} E^*[H] \leq x \}$$

– for a given cost constraint x

(see also: Barrieu – El Karoui, ...)

<u>complete case</u>: $\mathcal{P}^* = \{P^*\}$

a) standard situation: $\mathcal{Q} = \{Q\}$
solution is well known:

$$H = (u')^{-1}\left(\lambda \frac{dP^*}{dQ}\right)$$

b) <u>robust case</u>: $|\mathcal{Q}| > 1$

solution via

$Q_0 :=$ "least favorable measure" in $\mathcal{Q}$ w.r.t. P^*

$:\Longleftrightarrow$ $\varphi := \frac{dP^*}{dQ_0}$ satisfies

$$Q_0[\varphi \le c] = \inf_{Q \in \mathcal{Q}} Q[\varphi \le c] \quad \forall c$$

$$\Longleftrightarrow \quad E_{Q_0}[f(\varphi)] = \inf_{Q \in \mathcal{Q}} E_Q[f(\varphi)] \quad \forall f \downarrow$$

Existence ? via

Neyman-Pearson Lemma for capacities (Huber-Strassen 1973)

$$\gamma(A) := \sup_{Q} Q[A]$$

assume:

i) $\gamma(A \cup B) + \gamma(A \cap B) \leq \gamma(A) + \gamma(B)$
2-alternating capacity (Choquet)
(Schmeidler: $\succsim$ satisfies "comonotonic independence")

ii) $\mathcal{Q} = \{Q \mid Q \leq \gamma\}$
saturated

Define

$$\frac{dP^*}{d\gamma}(\omega) := \inf\{t \mid \omega \in A_t\}$$

A_t maximizes $t\gamma(A) - P^*[A]$

$\Longrightarrow$

$$dQ_0 = \left(\frac{dP^*}{d\gamma}\right)^{-1} dP^*$$

is least favorable

Application to robust optimization:

A. Schied

"On the Neyman-Pearson problem for law-invariant risk measures and robust utility functionals"

see

www.math.tu-berlin.de/~schied

A scetch:

<u>Theorem (Schied)</u>: Q_0 <u>least favorable</u>

$\Rightarrow$ robust utility maximization problem is solved by

$$H_0 = (u')^{-1}\left(\lambda \frac{dP^*}{dQ_0}\right)$$

Proof:

$$\inf_{Q \in \mathcal{Q}} E_Q[u(H)] \leq E_{Q_0}[u(H)]$$

$$\underset{\text{classical}}{\leq} E_{Q_0}[u(H_0)]$$

$$= E_{Q_0}\left[f\left(\frac{dP^*}{dQ_0}\right)\right]$$

$f := u \circ (u')^{-1}(c\,\cdot) \searrow$

$$\underset{Q_0 \text{ least favorable}}{=} \inf_{Q \in \mathcal{Q}} E_Q\left[f\left(\frac{dP^*}{dQ_0}\right)\right]$$

$$= \inf_{Q \in \mathcal{Q}} E_Q[H_0]$$

incomplete case: Anne Gundel (2004)

$|\mathcal{P}^*| > 1$

a) standard case: $\mathcal{Q} = \{Q\}$

Kramkov - Schachermayer (1999)
Bellini - Fritelli (1999)
Goll - Rüschendorf (2001)
Th. Zariphopoulou, Hobson, ...

f convex:

"f-divergence": $f(P^*|Q) = E_Q\left[f\left(\frac{dP^*}{dQ}\right)\right]$

($f(x) = x\log x$: relative entropy)

"f-projection" of Q on $\mathcal{P}^*$:

$$f(P_0^*|Q) = \inf_{\mathcal{P}^*} f(P^*|Q) =: f(\mathcal{P}^*|Q)$$

(e.g.: minimal entropy measure)

Define

$$U_{P^*}(x) := \sup_{E^*[H] \le x} E_Q[u(H)]$$

$$v(y) := \sup\,(u(x) - xy), \quad v_\lambda(y) = v(\lambda y)$$

b) <u>robust case:</u> $|\mathcal{Q}|>1$

Q_0 := "<u>reverse f-projection</u>" of P^* on $\mathcal{Q}$

$\Leftrightarrow$ Q_0 minimizes $f(P^*|Q)$ over $\mathcal{Q}$

i.e.

$$f(P^*|Q_0) = f(P^*|\mathcal{Q}) := \inf_{Q\in\mathcal{Q}} f(P^*|Q)$$

Lemma: $f(P^*|Q) = \hat{f}(Q|P^*)$

$= \hat{f}$ - projection of P^* on $\mathcal{Q}$

$$\hat{f}(x) := x f\left(\tfrac{1}{x}\right)$$

P^* <u>robust minimax measure</u>

$$:\Leftrightarrow \quad U_{P^*}(x) = \inf_{Q\in\mathcal{Q}} U^Q_{a}(x)$$

$$= \inf_{P^*\in\mathcal{P}^*} \sup_{H\in\mathcal{H}_{P^*}(x)} \inf_{Q\in\mathcal{Q}} E_Q[U(X)]$$

$$=: U(x)$$

Theorem (e.g. Goll–Rüschendorf)

