Volatility Time Scales and Perturbations

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References:

Derivatives in Financial Markets with Stochastic Volatility

Cambridge University Press, 2000

Short time-scale in S&P 500 volatility

Journal of Computational Finance, Summer 2003

Multiscale Stochastic Volatility Asymptotics

SIAM Journal on Multiscale Modeling and Simulation, 2(1), 2003

Timing the Smile

Wilmott Magazine, March 2004

Volatility Perturbations in Financial Markets

In preparation: Cambridge University Press, 2004

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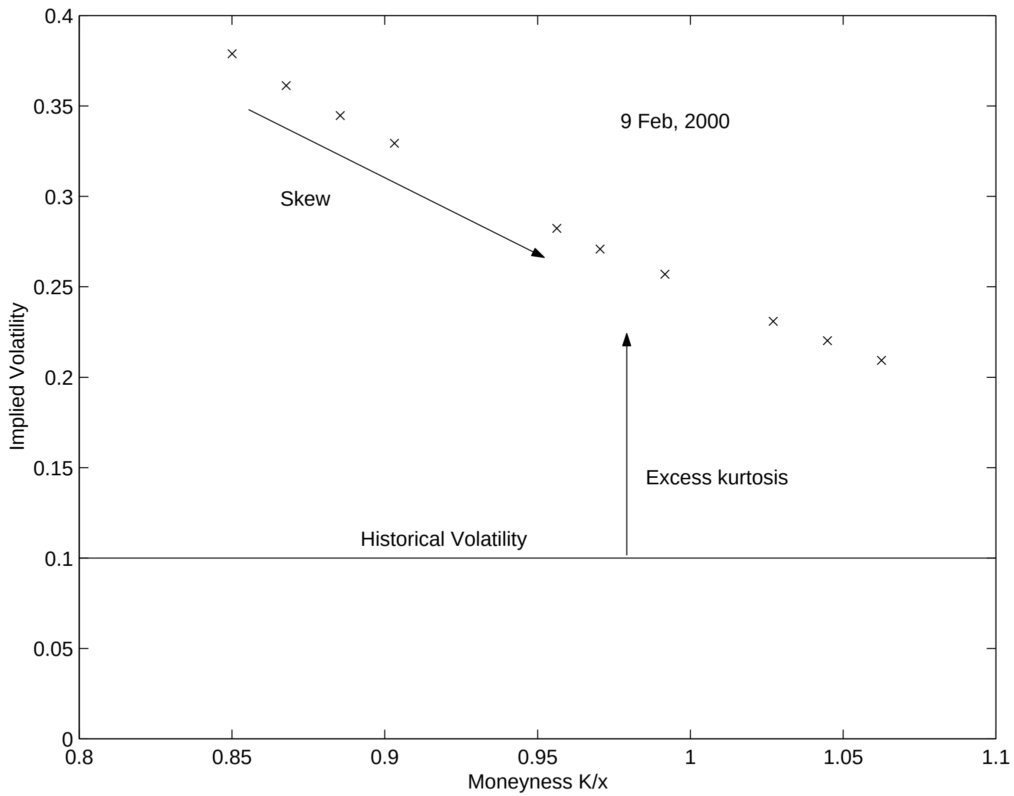


Figure 1: *S&P 500 Implied Volatility Curve as a function of moneyness from S&P 500 index options on February 9, 2000. The current index value is* $x = 1411.71$ *and the options have over two months to maturity. This is typically described as a downward sloping* **skew**.

“Parametrization” of the Implied Volatility Surface $I(t\,;T,K)$

REQUIRED QUALITIES

- **Universal Parsimonous Parameters**: *Model Independence*
- **Stability in Time**: *Predictive Power*
- **Easy Calibration**: *Practical Implementation*
- **Compatibility with Price Dynamics**: *Applicability to Pricing other Derivatives and Hedging*

At least three approaches:

- **Local Volatility Models**: $\sigma_t = \sigma(t, S_t)$

  +'s: market is complete (no additional randomness), Dupire formula

  -'s: stability of calibration

- **Implied Volatility Surface Models**: $dI_t(T, K) = \cdots$

  +'s: predictive power

  -'s: no-arbitrage conditions not easy. Which underlying?

- **Stochastic Volatility Models**: $\sigma_t = f(Y_t)$

Stochastic Volatility Framework

WHY?

- Distributions of returns are **not log-normal**
- Smile (Skew) effect observed in implied volatilities

HOW?

$$dS_t = \mu S_t dt + \sigma_t S_t dW_t$$

with, for instance:

$$\sigma_t = f(Y_t)$$

$$dY_t = \alpha(m - Y_t)dt + \nu\sqrt{2\alpha}\, dW_t^{(1)}$$

$$d\langle W, W^{(1)} \rangle_t = \rho dt$$

Normalized Fluctuations

Normalized fluctuations process:

$$\bar{D}_n = \sigma_n \epsilon_n$$

Volatility process: σ_n

Continuous analog:

$$\frac{1}{\sqrt{\Delta t}}\left(\frac{\Delta S_t}{S_t} - \mu \Delta t\right) = \sigma_t \frac{\Delta W_t}{\sqrt{\Delta t}}$$

White noise process: i.i.d. $\mathcal{N}(0,1)$ random variables ϵ_n

Log absolute value normalized fluctuations

$$X_n = \log|\bar{D}_n| = \log \sigma_n + \log|\epsilon_n|$$

log-volatility + white noise

Variogram Analysis

Empirical Structure Function or Variogram of X_n:

$$V_j^N = \frac{1}{N} \sum_n (X_{n+j} - X_n)^2$$

where j is the total **lag** and N is the total number of points.

Model variogram:

$$V_j^N \approx 2\gamma^2 + 2\nu_f^2 (1 - e^{-\alpha j \Delta t})$$

Three parameters:

- Vertical intercept: $2\gamma^2 = 2\text{var}(\log |\epsilon_n|)$
- Long-run log-volatility variance: ν_f^2
- **Characteristic volatility time-scale:** $1/\alpha$

Exact formula if: $\sigma_t = \exp(Y_t)$ and $dY_t = \alpha(m - Y_t)\,dt + \nu\sqrt{2\alpha}\,dZ_t$

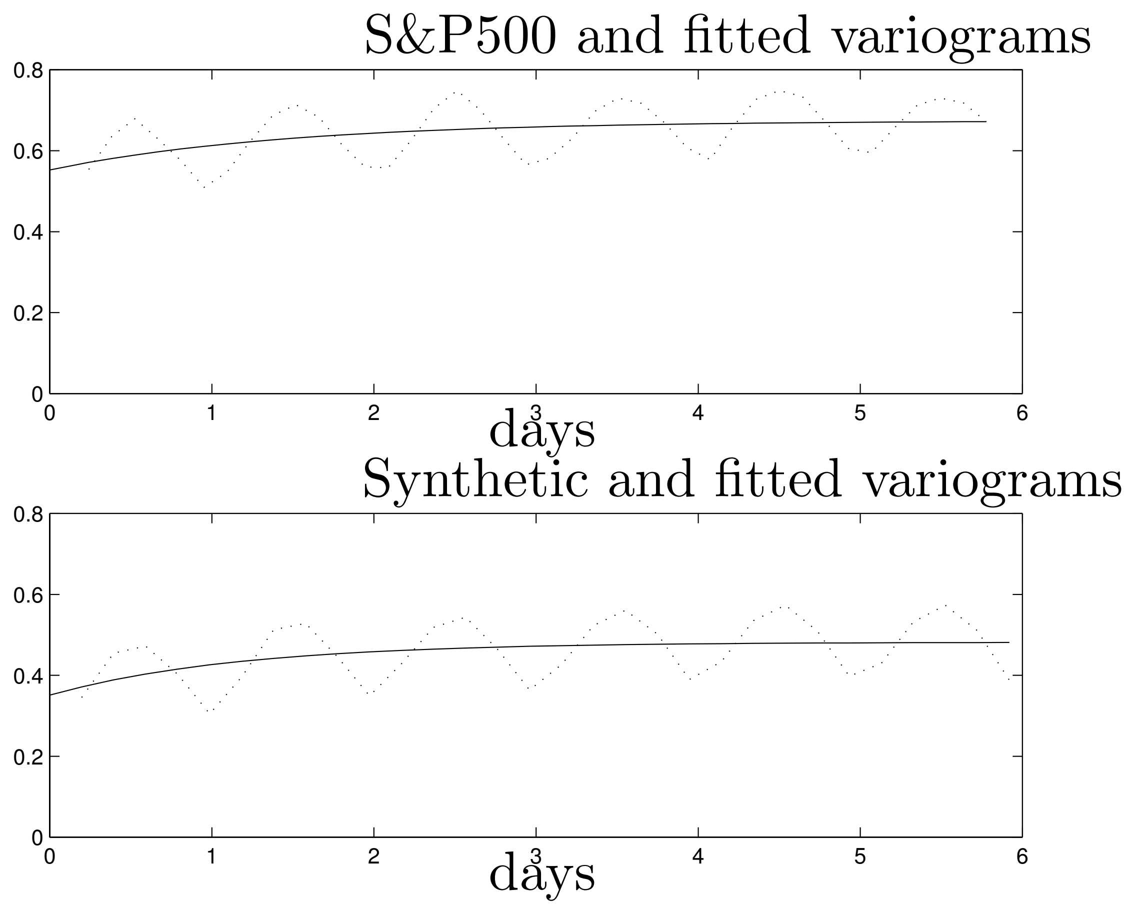


Figure 2: *The empirical variogram for the 1994 S&P* 500 *and synthetic variogram.*

Implications

Day effect:

$$\bar{D}_n = f(Y_n) g_n \epsilon_n$$

where g_n a deterministic periodic function that models the systematic intraday variations in the volatility. It can be easily estimated from the S&P 500 data.

