Reduction of Asian Options

Sean Han
Institute for Mathematics and its Applications
University of Minnesota

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Joint work with Jean-Pierre Fouque

Outline

- Introduction to Multi-factor Stochastic Volatility Model
- Pricing Arithmetic Average Asian Options
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Multi-factor Stochastic Volatility Model

Under the pricing measure $I\!P^{\star}$, a two-factor stochastic volatility model is given as:

$$\begin{aligned}
dS_t &= rS_t dt + \sigma_t S_t dW_t^{(0)}, \\
\sigma_t &= f(Y_t, Z_t), \\
dY_t &= \left(\alpha(m_1 - Y_t) - \nu_1\sqrt{2\alpha}\,\Lambda_1(Y_t, Z_t)\right) dt \\
&\quad +\nu_1\sqrt{2\alpha}\left(\rho_1 dW_t^{(0)} + \sqrt{1-\rho_1^2}\, dW_t^{(1)}\right), \\
dZ_t &= \left(\delta(m_2 - Z_t) - \nu_2\sqrt{2\delta}\,\Lambda_2(Y_t, Z_t)\right) dt \\
&\quad +\nu_2\sqrt{2\delta}\left(\rho_2 dW_t^{(0)} + \rho_{12} dW_t^{(1)} + \sqrt{1-\rho_2^2-\rho_{12}^2}\, dW_t^{(2)}\right),
\end{aligned}$$

where $\left(W_t^{(0)}, W_t^{(1)}, W_t^{(2)}\right)$ are independent standard Brownian motions. Assume time scales are separated: $\alpha^{-1} < 1 < \delta^{-1}$.

The Pricing of Arithmetic Average Asian Options: PDE Approach

Introduce the running sum process:

$$dI_t = S_t dt.$$

The price of an Asian call option at time $0 \le t \le T$ defined by

$$P(t, s, y, z, I) = E^*_{t,x,y,z,I} \left\{ e^{-r(T-t)} h(\frac{I_T}{T} - K_1 \, S_T) \right\}$$

where $h(v) = v^+$, solves a four-dimensional PDE

$$\mathcal{L}^{\alpha,\delta} P = 0,$$
$$P(T, x, y, z, I) = h(I/T - K_1 x),$$

where the partial differential operator $\mathcal{L}^{\alpha,\delta}$ is defined by

$$\mathcal{L}^{\alpha,\delta} = \alpha \mathcal{L}_0 + \sqrt{\alpha} \mathcal{L}_1 + \mathcal{L}_2 + \sqrt{\delta} \mathcal{M}_1 + \delta \mathcal{M}_2 + \sqrt{\alpha \, \delta} \mathcal{M}_3.$$

Dimension Reduction Technique: Change of Numeraire

Dynamically investing the amount of units $q(t) = \frac{1-e^{-r(T-t)}}{rT}$ of your wealth in stocks and put the rest in bonds, your final wealth will replicate the stock price average by self-financing trading, i.e.,

$$X_T = \frac{1}{T}\int_0^T S_t dt,$$

if your initial wealth is $X_0 = q(0)S_0$. Change of numeraire $\Psi_t = \frac{X_t}{S_t}$ results in

$$d\Psi_t = (q(t) - \Psi_t)f(Y_t, Z_t)d(W_t^* - \int_0^t f(Y_s, Z_s)ds).$$

Assume that f is bounded.

Change of Pricing Measure

An equivalent probability measure $\tilde{P}^*$ is defined by

$$\frac{d\tilde{P}^*}{dP^*} = \exp\left(\int_0^T f(Y_t, Z_t)dW_t^* - \frac{1}{2}\int_0^T f(Y_t, Z_t)^2 dt\right) = e^{-rT}\frac{S_T}{S_0}.$$

Under this new measure $\tilde{P}^*$, the Asian option price can be expressed as $P(0, x, y, z; T, K) = xu(0, \psi, y, z; T, K)$, where $u(t, \psi, y, z; T, K)$ solves the three-dimensional PDE

$$\left(\alpha\mathcal{L}_0 + \sqrt{\alpha}\mathcal{L}_1 + \hat{\mathcal{L}}_2 + \sqrt{\delta}\hat{\mathcal{M}}_1 + \delta\mathcal{M}_2 + \sqrt{\alpha\delta}\mathcal{M}_3\right)u = 0, \qquad (1)$$

$$\begin{aligned}
\hat{\mathcal{L}}_2(f(y,z)) &= \frac{\partial}{\partial t} + \frac{1}{2}(\psi - q_{t-})^2 f(y,z)^2 \frac{\partial^2}{\partial \psi^2}, \\
\hat{\mathcal{M}}_1 &= -\left(g(z)\Gamma(y,z) - \rho_2 g(z) f(y,z)\right)\frac{\partial}{\partial z} \\
&\quad + \rho_2 g(z) f(y,z)(q_{t-} - \psi)\frac{\partial^2}{\partial x \partial z}
\end{aligned}$$