P_0^* **minimax martingale measure**

i.e.

$$U_{P_0^*}(x) = \inf_{P^* \in \mathcal{P}^*} U_{P^*}(x) =: U^Q(x)$$

(most unfavorable pricing system)

$$\iff \quad P_0^* = \nu_\lambda(\mathcal{P}^* \mid Q)$$

$= \nu_\lambda$-projection of Q on $\mathcal{P}^*$

In this case:

$$U_{P_0^*}(x) = \sup\left\{ E_Q[u(H)] \;\middle|\; \sup_{P^* \in \mathcal{P}} E^*[H] \le x \right\}$$

solution:

$$H_0 = (u')^{-1}\left(\lambda \frac{dP_0^*}{dQ}\right)$$

= classical solution for pair (P_0^*, Q)

Theorem (A. Gundel)

P^* robust minimax

$\Longleftrightarrow$ P^* is a robust $v_{\lambda(\bar{x})}$ projection for some $\lambda \in \partial U(x)$

i.e. for $f = v_{\lambda(x)}$:

$$f(P^* | Q) = \inf_{Q} f(P^* | Q) =: f(Q^* | Q)$$

In that case:

$$H_0 = (u')^{-1}\left(\lambda \frac{dP^*}{dQ^*}\right)$$

Q^* = reverse v_λ - projection of P^*

Case studies: in progress

so far: risk measures for one period!

Back to path space:

Dynamics of risk measures ?

(position / information)

Artzner, Delbaen, Eber, Heath, Ku
Cheridito, Delbaen, Kupper

here:
- fixed position X
- increasing information $\mathcal{F}_t$ $(0 \leq t \leq T)$

$\forall t \in [0, T]$:

$$\rho_t(X) = \underset{Q \ll P}{\text{ess.sup}} \left(E_Q[-X \mid \mathcal{F}_t] - \alpha_t(Q) \right)$$

(or rather in terms of stochastic kernels $K_t(\omega, \cdot)$: Detlefsen (2003))

+ <u>time consistency</u>: e.g.

$$\rho_t(X) = \rho_t(-\rho_{t+h}(X))$$

(cf. Peng, Hémin, .., El Karoui, ...) "filtration-consistent"

<u>dynamics of $(\alpha_t(Q))$?</u>

(in analogy to "pasting conditions"

Theorem (H.F., Irina Penner):

$\forall Q$:

$$\rho_t(X) + \alpha_t(Q) \qquad (0 \le t \le T)$$

<u>supermartingale</u> under Q

martingale $\iff Q = Q_X :=$ <u>"worst case model" for X</u>

(cf. Bellman principle
coherent case: ADEHK)

Illustration: <u>"entropic" case</u>

$\alpha_t(Q) =$ <u>remaining relative entropy</u>

$$\hat{H}_t(Q|P) = E_Q\left[\log \frac{dQ[\cdot|\mathcal{F}_t]}{dP[\cdot|\mathcal{F}_t]} \,\middle|\, \mathcal{F}_t\right]$$

$$\varphi := \frac{dQ}{dP}\Big|_{\mathcal{F}_T}$$

$$= \varphi_t \, \hat{\varphi}_t \,, \qquad \varphi_t = \frac{dP}{dQ}\Big|_{\mathcal{F}_t}$$

$$\hat{\varphi}_t := \frac{\varphi}{\varphi_t}$$

$\Longrightarrow$

$$\hat{H}_t(Q|P) = E_Q[\log \hat{\varphi}_t \,|\, \mathcal{F}_t]$$

$$= E_Q[\log \varphi \,|\, \mathcal{F}_t] + \underbrace{\log \varphi_t^{-1}}_{\substack{Q\text{-super-}\\\text{martingale}\\(\text{martingale if } Q \approx P)}}$$

<u>Q-supermartingale</u>

$$\rho_t(X) + \hat{H}_t(Q|P) = \cancel{\log E[e^{-X}|\mathcal{F}_t]}$$

$$+ \Big(\underbrace{\hat{H}_t(Q|Q_X)}_{\geq 0,\ =0 \iff Q = Q_X = \frac{e^{-X}}{E[e^{-X}]}dP} - E_Q[X|\mathcal{F}_t] - \cancel{\log E[e^{-X}|\mathcal{F}_t]}\Big)$$

"worst case

on a <u>Brownian filtration</u>:

$$\rho_t(X) = \log \underbrace{E[e^{-X} \mid \mathcal{F}_t]}_{E[e^{-X}] + \int_0^t H_s \, dB_s}$$

$$\underset{\text{Itô}}{=} \rho_0(X) + \int_0^t Z_s \, dB_s - \int_0^t g(Z_s) \, ds$$

$g(z) := \frac{1}{2} z^2$ convex

i.e. <u>**BSDE**</u>

$$d\rho_t = Z \, dB - g(Z) \, dt$$
$$\rho_T = -X$$

N. El Karoui, S. Peng, ...

~ <u>Th. Zariphopoulou</u> (Beyond the case $|g(z)| \leq \mu \cdot |z|$)

\+ intertemporal preferences

optimization,

equilibrium, ----