The inverse problem of option pricing

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1 Introduction

The Black-Scholes formula [6] provides with an elegant and simple method to price financial derivatives under the assumption that the stock price is log-normally distributed. However, the actual distribution of most assets is rarely log-normal, and theoretical prices of options with different strikes generated by the Black-Scholes formula differ from observed market prices. One way to reconcile the differences is to replace the log-normal process with constant volatility by a more general diffusion model. While in the log-normal it is sufficient to calculate a single volatility number from each option quote to calibrate the model, in the general diffusion framework the whole volatility function must be restored from collection of simulteneous option quotes with different strikes. Several useful numerical algorithms [1], [5], [9] to recover time dependent volatility from time-space data have been proposed, these algorithms are mostly based on regularized least squares fitting and as for typically for nonconvex minimization convergence properties have not been satisfactory for practitioners. Recently, in the case of time-space data an interesting relation between implied and local volatilities was discovered [2] and one can expect a nice algorithm based on the Black-Scholes formula and theory of nonlinear parabolic equations. In [3] we proposed another extension of the Black-Scholes formula by using two terms of the standard construction of a fundamental solution of the initial value problem for a Black-Scholes equation. This algorithm works well for short maturities and for both time-space or only space data, but again it lacks a rigorous mathematical justification. In the recent paper [4] we considered a linearization of the inverse problem around constants and found conditions which ensure uniqueness of its solution.

A rigorous justification of numerical algorithms needs uniqueness and stability theory. We formulate available results for complete (nonlinear) calibration problems in section 2. In section 3 we focus on mathematical foundation of linearization around constants. Section 4 illustrates the accuracy and the robustness of our fast numerical linearization algorithm for volatility functions of various shapes.

For any stock price, $0 < s < \infty$, and time $t, 0 < t < T$, the price u for an option expiring at time T satisfies the following Black-Scholes partial differential equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}s^2\sigma^2(s)\frac{\partial^2 u}{\partial s^2} + s\mu\frac{\partial u}{\partial s} - ru = 0. \tag{1.1}$$

Here, $\sigma(s)$ is the volatility coefficient that satisfies $0 < m < \sigma(s) < M < \infty$ and is assumed to belong to the Hoelder space $C^\lambda(\bar{\omega}), 0 < \lambda < 1$ on some interval ω and outside this interval, and μ and r are, respectively, the risk-neutral drift and the risk-free interest rate assumed to be constants. The backward in time parabolic equation (1.1) is augmented by the final condition specified by the payoff of the call option with the strike price K

$$u(s,T) = (s-K)^+ = max(0, s-K),\ 0 < s. \tag{1.2}$$

It is known that there is a unique solution u to (1.1),(1.2) which belongs to $C^1((0,\infty)\times(0,T]$ and to $C((0,\infty)\times[0,T])$ and satisfies the bound $|u(s,t)| < C(s+1)$. The inverse problem of option pricing seeks for σ given

$$u(s^*, t^*; K, T) = u^*(K),\ x \in \omega^*. \tag{1.3}$$

Here s^* is market price of the stock at time t^*, and $u^*(K)$ denote market price of options with different strikes K for a given expiry T. We attempt to recover volatility in the same interval ω^*.

To obtain our results we will use that the option premium $u(.,.;K,T)$ satisfies the equation dual to the Black-Scholes equation (1.1):

$$\frac{\partial u}{\partial T} - \frac{1}{2}K^2\sigma^2(K)\frac{\partial^2 u}{\partial K^2} + \mu K\frac{\partial u}{\partial K} + (r-\mu)u = 0. \tag{1.4}$$

The equation (1.4) was found by Dupire [7] and rigorously justified, for example, in [3].