Presence of a short time-scale:

$$1/\alpha \approx 1.5 \text{ days}$$

obtained by fitting $2\gamma^2 + 2\nu_f^2(1 - e^{-\alpha j \Delta t})$.

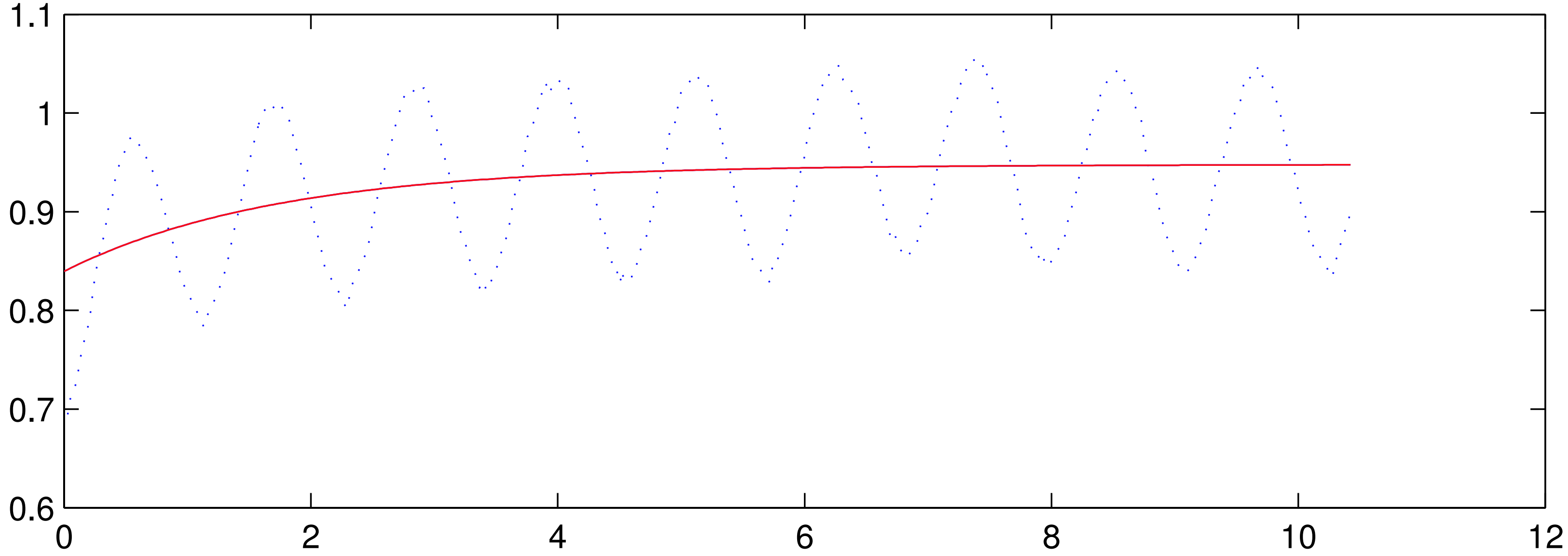


Figure 3: *The empirical variogram for the S&P* 500 , dotted line, for the 1999-2000 data. The solid line is the exponential fit from which the rate of mean-reversion can be obtained.

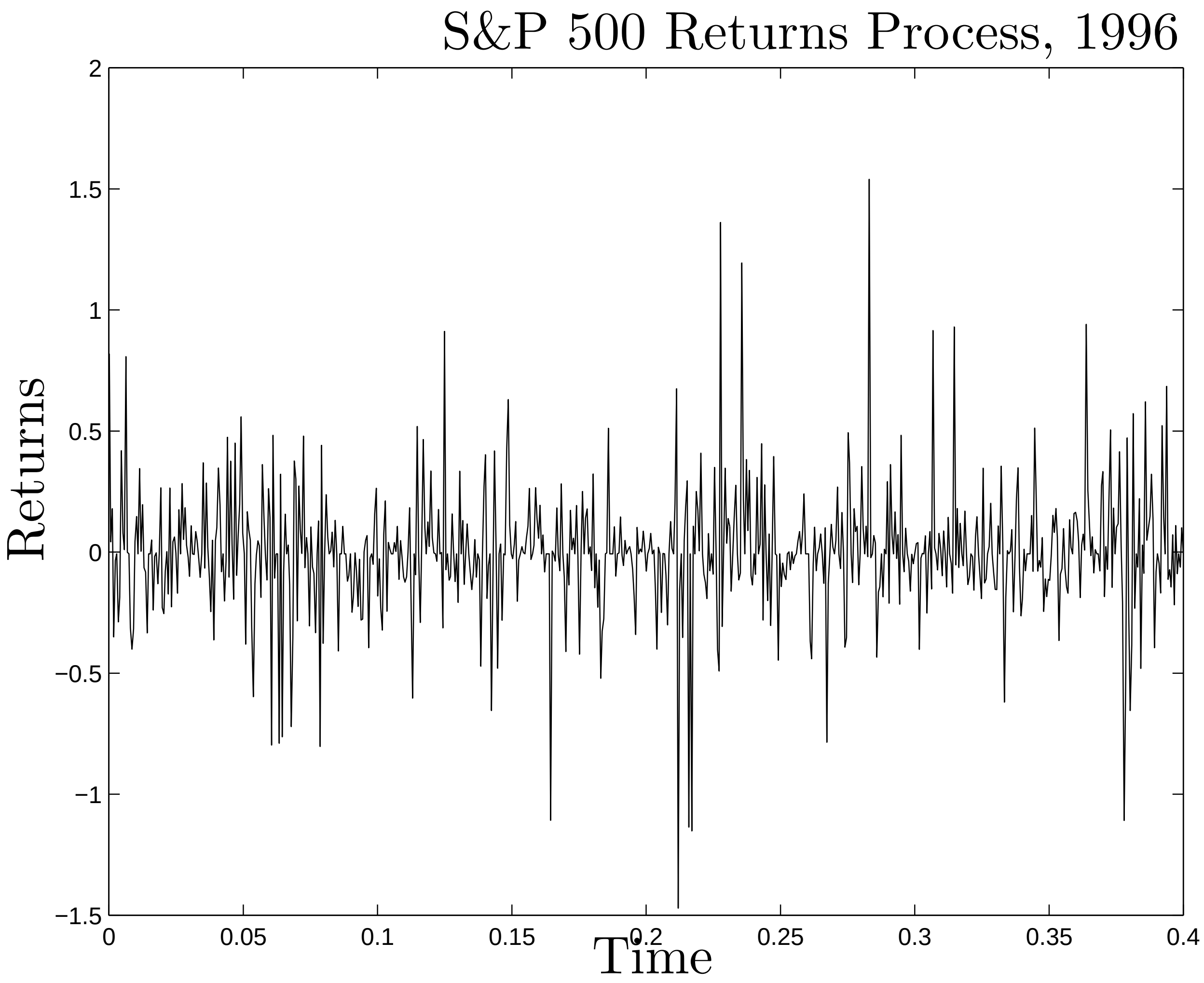


Figure 4: *1996 S&P* 500 *returns computed from half-hourly data*.