Asian Option Asymptotics: Solve Two 1-dim. PDEs

Denote $\varepsilon = 1/\alpha, \delta \ll 1$

$$u^{\varepsilon,\delta}(t,\psi,y,z) \approx u_0(t,\psi;z) + \sqrt{\varepsilon}u_{1,0}(t,\psi;z) + \sqrt{\delta}u_{0,1}(t,\psi;z)$$

where $u_0(t,\psi;z)$ solves

$$< \hat{\mathcal{L}}_2 > u_0 = \frac{\partial u_0}{\partial t} + \frac{1}{2}(\psi - q_{t-})^2 < f(y,z)^2 > \frac{\partial^2 u_0}{\partial \psi^2} = 0,$$

$$u_0(T,\psi;z) = h(\psi - K_1),$$

$$< \hat{\mathcal{L}}_2 > (\tilde{u}_{1,0} + \tilde{u}_{0,1})(t,\psi;z) \quad = \quad (\mathcal{A} + \mathcal{B})\, u_0,$$

with a zero terminal condition, where

$$\mathcal{A} \quad = \quad \overline{V}_2(z)(q_{t-} - \psi)^2 \frac{\partial^2}{\partial \psi^2} + \overline{V}_3(z)(q_{t-} - \psi)^3 \frac{\partial^3}{\partial \psi^3},$$

$$\mathcal{B} \quad = \quad \frac{1}{\overline{\sigma}} \left(\overline{V}_0^{\delta} \frac{\partial}{\partial \sigma} + \overline{V}_1^{\delta}(q_{t-} - \psi) \frac{\partial^2}{\partial \psi \partial \sigma} \right)$$

Numerical Illustration

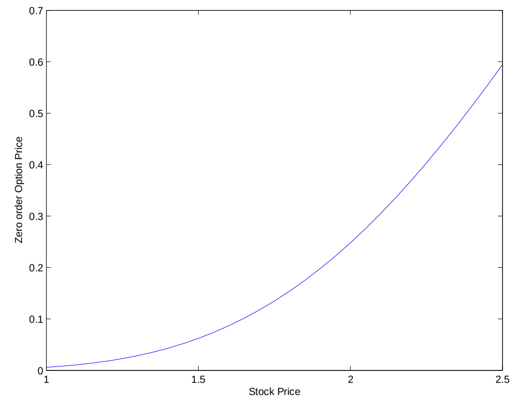

Figure 1: Finite difference numerical solution for the effective volatility price $P_0(0, S, z)$ of an arithmetic average Asian call option with parameters $\overline{\sigma}(z) = 0.5, r = 0.06, K_1 = 0, K_2 = 2$, and time to maturity $T = 1$. Note that z is fixed.

Correction and Calibration

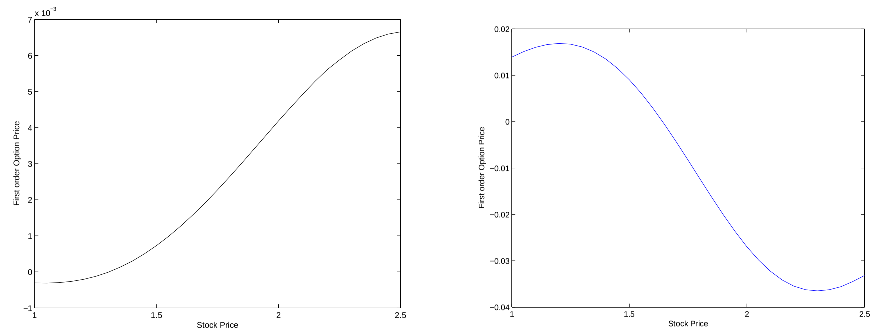

Figure 2: Finite difference numerical solution for the correction $\tilde{P}_1(0, S, z)$ for an arithmetic average Asian call option price with parameters $\overline{\sigma} = 0.5, r = 0.06, T = 1, V_0 = 0.001, V_1 = -0.05, V_2 = -0.01, V_3 = 0.004$. In practice the last four parameters should be calibrated to the observed implied volatility surface.

An Alternative Approach to Evaluate Asian Options: Monte Carlo Simulations

As an example, the price of an Asian fixed-strike call option with stochastic volatility at time $0 \le t \le T$ is given by

$$P(t,x,y,z,I) = E^*_{t,x,y,z,I}\left\{ e^{-r(T-t)} \left(\frac{I_T}{T} - K \right)^+ \right\}.$$

A basic Monte Carlo simulation for option prices is to build N independent trajectories governed by our model. Based on the law of large number, we calculate the sample mean of the payoffs for all trajectories by

$$P^{MC} \approx \frac{e^{-r(T-t)}}{N} \sum_{i=1}^{N} \left(\frac{I_T^{(i)}}{T} - K \right)^+ .$$

Control Variate Estimator

Denote by P^{CV} the control variate estimator for P^{MC} such that

$$P^{CV} \triangleq P^{MC} + \lambda(\hat{P}^\star - P^\star),$$

where $\hat{P}^\star$ is an unbiased Monte Carlo estimator of $P^\star$ computed by the same run as the run of P^{MC}. The company price $P^\star$ is often chosen to have an analytical solution , for example an Asian option with geometric average, and the parameter λ is chosen to minimize the sample variance. For Asian options, λ is often chosen as -1.