The logarithmic substitution

$$y = ln\frac{K}{s^*},\ \tau = T - t,\ U(y,\tau) = u(;K,T) \tag{1.5}$$

transforms the dual problem (1.4) and the additional (market) data into the following inverse parabolic problem with the final observation

$$\frac{\partial U}{\partial \tau} = \frac{1}{2}a^2(y)\frac{\partial^2 U}{\partial y^2} - (\frac{1}{2}a^2(y)+\mu)\frac{\partial U}{\partial y} + (\mu - r)U \tag{1.6}$$

$$U(y,0) = s^*(1-e^y)^+,\ y \in \mathbf{R} \tag{1.7}$$

$$U(y,\tau^*) = U^*(y),\ y \in \omega. \tag{1.8}$$

2 Uniqueness and stability for the inverse problem

We list available theoretical results. If $\sigma(s,t)$ can be found from the data $U^*(K,T)$ due to the Dupire's equation (1.4). In many important cases temporal data are not available or sparse. In any case these data are not available for future, when knowledge of volatility is most important for useful predictions. This is why we think it is reasonable to look for $\sigma = \sigma(s)$.

Theorem 2.1 *Let U_1 and U_2 be two solutions to the initial value problem with $a = a_1$ and $a = a_2$ and let U_1^*, U_2^* be the corresponding final data (1.2). Let ω_0 be a non-void open subinterval of ω.*

If $U_1^*(y) = U_2^*(y)$ for $y \in \omega$ and $a_1(y) = a_2(y)$ for $y \in \omega_0$, then $a_1(y) = a_2(y)$ when $y \in \omega$.

If, in addition, $a_1(y) = a_2(y)$ when $y \in \omega \cup (\mathbf{R} \setminus \omega)$ and if ω is bounded, then there is a constant C depending only on $|a_1|_1(\omega), |a_2|_2(\omega), \omega, \omega_0, \tau^*, \lambda$ such that

$$|a_2 - a_1|_\lambda(\omega) \leq C|U_2^* - U_1^*|_{2+\lambda}(\omega) \tag{2.9}$$

To show uniqueness we subtract two equations for U_2 and U_1 to get

$$\frac{\partial U}{\partial \tau} = \frac{1}{2}a_2^2(y)\frac{\partial^2 U}{\partial y^2} - (\frac{1}{2}a_2^2(y) + \mu)\frac{\partial U}{\partial y} + (\mu - r)U + \alpha_1(y,\tau)f(y)$$

where

$$U = U_2 - U_1,\ \alpha_1 = \frac{\partial^2 U_1}{\partial y^2} - \frac{\partial U_1}{\partial y},\ f(y) = \frac{1}{2}(a_2^2(y) - a_1^2(y)).$$

Besides, $U(y,0) = 0$. It is known that solutions of an initial value parabolic problem with time independent coefficients are time analytic. Since $U(,\tau^*) = 0, f = 0$ on ω_0 we conclude from the differential equation that $\frac{\partial U}{\partial \tau}(,\tau^*) = 0$ on ω_0. Repeating this argument we conclude that all τ-derivatives of U are zero on $\omega_0 \times \{\tau^*\}$. By analyticity $U = 0$ on $\omega_0 \times (0,\tau^*)$. Then applying the method of Carleman estimates we can prove that $f = 0$ on ω.

For detals of a proof we refer to [3].

The assumption that $a(y)$ is known on a subinterval of ω is probably not nesessary. Moreover it prevents from existence results since it severely overdetrmines the inverse problem. Masahiro Yamamoto observed that for $a \in C^\infty(\mathbf{R})$ this assumption can be removed.

A feature of the inverse options pricing problem is localization around the underlying price s^* resulting from singularity of the final data. Thus local results make sense.

Theorem 2.2 *Let* $|a|_\lambda(\omega) < M$ *and* $a^2(y) = \sigma_0$ *on* $\mathbf{R} \setminus \omega$.

Then there is $\varepsilon > 0$ *(depending only on* s^*, τ^*, σ_0*) and* M *such that under the condition* $\omega \subset (-\varepsilon, \varepsilon)$ *a solution* $a(y)$ *to the inverse problem (1.6), (1.7), (1.8) is unique.*

A proof follows standard contraction arguments augmented by careful study of singularities of solution and it is based on the study of the linearized inverse problem in section 3.