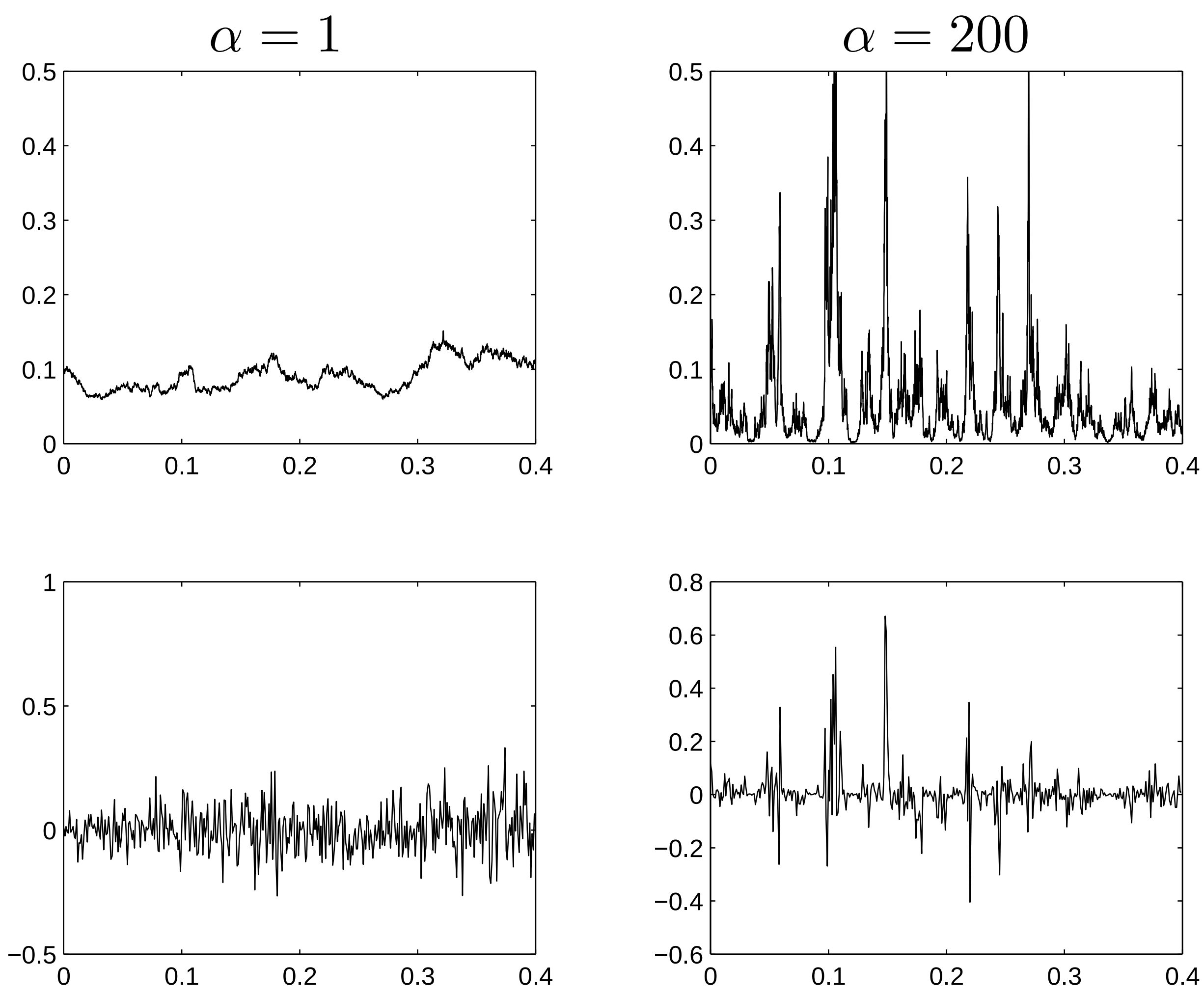


Figure 5: *Simulated volatility and corresponding returns paths for small and large rates of mean-reversion for the OU model with* $f(y) = e^y$.

Effect of Longer Scales

$$\sigma_t = f(Y_t, Z_t)$$

where (Y_t) and (Z_t) are two OU processes, and (Z_t) has a longer mean-reversion time $1/\alpha_L$ of the order of months.
The variogram becomes

$$V_j^N \approx 2\gamma^2 + 2\nu^2(1 - e^{-\alpha j \Delta t}) + 2\nu_L^2(1 - e^{-\alpha_L j \Delta t})$$

For lags such that $j\Delta t$ is up to a week, $\alpha_L j \Delta t$ is small and the last term is negligible.

For longer lags (and longer data), the first term is constant $(2\nu^2)$.

Two-Scale Stochastic Volatility Models

$$\varepsilon << T << 1/\delta$$

$$\begin{aligned} dX_t &= rX_t dt + f(Y_t, Z_t) X_t dW_t^{(0)\star} \\ dY_t &= \left(\frac{1}{\varepsilon}(m - Y_t) - \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}} \Lambda(Y_t, Z_t) \right) dt + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}} dW_t^{(1)\star} \\ dZ_t &= \left(\delta \; c(Z_t) - \sqrt{\delta} \; g(Z_t) \Gamma(Y_t, Z_t) \right) dt + \sqrt{\delta} \; g(Z_t) dW_t^{(2)\star} \end{aligned}$$

$$\begin{aligned} d < W^{(0)\star}, W^{(1)\star} >_t &= \rho_1 dt \\ d < W^{(0)\star}, W^{(2)\star} >_t &= \rho_2 dt \end{aligned}$$

Pricing Equation

$$P^{\varepsilon,\delta}(t,x,y,z) \quad = \quad \mathbb{E}^{\star}\left\{e^{-r(T-t)}h(X_T)|X_t = x, Y_t = y, Z_t = z\right\}$$

$$\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2 + \sqrt{\delta}\mathcal{M}_1 + \delta\mathcal{M}_2 + \sqrt{\frac{\delta}{\varepsilon}}\mathcal{M}_3\right)P^{\varepsilon,\delta} = 0$$

$$P^{\varepsilon,\delta}(T,x,y,z) = h(x)$$

$$\mathcal{L}_0 = (m-y)\frac{\partial}{\partial y} + \nu^2\frac{\partial^2}{\partial y^2} \qquad \mathcal{M}_1 = -g\Gamma\frac{\partial}{\partial z} + \rho_2 g f x\frac{\partial^2}{\partial x\partial z}$$

$$\mathcal{L}_1 = \nu\sqrt{2}\left(\rho_1 f x\frac{\partial^2}{\partial x\partial y} - \Lambda\frac{\partial}{\partial y}\right) \qquad \mathcal{M}_2 = c\frac{\partial}{\partial z} + \frac{g^2}{2}\frac{\partial^2}{\partial z^2}$$

$$\mathcal{L}_2 = \frac{\partial}{\partial t} + \frac{1}{2}f^2x^2\frac{\partial^2}{\partial x^2} + r\left(x\frac{\partial}{\partial x} - \cdot\right) \qquad \mathcal{M}_3 = \nu\sqrt{2}\;\tilde{\rho}_{12}g\frac{\partial^2}{\partial y\partial z}$$

Double Expansion

$$P^{\varepsilon,\delta} = P_0^\varepsilon + \sqrt{\delta} P_1^\varepsilon + \delta P_2^\varepsilon + \cdots$$

$$\left(\frac{1}{\varepsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}} \mathcal{L}_1 + \mathcal{L}_2 \right) P_0^\varepsilon = 0$$

$$P_0^\varepsilon = P_0 + \sqrt{\varepsilon} P_{1,0} + \varepsilon P_{2,0} + \cdots$$

Leading order term: $P_0(t,x,z) = P_{BS}(t,x;\bar{\sigma}(z))$

Correction: $\tilde{P}_1 = \sqrt{\varepsilon} P_{1,0}$ with $V_2^\varepsilon, V_3^\varepsilon$ (z-dependent):

$$\mathcal{L}_{BS}(\bar{\sigma}) \tilde{P}_1 + \left(V_2^\varepsilon x^2 \frac{\partial^2 P_{BS}}{\partial x^2} + V_3^\varepsilon x \frac{\partial}{\partial x} \left(x^2 \frac{\partial^2 P_{BS}}{\partial x^2} \right) \right) = 0$$

$$\tilde{P}_1(T,x,z) = 0$$

$$\tilde{P}_1(t,x,z) = (T-t) \left(V_2^\varepsilon x^2 \frac{\partial^2 P_{BS}}{\partial x^2} + V_3^\varepsilon x \frac{\partial}{\partial x} \left(x^2 \frac{\partial^2 P_{BS}}{\partial x^2} \right) \right)$$

