Insiders' Hedging in a Jump Diffusion Model

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<Introduction>

- In financial markets, there are traders with several different levels of information, and their behaviours and strategies heavily depend on amount of information they have.

- We call general public who can only observe the price of an underlying asset an honest trader. Her information is mainly from newspapers and media, available to anyone. Insiders are those who have some exclusive information not available to honest traders. Examples of insiders are executives or employees of a company.

- This exclusive information often causes bigger movements than those usual diffusion can explain, and it is natural to involve this information to jump terms. We model these jumps by a doubly stochastic Poisson process, whose intensity is a function of the exclusive information X.

- The purpose of this study is to obtain a closed form local risk minimization hedging strategy for insiders.

<Model>

- We consider a market with one risky asset(S_t) and one riskless asset(R_t).
- Portfolio : a pair of processes (ξ_t, η_t),
  $V_t = \xi_t S_t + \eta_t R_t$
- Contingent Claim : $H = H(S_T)$
- cost process of (ξ_t, η_t): $C_t = V_t - \int_0^t \xi_s dS_s$, $(0 \leq t \leq T)$
- optimal strategy : The portfolio (ξ_t, η_t) whose cost process C_t is a square integrable martingale orthogonal to M under $\mathbf{P}$ and $V_T = H$.

The price process of the risky asset S follows the SDE:

$$dS_t = f(S_{t-})dB_t + g(S_{t-})dR_t + h(S_{t-})dt \qquad (1)$$

$$0 \leq t \leq T$$

where

- $R_t = \sum_{n=1}^{N_t} U_n$
- $N_t - \int_0^t \lambda(X_s)ds =$ a local martingale under $\mathbf{P}$
- X, which is a firm specific information available only to insiders, satisfies the stochastic differential equation $dX_t = \alpha(X_t)dt + \beta(X_t)dB'_t$ for $0 \leq t \leq T$.

- B' is another standard Brownian motion under $\mathbf{P}$, which can be the same as, correlated to, or independent of B.

- $U_n s$ are i.i.d and have a density ν

- $U_n s$ denote the jump sizes of S_t and has mean 0 and a finite second moment σ^2.

- For all x, y, $\frac{g(x)h(x)}{f(x)^2+g(x)^2\lambda(y)\sigma^2} < 1$ and $\frac{g(x)f(x)}{f(x)^2+g(x)^2\lambda(y)\sigma^2} < 1$.

- We assume that S_t is a $\mathcal{H}^2$ semimartingale with the decomposition $S_t = M_t + A_t$, where $M_t = \int_0^t (f(S_{s-})dB_s + g(S_{s-})dR_s)$ is a local martingale and $A_t = \int_0^t h(S_{s-})ds$ is the predictable part.

- Since S_t is a $\mathcal{H}^2$ semimartingale, M_t is not only a local martingale, but also a square integrable martingale under $\mathbf{P}$.

- insiders observe both S and X, while general public observes only S, which means an insider has a larger filtration $\mathcal{G}_t = \sigma(S_s, X_s, 0 \leq s \leq t)$ and honest traders have only $\mathcal{F}_t = \sigma(S_s, 0 \leq s \leq t)$.

<The Minimal Martingale Measure>

Föllmer -Schweizer proposed a method to choose a good martingale measure.

Definition 1 *A martingale measure* $\widehat{\mathbf{P}} \approx \mathbf{P}$ *will be called* minimal *if*

$$\widehat{\mathbf{P}} = \mathbf{P} \, on \, \mathcal{F}_0,$$

and if any square-integrable $\mathbf{P}$ *martingale* L_t *which satisfies* $\langle L, M \rangle = 0$ *where* M_t *is the martingale part of* S_t *in the canonical decomposition under* $\mathbf{P}$*, remains a martingale under* $\widehat{\mathbf{P}}$*.*

Theorem 1 *Let*

$$Y_t = -\int_0^t \frac{h(S_{s-})}{f(S_{s-})^2 + g(S_{s-})^2 \lambda(X_s)\sigma^2} (f(S_{s-})dB_s + g(S_{s-})dR_s).$$

and assume that $E\exp(2Y_t)$ *is bounded for every* $t \leq T$*. Define*

$$Z_t = 1 - \int_0^t \frac{Z_{s-}h(S_{s-})}{f(S_{s-})^2 + g(S_{s-})^2 \lambda(X_s)\sigma^2} (f(S_{s-})dB_s + g(S_{s-})dR_s).$$

Then, $Z_t > 0$ *and* $E(Z_t) = 1$ *for all* $t \in (0, T]$*. Furthermore,* $\mathbf{Q}$ *defined by* $\frac{d\mathbf{Q}}{d\mathbf{P}} = Z_T$ *is the unique minimal martingale measure of* S*.*

Proof : Doob-Meyer decomposition of M_t, Girsanov-Meyer theorem, Kunita-Watanabe inequality, Uniqueness of SDE, Stochastic Exponential, a condition on a local martingale to be a martingale