3 Linearization around constant volatility

In this section we assume that

$$\frac{1}{2}\sigma^2(s) = \frac{1}{2}\sigma_0^2 + f_*(s) \tag{3.10}$$

where f_* is a small $C(\bar{\omega})$-perturbation of constant σ_0^2 and $f_* = 0$ outside ω_*. Then the option price can be computed (up to a quadraticaly small error) as the sum of the Black-Scholes formula with volatility σ_0 and of a certain linear operator $V(f_*)$.

The substitution

$$y = \ln(\frac{K}{s^*}), \quad \tau = T - t, \ a(y) = \sigma(s^* e^y), \ U(y,\tau) = u(s^* e^y, \tau + t) \tag{3.11}$$

transforms the equation (1.4) and the initial data (1.2) into

$$\frac{\partial U}{\partial \tau} - \frac{1}{2} a^2(y) \frac{\partial^2 U}{\partial y^2} + \left(\frac{1}{2} a^2(y) + \mu \right) \frac{\partial U}{\partial y} + (r - \mu) U = 0 \tag{3.12}$$

$$U(y, 0) = s^* (1 - e^y)^+, \quad y \in \mathbf{R}$$

$$U(y, \tau^*) = U^*(y), \quad y \in \omega \tag{3.13}$$

Here ω is the transformed interval ω^* (ω^* in y- variables (3.11)). Observe that $\tau^* = T - t^*$.

To derive the linearized inverse problem we observe that due to the assumption (3.10),

$$\frac{1}{2} a^2(y) = \frac{1}{2} \sigma_0^2 + f(y)$$

where f is $C(\bar{\omega})$-small and $f = 0$ outisde ω. So

$$U = V_0 + V + v. \tag{3.14}$$

Here V_0 solves (3.12) with $a = \sigma_0$ and v is quadratically small with respect to f_*, while the principal linear term V satisfies the equations

$$\begin{aligned}
\frac{\partial V}{\partial \tau} - \frac{1}{2} \sigma_0^2 \frac{\partial^2 V}{\partial y^2} + \left(\frac{\sigma_0^2}{2} + \mu \right) \frac{\partial V}{\partial y} + (r - \mu) V &= \alpha_0 f, \\
V(y, 0) = 0, \quad y &\in \mathbf{R}, \\
V(y, \tau^*) = V^*(y), \quad y &\in \omega,
\end{aligned} \tag{3.15}$$

where

$$\begin{aligned}
\alpha_0(y, \tau) &= s^* \frac{1}{\sigma_0 \sqrt{4 \pi a \tau}} e^{-\frac{y^2}{2\tau\sigma_0^2} + cy + d\tau}, \\
c &= \frac{1}{2} + \frac{\mu}{\sigma_0^2}, \quad d = -\frac{1}{2\sigma_0^2} \left(\frac{\sigma_0^2}{2} + \mu \right)^2 + \mu - r
\end{aligned}$$

and V^* is the principal linear part of U^*.

The new substitution

$$V = e^{cy+d\tau}W \tag{3.16}$$

simplifies (3.15) to

$$\begin{aligned} \frac{\partial W}{\partial \tau} - \frac{1}{2}\sigma_0^2 \frac{\partial^2 W}{\partial y^2} &= \alpha f, \quad 0 < \tau < \tau^*, \quad y \in \mathbf{R}, \\ \alpha(\tau, y) &= \frac{s^*}{\sqrt{2\pi\tau}\sigma_0} e^{-\frac{y^2}{2\tau\sigma_0^2}}, \\ W(y, 0) &= 0, \quad y \in \mathbf{R}, \\ W(y, \tau^*) &= W^*(y), \quad y \in \omega, \end{aligned} \tag{3.17}$$

with

$$W^*(y) = e^{-cy-d\tau^*} V^*(y)$$

In the remaining part of this section we will assume that $f = 0$ outside ω. From numerical experiments in section 4 and in [9] we can see that values of f outside ω are not essential, due to a very fast decay of the Gaussian kernel α in s.

Let us denote by Af the solution to (3.17) on ω : $Af(y) = W(y, \tau^*)$, $\quad y \in \omega$.