$\sqrt{\delta}$ Correction

$$\mathcal{M}_1 P_0^\varepsilon + \left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2\right) P_1^\varepsilon + \sqrt{\frac{\delta}{\varepsilon}}\mathcal{M}_3 P_0^\varepsilon = 0$$

$$P_0^\varepsilon(t,x,y,z) = P_0(t,x,z) + \sqrt{\varepsilon}P_{1,0}(t,x,z) + \mathcal{O}(\varepsilon)$$

$$P_1^\varepsilon = P_{0,1} + \sqrt{\varepsilon}P_{1,1} + \varepsilon P_{2,1} + \cdots \quad \Longrightarrow \quad \langle\mathcal{L}_2\rangle P_{0,1} + \langle\mathcal{M}_1\rangle P_0 = 0$$

$$\tilde{Q}_1(t,x,z) = \sqrt{\delta}P_{0,1} : \;\; \mathcal{L}_{BS}(\bar{\sigma}(z))\tilde{Q}_1 + \sqrt{\delta}\langle\mathcal{M}_1\rangle P_0 = 0$$

$$\mathcal{L}_{BS}(\bar{\sigma})\tilde{Q}_1 + 2\left(V_0^\delta \frac{\partial P_{BS}}{\partial \sigma} + V_1^\delta x \frac{\partial^2 P_{BS}}{\partial x \partial \sigma}\right) = 0$$

$$\tilde{Q}_1(T,x) = 0$$

Price Approximation: order $\varepsilon, \delta, \sqrt{\varepsilon\delta}$

$$
\begin{aligned}
P^{\varepsilon,\delta}(t,x,y,z) \approx P_{BS}(t,x;T,\bar{\sigma}) \quad &+ \quad (T-t)\left(V_2^{\varepsilon} x^2 \frac{\partial^2 P_{BS}}{\partial x^2} + V_3^{\varepsilon} x \frac{\partial}{\partial x}\left(x^2 \frac{\partial^2 P_{BS}}{\partial x^2}\right)\right) \\
&+ \quad (T-t)\left(V_0^{\delta} \frac{\partial P_{BS}}{\partial \sigma} + V_1^{\delta} x \frac{\partial^2 P_{BS}}{\partial x \partial \sigma}\right)
\end{aligned}
$$

$$
\mathcal{L}_{BS}(\bar{\sigma})\tilde{Q}_1 + 2\left(V_0^{\delta} \frac{\partial P_{BS}}{\partial \sigma} + V_1^{\delta} x \frac{\partial^2 P_{BS}}{\partial x \partial \sigma}\right) = 0
$$

$$
\tilde{Q}_1(T,x) = 0
$$

KEY:

$$
\frac{\partial P_{BS}}{\partial \sigma} \quad = \quad (T-t)\sigma x^2 \frac{\partial^2 P_{BS}}{\partial x^2}
$$

Term Structure of Implied Volatility

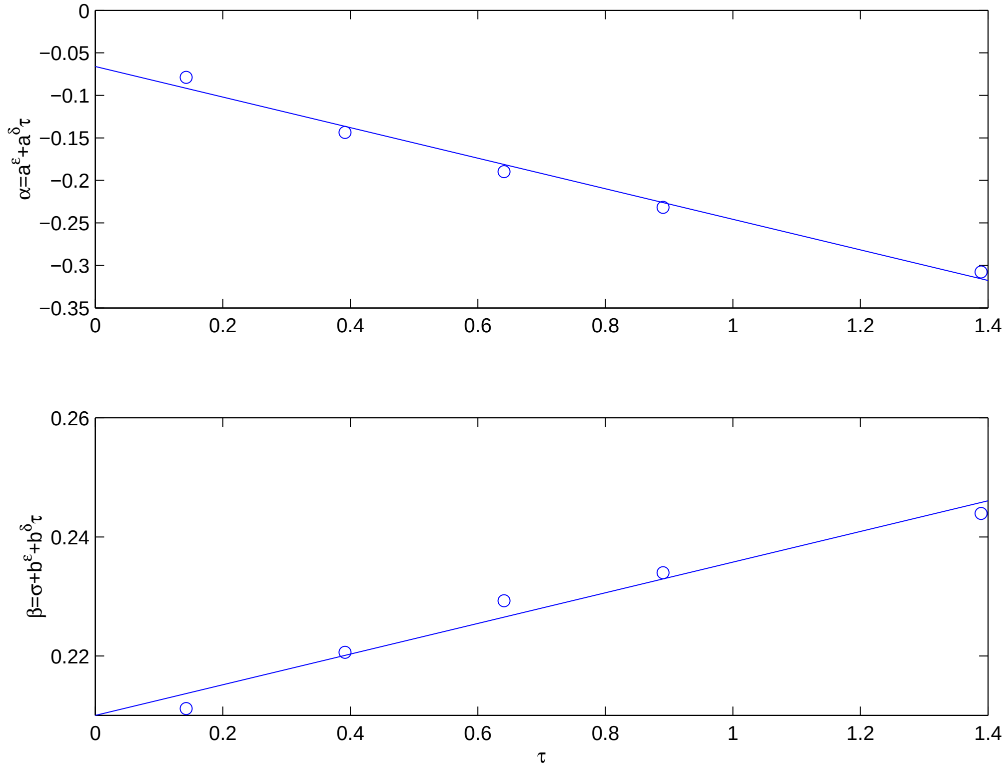
$$I_0 + I_1^\varepsilon + I_1^\delta =$$

$$\bar{\sigma} + \left[b^\varepsilon + b^\delta (T-t)\right] + \left[a^\varepsilon + a^\delta (T-t)\right] \frac{\log(K/x)}{T-t},$$

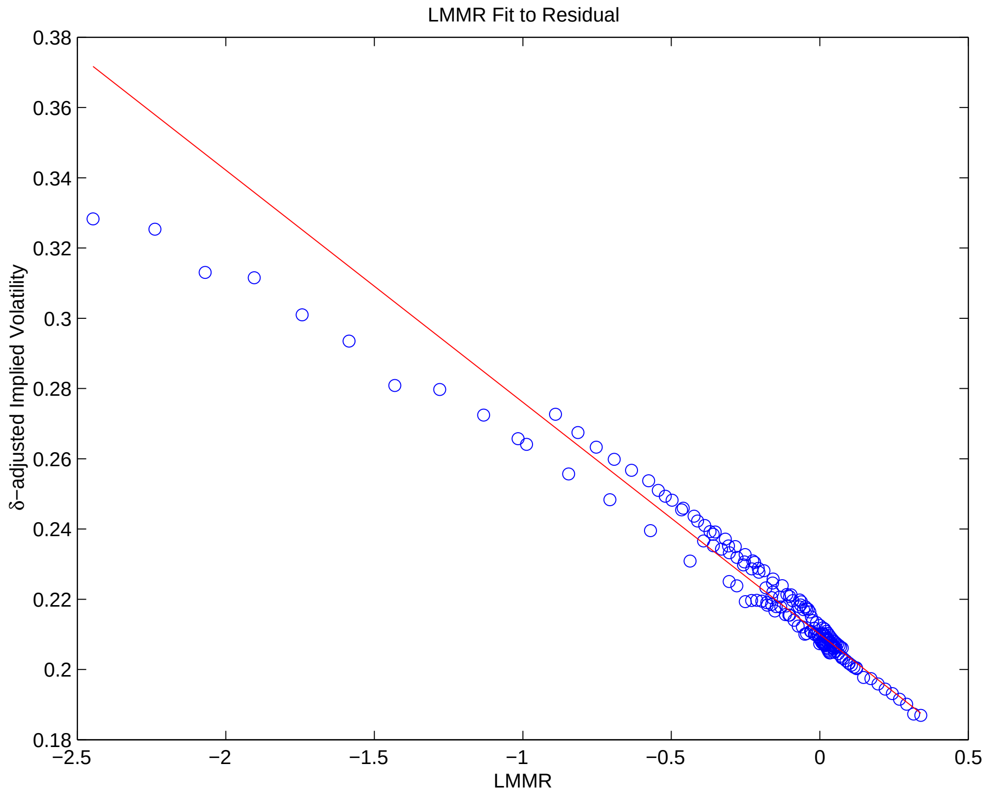
where the parameters $(\bar{\sigma}, a^\varepsilon, a^\delta, b^\varepsilon, b^\delta)$ depend on z and are related to the group parameters $(V_0^\delta, V_1^\delta, V_2^\varepsilon, V_3^\varepsilon)$ by

$$a^\varepsilon = \frac{V_3^\varepsilon}{\bar{\sigma}^3} \quad , \quad b^\varepsilon = \frac{V_2^\varepsilon}{\bar{\sigma}} - \frac{V_3^\varepsilon}{\bar{\sigma}^3}\left(r - \frac{\bar{\sigma}^2}{2}\right)$$

$$a^\delta = \frac{V_1^\delta}{\bar{\sigma}^2} \quad , \quad b^\delta = V_0^\delta - \frac{V_1^\delta}{\bar{\sigma}^2}\left(r - \frac{\bar{\sigma}^2}{2}\right)$$