<$\mathbf{Q}$-dynamics of R, B >

Lemma 1 *Assume that $\mathbf{Q} \ll \mathbf{P}$ and let Z be the density process. Let $\mu = \mu(\omega; dt, dx)$ be an integer-valued random measure on $\mathbb{R} \times E$, and denote by $\upsilon = \upsilon(\omega; dt, dx)$ its $\mathbf{P}$-compensator of μ. Let Y be any nonnegative version of $M_\mu^{\mathrm{P}}(\frac{Z}{Z_-} 1_{\{Z_- > 0\}} | \widetilde{\mathcal{P}})$ and υ' be a version of the $\mathbf{Q}$ compensator. Then*

$$\upsilon'(\omega; dt, dx) = Y(\omega; t, x) \upsilon(\omega; dt, dx)$$

$\mathbf{P}$-*a.s.*

Proposition 1 *Under $\mathbf{Q}$,*

$$\widetilde{B}_t = B_t + \int_0^t \frac{h(S_{s-})}{f(S_{s-})^2 + g(S_{s-})^2 \lambda(X_s) \sigma^2} f(S_{s-}) ds$$

is a Brownian Motion.[*]

Proof : Girsanov-Meyer theorem, Lévy theorem

Proposition 2 *The compensated measure of $p^R(dt, dx)$ is given by*

$$\begin{aligned} q^*(dt, dx) = {} & p^R(dt, dx) \\ & - (1 - \frac{h(S_{t-}) g(S_{t-})}{f(S_{t-})^2 + g(S_{t-})^2 \lambda(X_t) \sigma^2} x) \lambda(X_t) \nu(dx) dt. \end{aligned}$$

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[*] **We can similarly construct $\tilde{B}'$ for B'.**

<Sufficient Conditions of the Existence of an Optimal Strategy>

Proposition 3 *The existence of an optimal strategy is equivalent to a decomposition*

$$H = V_0 + \int_0^T \xi_s^H dS_s + L_T^H \tag{2}$$

where $L^H = L_t^H$ is a square integrable martingale orthogonal to M. For such a decomposition, the associated optimal strategy (ξ_t, η_t) is given by $\xi_t = \xi_t^H$, $\eta_t = V_t - \xi_t S_t$, where $V_t = V_0 + \int_0^t \xi_s^H dS_s + L_t^H$.

Proposition 4 *Suppose that $V_t = E_{\mathbf{Q}}[H(S_T)|\mathcal{F}_t]$ has a decomposition $V_t = V_0 + \int_0^t \xi_s^H dS_s + L_t$ where L_t is a square integrable $\mathbf{P}$ martingale such that $\langle L, M \rangle_t = 0$ under $\mathbf{P}$. Then ξ_t^H is given by*

$$\xi_t^H = \frac{d\langle V, S\rangle_t}{d\langle S, S\rangle_t} \tag{3}$$

where the conditional quadratic variations are calculated under $\mathbf{P}$.

$<$A closed form of the local risk minimization(special case) $>$

Let us consider the case where two Brownian motions, B and B', are either same or independent. Define $I = 1$ if two Brownian motions are same, and $I = 0$ when they are independent, $[B, B'] = 0$.

We can easily see that the vector process (S, X) is Markov. Using this property, let us assume $v(t, S_t, X_t) = E_{\mathbf{Q}}[H(S_T)|S_t, X_t]$ is $C^{1,2,2}$. Then we expand it using Itô's formula. Then, using a martingale property of $v(t, S_t, X_t)$ under $\mathbf{Q}$, we get the following integro-differential equation

$$
\begin{aligned}
&\int_0^t v_t(u,S_{u-},X_u)du + \int_0^t v_s(u,S_{u-},X_u)h(S_u)du \\
&\quad + \int_0^t v_x(u,S_{u-},X_u)\alpha(X_u)du \\
&\quad + \frac{1}{2}\int_0^t v_{ss}(u,S_{u-},X_u)f(S_u)^2du + \frac{1}{2}\int_0^t v_{xx}(u,S_{u-},X_u)\beta(S_u)^2du \\
&\quad + \int_0^t v_{xs}(u,S_{u-},X_u)f(S_u)\beta(X_u)Idu \\
&\quad + \int_0^t f(S_{u-})v_s(u,S_{u-},X_u)\frac{h(S_{u-})}{f(S_{u-})^2+g(S_{u-})^2\lambda(X_u)\sigma^2}f(S_{u-})du \\
&\quad + \int_0^t \beta(X_u)v_x(u,S_{u-},X_u)\frac{h(S_{u-})}{f(S_{u-})^2+g(S_{u-})^2\lambda(X_u)\sigma^2}f(S_{u-})du \\
&\quad - \int_0^t \int_{\mathbb{R}} \{v(u,S_{u-}(1+x\frac{g(S_{u-})}{S_{u-}}),X_u) - v(u,S_{u-},X_u)\} \\
&\quad \times (1-\frac{g(S_{u-})h(S_{u-})}{f(S_{u-})^2+g(S_{u-})^2\lambda(X_u)\sigma^2}x)\lambda(X_u)\nu(dx)du = 0, \qquad (4)
\end{aligned}
$$

for almost all t almost surely.