Lemma 3.1 *We have*

$$Af(x) = \int_\omega B(x, y; \tau^*) f(y) dy, \quad x \in \omega, \tag{3.18}$$

where

$$B(x, y; \tau^*) = \frac{s^*}{\sigma_0^2 \sqrt{\pi}} \int_{\frac{|x-y|+|y|}{\sigma_0\sqrt{2\tau^*}}}^{\infty} e^{-\tau^2} d\tau.$$

Proof: The well-known representation of the solution to the Cauchy problem (3.17) for the heat equation yields

$$\begin{aligned} W(x, \tau) &= \int_{\mathbf{R}} B(x, y; \tau) f(y) dy, \\ B(x, y; \tau) &= \int_0^\tau \frac{1}{\sqrt{2\pi(\tau-\theta)}\sigma_0} e^{-\frac{|x-y|^2}{2\sigma_0^2(\tau-\theta)}} \frac{s^*}{\sqrt{2\pi\theta}\sigma_0} e^{-\frac{|y|^2}{2\sigma_0^2\theta}} d\theta. \end{aligned}$$

We will simplify $B(x,y;\tau)$ by using the Laplace transform $\Phi(p) = \mathcal{L}(\phi)(p)$ of $\phi(\tau)$ with respect to τ. Since the Laplace transform of the convolution is the product of Laplace transforms of convoluted functions, we have

$$\begin{aligned}
\mathcal{L}B(x,y;)(p) &= \frac{s^*}{2\pi\sigma_0^2}\mathcal{L}(\frac{1}{\sqrt{\tau}}e^{-\frac{|x-y|^2}{2\sigma_0^2\tau}})\mathcal{L}(\frac{1}{\sqrt{\tau}}e^{-\frac{|y|^2}{2\sigma_0^2\tau}}) \\
&= \frac{s^*}{2\pi\sigma_0^2}\sqrt{\frac{\pi}{p}}e^{-\frac{\sqrt{2}|x-y|}{\sigma_0}\sqrt{p}}\sqrt{\frac{\pi}{p}}e^{-\frac{\sqrt{2}|y|}{\sigma_0}\sqrt{p}} \\
&= \frac{s^*}{2\sigma_0^2}\frac{1}{p}e^{-\frac{\sqrt{2}(|x-y|+|y|)}{\sigma_0}\sqrt{p}},
\end{aligned}$$

where we used the formula for the Laplace transform of $\tau^{-\frac{1}{2}}e^{-\frac{\beta}{\tau}}$. Applying the formula for the inverse Laplace transform of the function $\frac{1}{p}e^{-\alpha\sqrt{p}}$ we arrive at the conclusion of Lemma 2.1.

Corollary 3.2 *The linearized inverse problem implies the following Fredholm integral equation*

$$f(x) - \frac{1}{2\tau^*\sigma_0^2}\int_\omega e^{-\frac{(|x-y|+|y|)^2-|x|^2}{2\tau^*\sigma_0^2}}(|x-y|+|y|)f(y)dy = -\frac{\sqrt{\pi\tau^*}}{\sqrt{2}s^*}\sigma_0^3 e^{\frac{|x|^2}{2\tau^*\sigma_0^2}}\frac{\partial^2}{\partial x^2}W(x,\tau^*),\ x\in\omega \tag{3.19}$$

A proof can be obtained by differentiating the equation $Af = W(,\tau^*)$ on ω.

Theorem 3.3 *Let* $\omega = (-b,b)$. *Let* θ_0 *be the root of the equation* $2\theta - e^{-4\theta} = 3$.

If

$$\frac{b^2}{\tau^*\sigma_0^2} < \theta_0, \tag{3.20}$$

then a solution $f \in L^\infty(\omega)$ *to the integral equation (3.19) and hence to the inverse option pricing problem (3.15) is unique.*

One can check numerically that $1.5012 < \theta_0 < 1.5013$.

We remind that $\|f\|_\infty(\omega)$ is essential supremum of $|f|$ over ω. In particular, when f is continuous on $\overline{\omega}$ is it just $max|f(x)|$ over $x \in \overline{\omega}$.