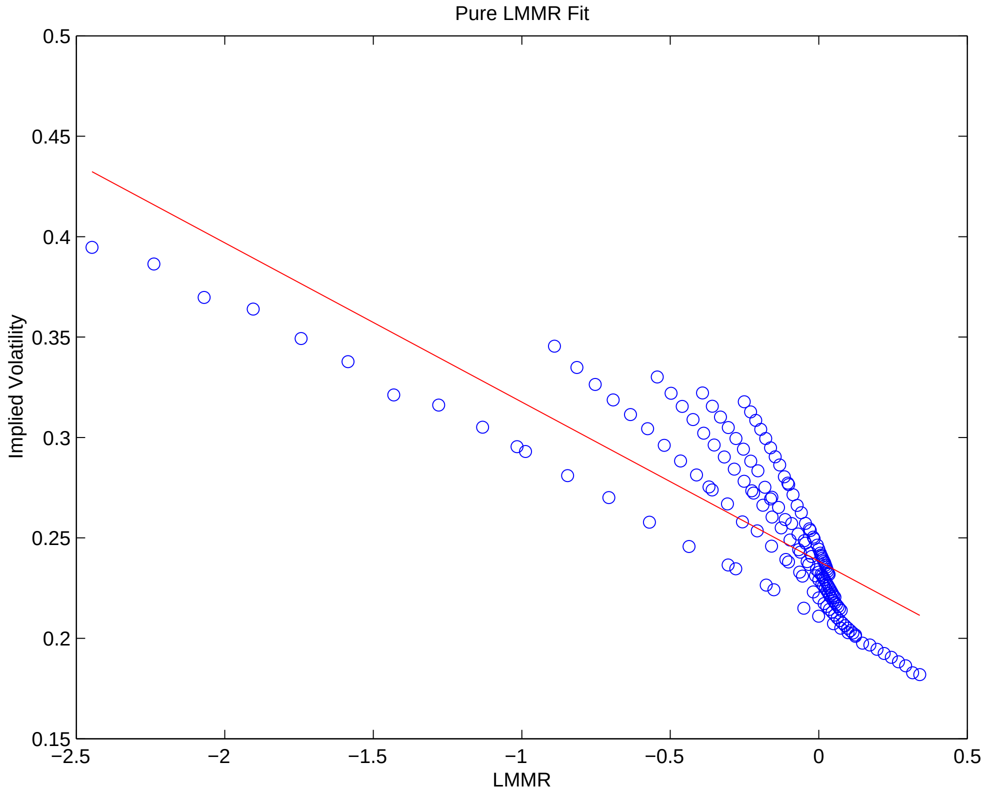


Term-structures fits



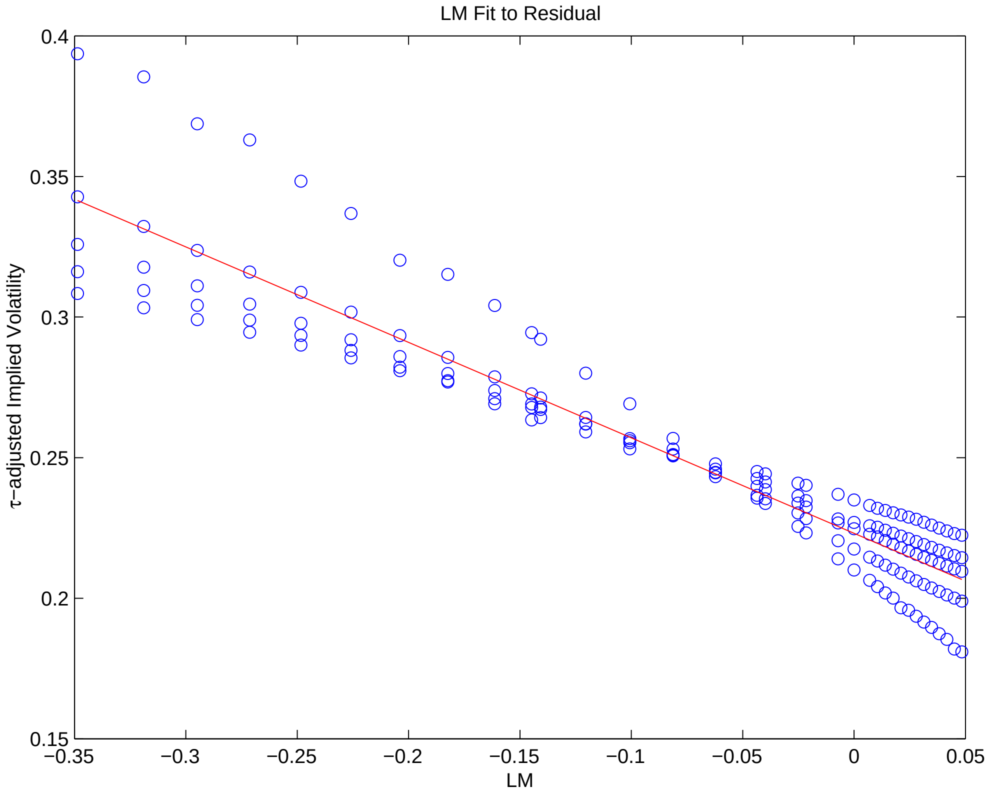
δ-adjusted implied volatility $I - b^{\delta}\tau - a^{\delta}(LM)$ as a function of LMMR. The circles are from S&P 500 data, and the line $R + a^{\varepsilon}(LMMR)$ shows the fit using the estimated parameters.

A slow volatility factor is needed



Implied volatility as a function of LMMR. The circles are from S&P 500 data, and the line $a(LMMR) + b$ shows the fit using maturities up to two years.

A fast volatility factor is needed



The circles are from S&P 500 data, and the line $a^{\delta}(LM) + \bar{\sigma}$ shows the fit using the estimated parameters from only a slow factor fit.

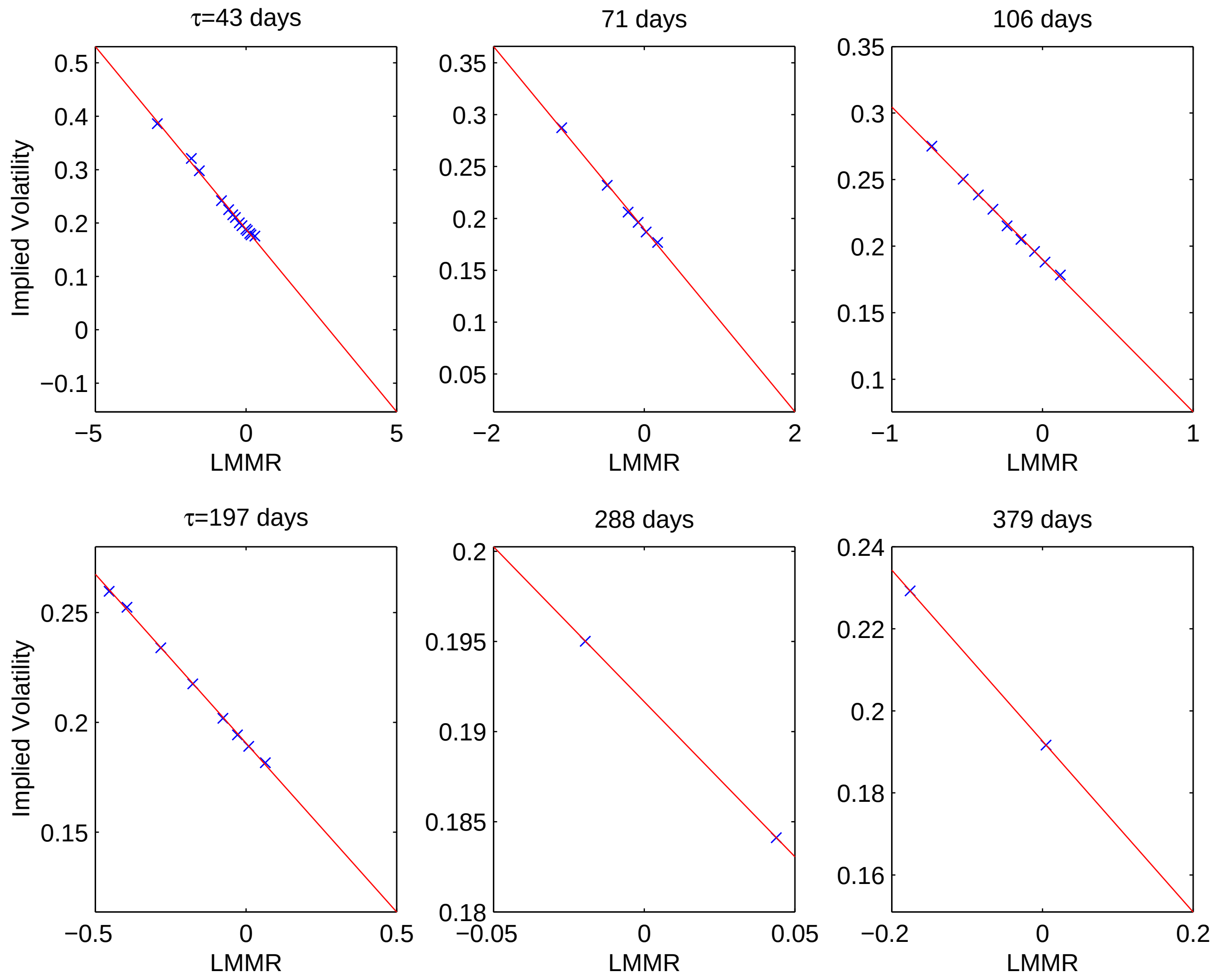


Figure 6: *S&P* 500 *Implied Volatility data on June 5, 2003 and fits to the affine LMMR approximation for six different maturities.*