Geometric Average Asian Options

Recall the GAO price

$$P^\star(t,x,y,z,\hat{I}) = E_{x,y,z,\hat{I}}\left\{e^{-r(T-t)}\left(exp(\hat{I}_T/T) - K\right)^+\right\}.$$

$$P^\star(t,x,y,z,\hat{I}) \approx \tilde{P}_G(t,x,z,\hat{I}),$$

where

$$\begin{aligned}\tilde{P}_G \quad = \quad & P_0^{fix} - (T-t)\sqrt{2}V_0\frac{\partial P_0^{fix}}{\partial\sigma} + (T+t)V_1 x\frac{\partial^2 P_0^{fix}}{\partial x\partial\sigma} \\ & -\frac{(T-t)^2}{2}V_2\frac{\partial P_0^{fix}}{\partial x} + \frac{(T-t)^3}{3}(V_2 - V_3)\frac{\partial^2 P_0^{fix}}{\partial x^2} + \frac{(T-t)^4}{4}V_3\frac{\partial^3 P_0^{fix}}{\partial x^3}.\end{aligned} \tag{2}$$

Importance Sampling

Monte Carlo simulation for approximation of $P^\star$ under the new probability measure $\tilde{\mathcal{P}}$ is given by

$$P^\star(t,x,y,z,\hat{I}) \approx \frac{1}{N}\sum_{i=1}^{N}(exp(\hat{I}_T^{(i)}/T) - K)^+ Q_T^{(i)}$$

where

$$Q_T = exp\left\{\int_0^T H(t,P_t^\star)d\tilde{\eta}_t - \frac{1}{2}\int_0^T |H(t,P_t^\star)|^2 dt\right\}$$

The important sampling technique consists of determining H such that the variance of $P^\star$ can be reduced. The optimal choice of the function H is $H = -\frac{1}{P^\star}a^T\nabla P^\star$ which can lead to a zero variance.

BUT $P^\star(t,x,y,z,\hat{I})$ is unknown of course!

Variance Reduction

Table 1: Comparison of simulated option prices and their variances for various values of α and δ; P^{MC} is obtained by basic Monte Carlo simulation and $P_G^{IS}(\tilde{P}_G)$ are computed by Monte Carlo simulations using importance sampling with $\tilde{P}_G$ (means are shown in parenthesis next to the variances).

α	δ	P^{MC}	$P_G^{IS}(\tilde{P}_G)$
100	0.05	0.048341 (7.97)	0.006334 (7.76)
75	0.1	0.043363 (7.57)	0.007707 (7.46)
50	0.5	0.051290 (7.45)	0.009676 (7.17)
25	1	0.058433 (7.31)	0.014814 (6.96)

Control Variates + Importance Sampling

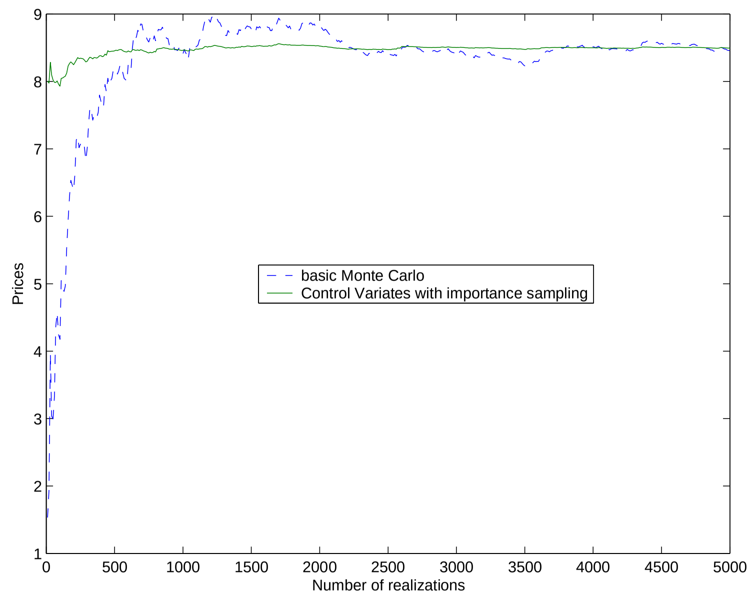

Figure 3: Mone Carlo simulations for the price of an arithmetic average Asian option. Rates of mean-reversion are chosen as $\alpha = 75$ and $\delta = 0.1$.

References

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