Proof: Due to Corollary 3.2 to prove Theorem 3.3 it suffices to show uniqueness of solution f of (3.19), i.e. to assume that the right side is zero and conclude that $f = 0$. To do it we observe that

$$\int_\omega e^{-\frac{(|x-y|+|y|)^2}{2\tau^*\sigma_0^2}}(|x-y|+|y|)dy =$$
$$e^{-\frac{x^2}{2\tau^*\sigma_0^2}}(\tau^*\sigma_0^2+x^2) - \frac{\tau^*\sigma_0^2}{2}(e^{-\frac{(2b-x)^2}{2\tau^*\sigma_0^2}} + e^{-\frac{(2b+x)^2}{2\tau^*\sigma_0^2}}). \quad (3.21)$$

Returning to uniqueness of f we assume that f is not zero. We can assume that $\|f\|_\infty(\omega) = f(x_0) > 0$ at some $x_0 \in [-b, b]$. From (3.19) at $x = x_0$ (with zero right side) we have

$$\|f\|_\infty \leq \frac{1}{2\tau^*\sigma_0^2}\int_\omega e^{-\frac{|x_0-y|+|y|)^2-x_0^2}{2\tau^*\sigma_0^2}}(|x_0-y|+|y|)\|f\|_\infty dy$$
$$= \|f\|_\infty(\frac{\tau^*\sigma_0^2+x_0^2}{2\tau^*\sigma_0^2} - \frac{1}{4}(e^{-\frac{(2b-x_0)^2-x_0^2}{2\tau^*\sigma_0^2}} + e^{-\frac{(2b+x_0)^2-x_0^2}{2\tau^*\sigma_0^2}}))$$

if we use (3.21).

One can show that

$$g(x) = \frac{x^2}{\tau^*\sigma_0^2} - \frac{1}{2}\left(e^{\frac{2b(x-b)}{\tau^*\sigma_0^2}} + e^{-\frac{2b(x+b)}{\tau^*\sigma_0^2}}\right) < 1, \quad -b \leq x \leq b. \quad (3.22)$$

Then the previous inequality yields

$$\|f\|_\infty < \|f\|_\infty\left(\frac{1}{2}+\frac{1}{2}\right) = \|f\|_\infty,$$

and hence $\|f\|_\infty(\omega) = 0$.

By Lemma 3.1 the linearized inverse option pricing problem implies the integral equation

$$Af(x) = W^*(x),\ x \in \omega = (-b, b), \quad (3.23)$$

where W^* is the function defined after (3.17). Corollary 3.2 and Theorem 3.3 guarantee uniqueness a solution $f \in C(\overline{\omega})$ to this equation under the condition (3.20). It is not known whether this condition is necessary for uniqueness in the linearized inverse problem.

The integral equation (3.23), Theorem 3.3 and simple properties of integral operators imply the following stability estimate

$$\|f\|_\infty(\omega) \leq C\|W^{*\prime\prime}\|_\infty(\omega). \tag{3.24}$$

One can show [4] that the range of A has the codimension 2 in $C^{2+\lambda}(\bar{\omega})$. At present we do not know an exact description of the range of A.

4 Numerical algorithm and its testing

We will solve the integral equation (3.23) with the data $W^*(x)$ equal to the difference of the final states $e^{-cy-d\tau^*}(U^*(y) - U_0^*(y))$ where U solves the parabolic equation (3.12) with $a^2(y) = \sigma_0^2 + 2f(y)$ and U_0 solves the unperturbed equation (3.12). We consider the interval $\omega^* = (-1, 1)$, $s^* = 20$ and we let $\mu = 0$ and $r = 0.05$. On this interval we will use uniformly distributed grid points. Observe that we are solving numerically a linear inverse problem, using the data generated by the original nonlinear problem (3.12). Of course, it generates data errors, due to the linearization. Otherwise, the data are (numerically) exact.

The direct problem (3.12) was solved numerically by the finite differences method (the Crank-Nicholson scheme with 80 grid points on the interval $(-1.5, 1.5)$ with artificial zero (Dirichlet) boundary conditions at $y = -1.5$ and $y = 1.5$).