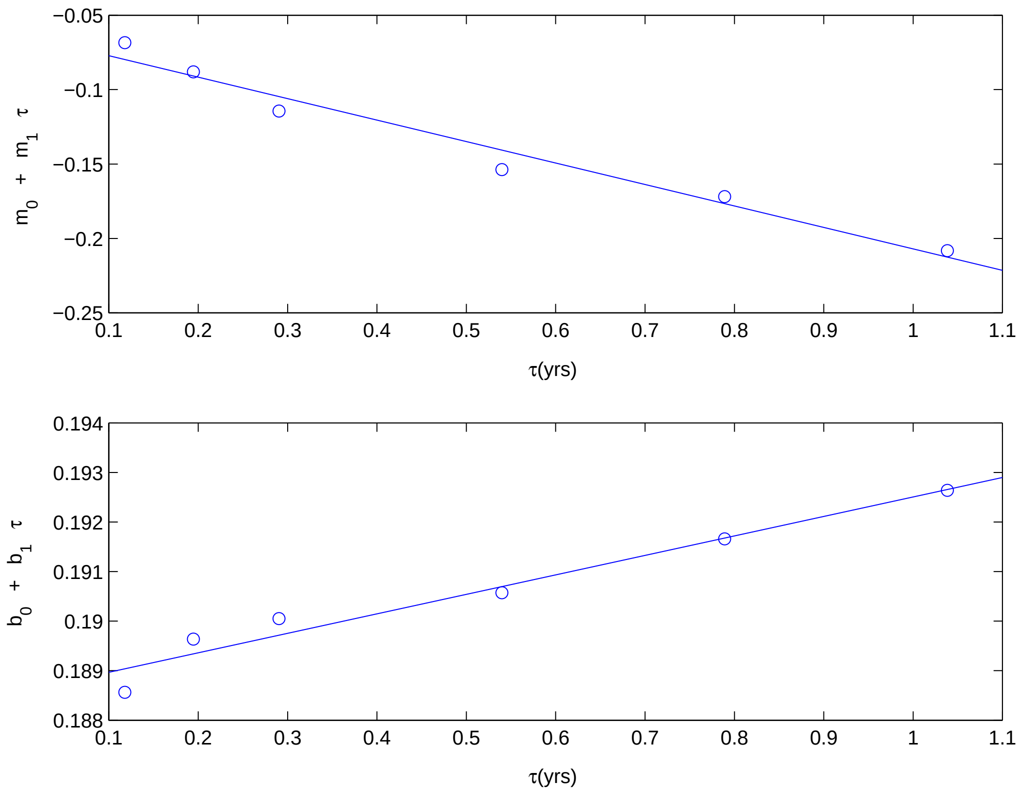


Figure 7: *S&P* 500 *Implied Volatility data on June 5, 2003 and fits to the two-scales asymptotic theory. The bottom (rep. top) figure shows the linear regression of* b *(resp.* a*) with respect to time to maturity* $\tau = T - t$*.*

Higher order terms in ε, δ and $\sqrt{\varepsilon\delta}$

$$I \approx \sum_{j=0}^{4} a_j(\tau)\,(LM)^j + \frac{1}{\tau}\Phi_t,$$

where

τ denotes the time-to maturity $T-t$,

LM denotes the moneyness $\log(K/S)$,

and Φ_t is a rapidly changing component that varies with the fast volatility factor

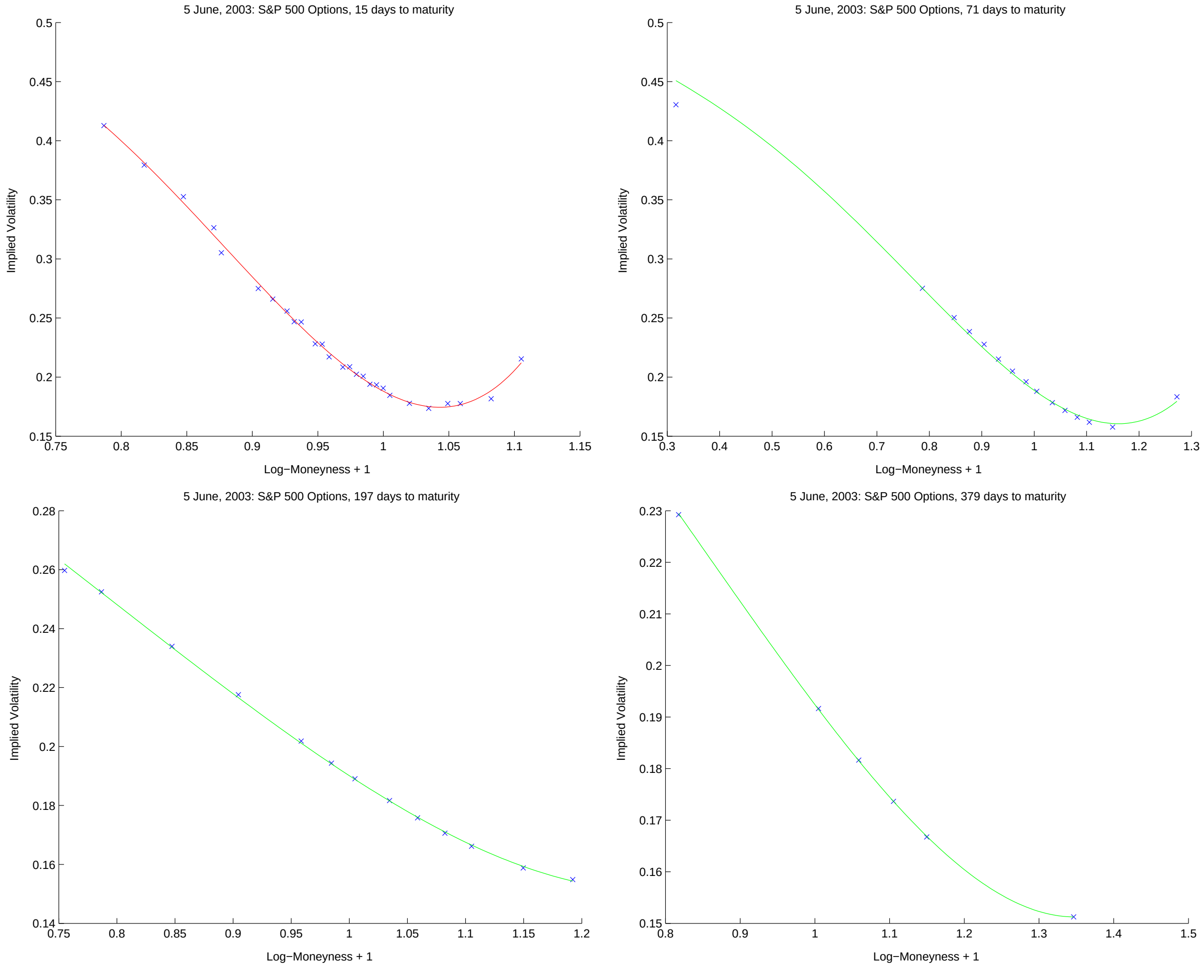


Figure 8: *S&P* 500 *Implied Volatility data on June 5, 2003 and quartic fits to the asymptotic theory for four maturities.*