We also obtain the relation from (4)

$$
\begin{aligned}
v(t,S_t,X_t) &= v(0,S_0,X_0) + \int_0^t v_s(u,S_{u-},X_u) f(S_u) d\widetilde{B}_u \\
&\quad + \int_0^t v_x(u,S_{u-},X_u)\beta(X_u) d\widetilde{B'}_u \\
&\quad + \int_0^t \int_{\mathbb{R}} \{v(u, S_{u-}(1+\frac{g(S_{u-})}{S_{u-}}x), X_u) - v(u,S_{u-},X_u)\} q^*(dx,du).
\end{aligned} \tag{5}
$$

Theorem 2 *Suppose that our contingent claim H is in $L^2(\mathbf{P})$. Let us consider the case when B and B' are independent or same. If the decomposition of $V_t = V_0 + \int_0^t \xi_s^H dS_s + L_t$ exists, then we get the local risk minimization strategy*

$$
\xi_t^H = \frac{j(t,S_{t-},X_t) + f(S_t)^2 v_s(t,S_{t-},X_t)}{f(S_{t-})^2 + g(S_{t-})^2\sigma^2\lambda(X_t)}, \tag{6}
$$

where

$$
\begin{aligned}
j(t,S_{t-},X_t) = g(S_{t-}) \int_{\mathbb{R}} &\{v(u, S_{u-}(1+\frac{g(S_{u-})}{S_{u-}}x), X_u) - v(u,S_{u-},X_u)\} \\
&x\nu(dx)\lambda(X_t) + f(S_{t-}) v_x(t,S_{t-},X_t)\beta(X_t) I.
\end{aligned}
$$

Proof: simple calculation of

$$
\xi_t^H = \frac{d\langle V,S\rangle_t}{d\langle S,S\rangle_t}
$$

<A closed form of the local risk minimization(general case) >

Define

$$A_t = B'_t - B_t$$

Then, from $\langle B', B' \rangle_t = \langle B + A, B + A \rangle_t = t$ and $\langle B, B \rangle_t = t$,,we get

$$\langle B, B' \rangle_t = t - \frac{1}{2} \langle A, A \rangle_t$$

So, we get the following theorem.

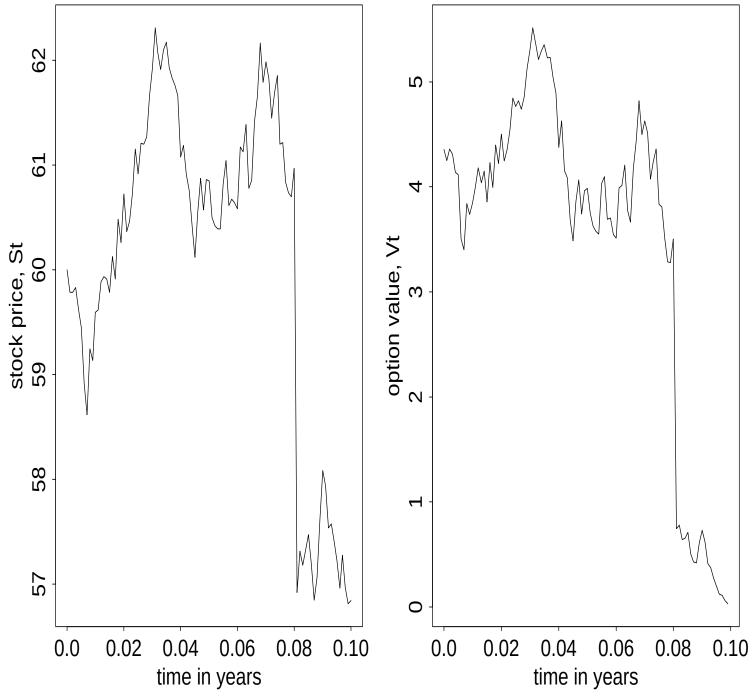
Theorem 3 *Suppose that our contingent claim H is in $L^2(\mathbf{P})$. If $d\langle A, A \rangle_t = D_t dt$, the local risk minimization strategy is given by*

$$\xi_t^H = \frac{j(t, S_{t-}, X_t) + f(S_t)^2 v_s(t, S_{t-}, X_t)}{f(S_{t-})^2 + g(S_{t-})^2 \sigma^2 \lambda(X_t)}, \tag{7}$$

where

$$j(t, S_{t-}, X_t) = g(S_{t-}) \int_{\mathbb{R}} \{ v(u, S_{u-}(1 + \frac{g(S_{u-})}{S_{u-}} x), X_u) - v(u, S_{u-}, X_u) \}$$
$$x\nu(dx)\lambda(X_t) + f(S_{t-}) v_x(t, S_{t-}, X_t) \beta(X_t)(1 - \frac{1}{2} D_t).$$

$<$Example of paths of S_t and V_t $>$



$f(x) = 0.15x$, $g(x) = x$ and $h(x) = 0.05x$

$dX_t = 0.3X_t dB'_t$, B' independent of B.

$U_n \sim \text{Unif}(-.25, .25)$

$\lambda(X_t) = \min(\max(10X_t, 1.3332), 10)$