The integral operator (3.23) is discretized by using standard tables for the error function $errfc$ as follows

$$K_{j,k} = s^* errfc(\frac{|x_j - y_k| + |x_j|}{\tau^*})\delta y$$

where $y_j, j = 1, ..., N$ are points of ω used in solving the direct problem. As a result, $N = 54$. The points x_j are the measurements points. Their collection coincides with the points $y_1, ..., y_{54}$.

We will consider 5 examples illustrated by figures 1,...,5, where we let $\sigma_0 = 1$ and we will use different observation times $\tau^* = 0.1, 0.3, 0.5$ and 0.7. As perturbations $f(y)$ of constant volatility we will take functions $f_1(y) = 0.3y$, $f_2(y) = 0.3y_2$, $f_3(y) = 0.5y^2 - 0.25y$, $f_4(y) = 0.3sin(2\pi y)$, and $f_5(y) = 0.3sin(4\pi y)$. The functions f_4, f_5 are oscillating functions and they are not typical for financial problems. We included them to test how robust is the numerical algorithm. Observe that perturbations can reach $0.3 - 0.7$ of the magnitude of the unperturbed constant coefficient.

¿From figures 1,2 we can see that the reconstruction is near perfect on the whole ω when τ^* is $0.5; 0.7$. This is with agreement with the condition (3.20), which in these examples simplifies to

$$0.6667 < \tau^*$$

The greater is τ^*, the better is the reconstruction on the whole interval $(-1, 1)$, so $\tau^* = 0.5$ or 0.7 correspond to the recovered volatilies closest to the given ones. On the other hand, for smaller time $\tau^* = 0.1$ the reconstructed f starts to deteriorate near endpoints (on the intervals $(-1, -0.6)$ and $(0.7, 1)$ on figure 1). For $\tau^* = 0.3$ the deterioration is visible, but not as strong.

The worst reconstruction is for asymmetric f_3 and for oscillating f_4 and f_5, but even in these cases the recovered function very much resembles its original. The deterioration of the images for oscillating functions is expected, and in the asymmetric case a possible explanation can be linked to a relatively large perturbation f_3.

An interesting feature of these examples is the space localization: most likely due to the fast decay of the Gaussian kernel α, the values of the data at distant points seem not to influence reconstruction near $y = 0$ ($s = s^*$). Indeed, for smaller τ^* recovered volatility practically coincides with the given one on a small interval $(-0.2, 0.2)$ on figures 1,2,4,5.

Similar local reconstruction was observed in [9]: it is remarkably better on the interval around s^* where the ratio of volatility to its value at s^* is less than 0.3. On the other hand, there is a difference between the numerical experiment in [9] and our tests: the data in [9] are exact, while our are approximate (due to linearization). Their reconstruction looks better on a smaller interval (after rescaling) and for smaller τ^*.

5 Conclusion

Uniqueness results can be certainly generalized to the many-dimensional case and probably to some parabolic equations with variable coefficients. The inverse option pricing problem is a particular case of the more general inverse diffusion problem which has a probabilistic interpretation. We are not aware of any uniqueness results about recovery of diffusion rate from probability of distribution at a fixed moment of time. The methods of proofs of Theorems 2.1, 3.3 can be applied at least to a linearized version of this inverse probabilistic problem.

The algorithm in section 4 is expected to perform very well when volatility is not changing fast with respect to stock price s and is changing very slow with respect to time. Probably, a minor modification of the proposed model (replacing $\mathbf{R}$ in (3.12) by a finite interval) can eliminate difficulties with existence theorem and generate better numerical algorithms. Observe, that to find continuous f the data F must be at least twice differentiable on ω, so the real market data are in need of a proper interpolation. A choice of an appropriate smoothing interpolation and an intensive numerical testing is a subject of future work.

For simplicity we considered only European options. We hope to adjust the linearization technique to American and more complicated options, which are in particular described by free boundary problems. So far there are actually no results in this very important practical case.

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