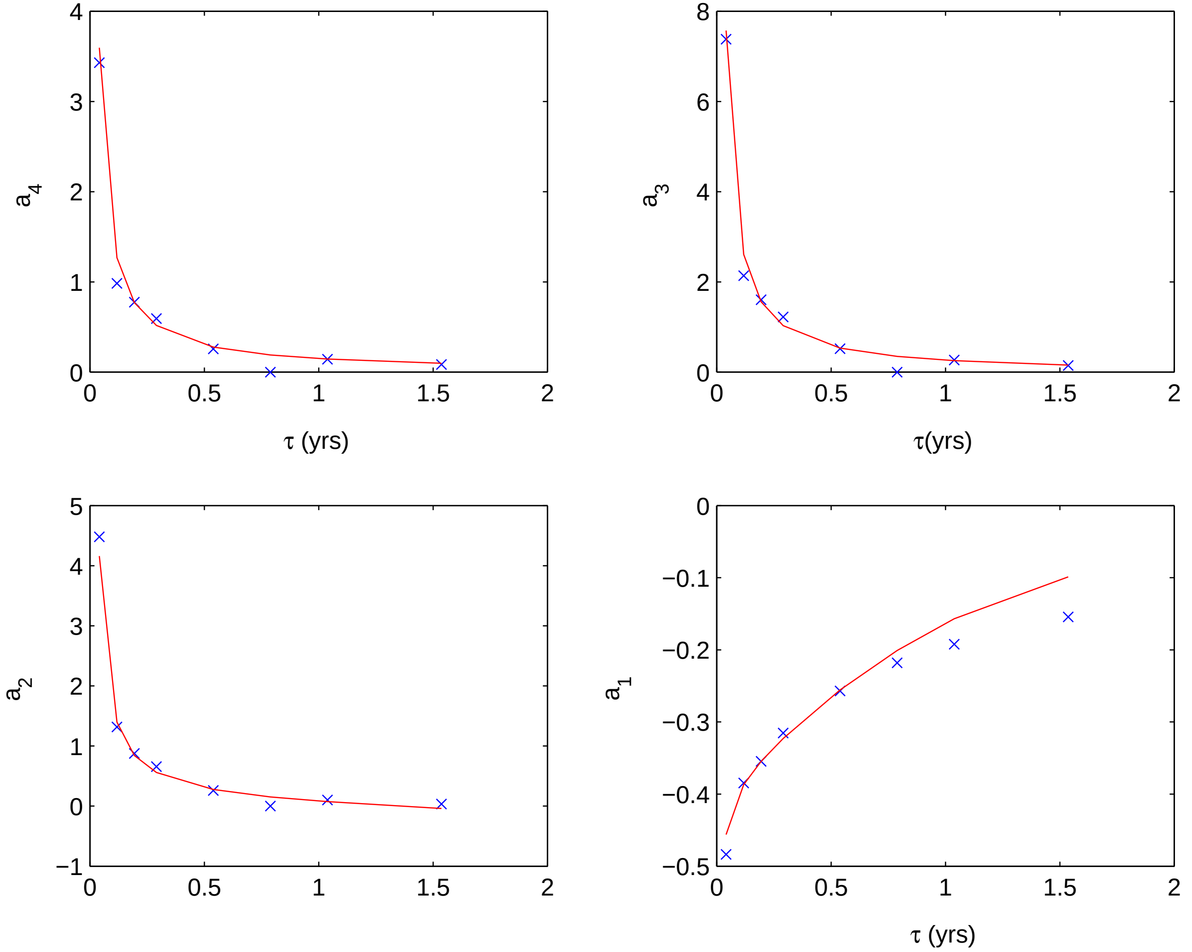


Figure 9: *S&P* 500 *Term-Structure Fit using second order approximation. Data from June 5, 2003.*

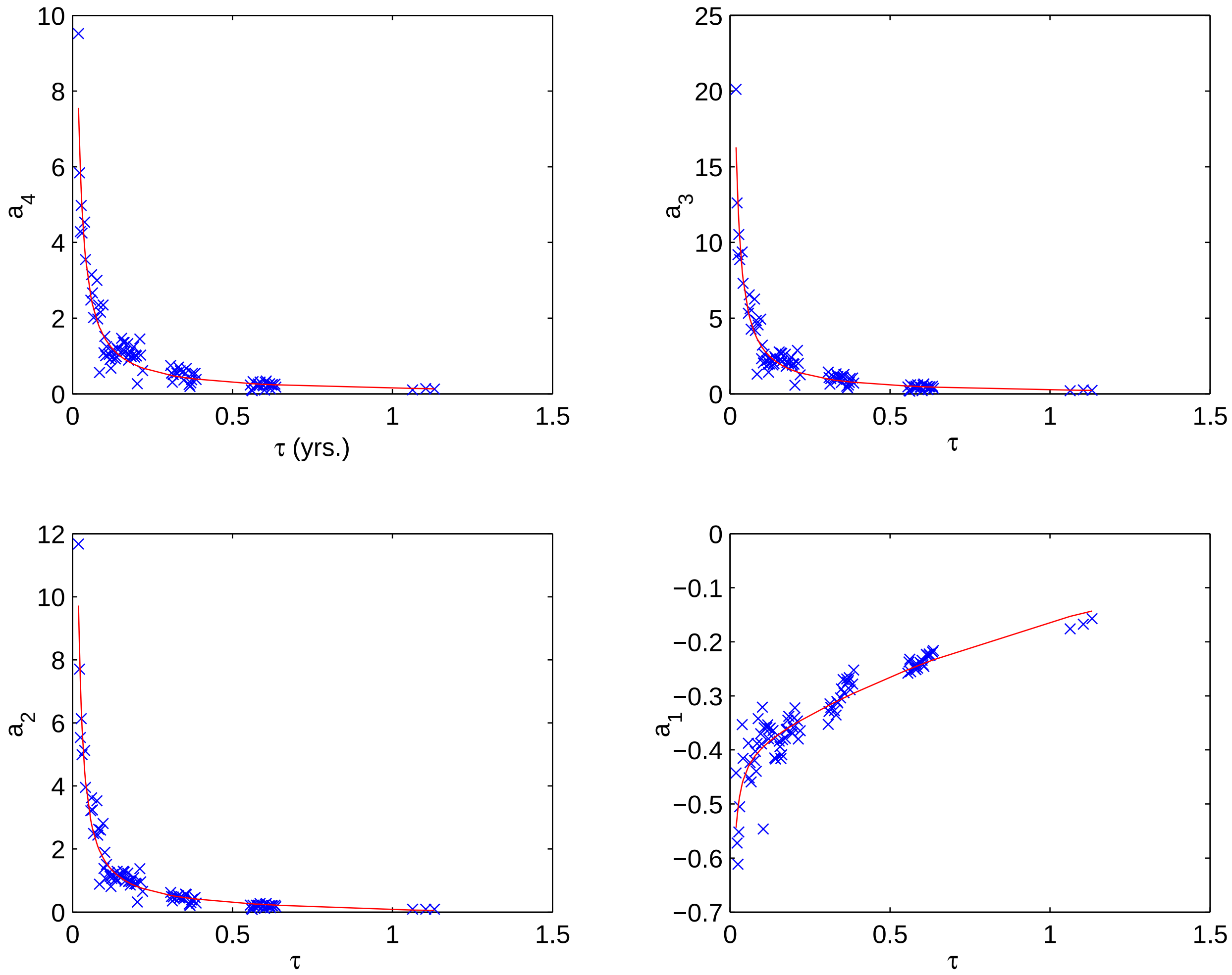


Figure 10: *S&P* 500 *Term-Structure Fit. Data from every trading day in May 2003.*

Parameter Reduction and Direct Calibration

$$\mathcal{L}_{BS}(\bar{\sigma})\left(\tilde{P}_1+\tilde{Q}_1\right) \quad + \quad \left(V_2 x^2\frac{\partial^2 P_{BS}}{\partial x^2}+V_3 x\frac{\partial}{\partial x}\left(x^2\frac{\partial^2 P_{BS}}{\partial x^2}\right)\right)$$

$$+ \quad 2\left(V_0\frac{\partial P_{BS}}{\partial \sigma}+V_1 x\frac{\partial^2 P_{BS}}{\partial x\partial\sigma}\right)=0$$

Set $\sigma^\star=\sqrt{\bar{\sigma}^2+2V_2}$. At the same order, the correction is:

$$(T-t)\left(V_0\frac{\partial P^\star_{BS}}{\partial\sigma}+V_1 x\frac{\partial^2 P^\star_{BS}}{\partial x\partial\sigma}+V_3 x\frac{\partial}{\partial x}\left(x^2\frac{\partial^2 P^\star_{BS}}{\partial x^2}\right)\right)$$

$I\approx b^\star+\tau b^\delta+\left(a^\varepsilon+\tau a^\delta\right)\text{LMMR}$

$$b^\star=\sigma^\star+\frac{V_3}{2\sigma^\star}\left(1-\frac{2r}{\sigma^{\star 2}}\right) \quad , \quad a^\varepsilon=\frac{V_3}{\sigma^{\star 3}}$$

$$b^\delta=V_0+\frac{V_1}{2}\left(1-\frac{2r}{\sigma^{\star 2}}\right) \quad , \quad a^\delta=\frac{V_1}{\sigma^{\star 2}}$$

Exotic Derivatives (Binary, Barrier, Asian,...)

- Calibrate $\sigma^\star, V_0, V_1$ and V_3 on the implied volatility surface
- Solve the corresponding problem with **constant volatility** $\sigma^\star$

$$\Longrightarrow P_0 = P_{BS}(\sigma^\star)$$

- Use V_0, V_1 and V_3 to compute the **source**

$$2\left(V_0 \frac{\partial P_{BS}^\star}{\partial \sigma} + V_1 x \frac{\partial^2 P_{BS}^\star}{\partial x \partial \sigma}\right) + V_3 x \frac{\partial}{\partial x}\left(x^2 \frac{\partial^2 P_{BS}^\star}{\partial x^2}\right)$$

- Get the **correction** by solving the SAME PROBLEM with **zero boundary conditions** and the **source**.

American Options

- Calibrate $\sigma^\star, V_0, V_1$ and V_3 on the implied volatility surface
- Solve the corresponding problem with **constant volatility** $\sigma^\star$

$$\Longrightarrow P^\star \textbf{ and the free boundary } x^\star(t)$$

- Use V_0, V_1 and V_3 to compute the **source**

$$2\left(V_0 \frac{\partial P^\star}{\partial \sigma} + V_1 x \frac{\partial^2 P^\star}{\partial x \partial \sigma}\right) + V_3 x \frac{\partial}{\partial x}\left(x^2 \frac{\partial^2 P^\star}{\partial x^2}\right)$$

- Get the **correction** by solving the corresponding problem with **fixed boundary** $x^\star(t)$, **zero boundary conditions** and the **source**.

First Conclusions

- **A short time-scale of order few days is present in volatility dynamics**
- **It cannot be ignored in option pricing and hedging**
- **It can be dealt with by using singular perturbation methods**
- **It is efficient as a parametrization tool for the term structure of implied volatilities when combined with a regular perturbation**

Accuracy of Approximation

For European options with **smooth payoffs** $h(x)$:

$$P^{\varepsilon,\delta} = P^{\star}_{BS} + \tilde{P}_1 + \tilde{Q}_1 + \mathcal{O}(\varepsilon + \delta)$$

For European **calls or puts**, $h(x)$ continuous piecewise smooth:

$$P^{\varepsilon,\delta} = P^{\star}_{BS} + \tilde{P}_1 + \tilde{Q}_1 + \mathcal{O}(\varepsilon \log|\varepsilon| + \delta)$$

For European **digital option**, $h(x) = Q\mathbf{1}_{\{x>K\}}$ discontinuous:

$$P^{\varepsilon,\delta} = P^{\star}_{BS} + \tilde{P}_1 + \tilde{Q}_1 + \mathcal{O}(\varepsilon^{\frac{2}{3}} \log|\varepsilon| + \delta)$$

Digital formulas $\longrightarrow$

Digital Options

$$P^{\star}_{BS} = Qe^{-r(T-t)}N(d_2),$$

where

$$d_2 = \frac{\log(x/K) + \left(r - \frac{1}{2}\sigma^{\star 2}\right)(T-t)}{\sigma^{\star}\sqrt{T-t}}$$

$$\begin{aligned}
\tilde{P}_1 + \tilde{Q}_1 &= (T-t)\left(V_0\frac{\partial}{\partial\sigma} + V_1 x\frac{\partial}{\partial x}\left(\frac{\partial}{\partial\sigma}\right) + V_3 x\frac{\partial}{\partial x}\left(x^2\frac{\partial^2}{\partial x^2}\right)\right)P^{\star}_{BS} \\
&= \frac{Q\tau e^{-r\tau - d_2^2/2}}{\sqrt{2\pi}}\left[-V_0\left(\frac{d_2}{\sigma^{\star}} + \sqrt{\tau}\right) + \left(\frac{V_1}{\sigma^{\star}} + \frac{V_3}{\sigma^{\star 2}\tau}\right)\left(\frac{d_2^2 - 1}{\sigma^{\star}\sqrt{\tau}} + d_2\right)\right]
\end{aligned}$$

$$(\tau = T - t)$$

Barrier Options

An example: **down-and-out call**

$$h = (X_T - K)^+ \mathbf{1}_{\{\inf_{\{s \le T\}} X_s > B\}}$$

Leading order term (*method of images*):

$$P^\star_{BS}(t,x) = C_{BS}(t,x) - \left(\frac{x}{B}\right)^{1-k} C_{BS}(t, B^2/x)$$

where $C_{BS}(t,x)$ is the Black-Scholes formula *for a call option*, with the z-dependent volatility parameter $\sigma^\star$, and where k is defined by $k = 2r/\sigma^{\star 2}$.

Correction $P_1 = \tilde{P}_1 + \tilde{Q}_1$**:**

$$\mathcal{L}_{BS}(\sigma^\star) P_1 + \left(2V_0 \frac{\partial}{\partial \sigma} + 2V_1 D_1 \left(\frac{\partial}{\partial \sigma}\right) + V_3 D_1 D_2 \right) P^\star_{BS} = 0, \qquad x > B, t < T$$

$$\mathbf{P_1(t, B, z) = 0}, \qquad P_1(T, x, z) = 0 \qquad\qquad (D_n = x^n \partial^n / \partial x^n)$$

Computation of Barrier Corrections

Set $P_1^{(B)} = P_1 - P_1^{(A)}$, where

$$P_1^{(A)} = (T-t)\left(2V_0\frac{\partial}{\partial\sigma} + 2V_1 D_1\left(\frac{\partial}{\partial\sigma}\right) + V_3 D_1 D_2\right) P_{BS}^{\star}$$

Use

$$\mathcal{L}_{BS}(\sigma^{\star})\left(\frac{\partial}{\partial\sigma}P_{BS}^{\star}\right) = -\sigma^{\star} D_2 P_{BS}^{\star},$$

$$\mathcal{L}_{BS}(\sigma^{\star})\left(D_1\frac{\partial}{\partial\sigma}P_{BS}^{\star}\right) = -\sigma^{\star} D_1 D_2 P_{BS}^{\star},$$

to deduce

$$\mathcal{L}_{BS}(\sigma^{\star})P_1^{(B)} = (T-t)\left(2V_0^{\delta}\sigma^{\star}D_2 + 2V_1^{\delta}\sigma^{\star}D_1D_2\right)P_{BS}^{\star}$$

$$\mathbf{P_1^{(B)}}(\mathbf{t},\mathbf{B},\mathbf{z}) = \mathbf{g}^{(\mathbf{B})}(\mathbf{t},\mathbf{z}), \qquad P_1^{(B)}(T,x,z) = 0,$$

$$\mathbf{g}^{(\mathbf{B})}(\mathbf{t},\mathbf{z}) = \lim_{x\downarrow B}\left(P_1(t,x,z) - P_1^{(A)}(t,x,z)\right) = -\lim_{x\downarrow B} P_1^{(A)}(t,x,z)$$

Computation of Barrier Corrections (continued)

We define $P_1^{(D)} = P_1^{(B)} - P_1^{(C)}$, where

$$P_1^{(C)} = -\frac{1}{2}(T-t)^2\left(2V_0\sigma^\star D_2 + 2V_1\sigma^\star D_1 D_2\right) P_{BS}^\star$$

Then $P_1^{(D)}$ solves the following problem in $x > B$:

$$\mathcal{L}_{BS}(\sigma^\star) P_1^{(D)} = \mathbf{0}$$
$$\mathbf{P_1^{(D)}(t, B, z) = g(t, z),}$$
$$P_1^{(D)}(T, x, z) = 0,$$

where the boundary condition function $\mathbf{g(t,z)}$ is given by

$$\mathbf{g(t, z) = g^{(B)}(t, z) - \lim_{x \downarrow B} P_1^{(C)}(t, x, z)}$$

Computation of Barrier Corrections (continued)

Log-variable: $\xi = \log x$, and define $u(t, \xi, z)$ by

$$P_1^{(D)}(t, x, z) = e^{-\frac{1}{8}\sigma^{\star 2}(1+k)^2(T-t)+\frac{1}{2}(1-k)\xi}\, u(t, \xi, z)$$

Define $L = \log B$, then we find that u solves the *heat equation*

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{1}{2}\sigma^{\star 2}\frac{\partial^2 u}{\partial \xi^2} &= \mathbf{0}\,, \quad \text{for} \quad \xi > L, t < T \\ u(T, \xi) &= 0\,, \\ u(t, L) &= \tilde{\mathbf{g}}(\mathbf{t}, \mathbf{z}) \end{aligned}$$

where the boundary term is given by

$$\tilde{\mathbf{g}}(\mathbf{t}, \mathbf{z}) = e^{\frac{1}{8}\sigma^{\star 2}(1+k)^2(T-t)} B^{(k-1)/2} g(t, z)$$

Computation of Barrier Corrections (continued)

The **probabilistic representation** of u is

$$u(t,\xi,z) = I\!E\left\{\tilde{g}(\tau_t,z)1_{\{\tau_t\le T\}} \mid B_t=\xi>L\right\}$$

where (B_t) is a Brownian motion with $\langle B\rangle_t = \sigma^{\star 2}t$, and τ_t is the first time after t that it hits L. Using the distribution for the **hitting time** τ_t this expectation is given by the integral

$$u(t,\xi,z) = \frac{1}{\sigma^\star\sqrt{2\pi}}\int_t^T \frac{(\xi-L)}{(s-t)^{3/2}}\exp\left(-\frac{(\xi-L)^2}{2\sigma^{\star 2}(s-t)}\right)\tilde{g}(s,z)\,ds$$

Finally, using the expression for $\tilde{g}$ we obtain

$$P_1^{(D)} \quad = \quad x^{\frac{1}{2}(1-k)}e^{-\frac{1}{8}\sigma^{\star 2}(1+k)^2(T-t)}u(t,\log x,z)$$

Computation of Barrier Corrections (end)

To summarize, the correction is given by

$$P_1 = P_1^{(A)} + P_1^{(C)} + P_1^{(D)}$$

with

$$\begin{aligned}
P_1^{(A)} &= (T-t)\left(2V_0\frac{\partial}{\partial\sigma} + 2V_1 D_1\left(\frac{\partial}{\partial\sigma}\right) + V_3 D_1 D_2\right) P^\star_{BS} \\
P_1^{(C)} &= -\frac{1}{2}(T-t)^2\left(2V_0\sigma^\star D_2 + 2V_1\sigma^\star D_1 D_2\right) P^\star_{BS} \\
P_1^{(D)} &= x^{\frac{1}{2}(1-k)} e^{-\frac{1}{8}\sigma^{\star 2}(1+k)^2(T-t)} u(t, \log x, z)
\end{aligned}$$

Explicit lengthy formulas in terms of the Greeks of $C^\star_{BS}$ are available.