Connecting univariate smiles and basket dynamics: a new multidimensional dynamics for basket options

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Abstract

A new approach to modelling and pricing derivative securities based on many underlying assets is developed, with the ultimate, practical aim to properly price such derivatives when each underlying shows a volatility smile/skew. We show that the proposed multidimensional model can indeed account for the observed implied volatility smiles for a range of single securities, when each single-asset volatility smile is modeled according to a density-mixture dynamical model. Extending in an intuitive way the model from the univariate to the multivariate setting, this theory allows to sample from an entirely new type of dynamics that still enjoys an internal consistency with the observed volatility surfaces for the individual securities.

The computational implications have a strong impact on the calculation of prices of European options on baskets of securities. Operative proposals for the use of this model to deduce skews on baskets of many stocks/FX rates will be presented. A natural extension to the case of the Libor Market Model would allow computing in a quasianalytical fashion the swap rates' smile given the smiles in the individual caplets.

Introduction

It is known that the Black–Scholes model [2] does not price all European options quoted on a given market in a consistent way. In fact, their model lies on the fundamental assumption that the asset price volatility is a constant. In reality, the implied volatility (*i.e.* the volatility parameter that, when plugged into the Black–Scholes formula, allows to reproduce the market price of an option) generally shows a dependence on both the option maturity and strike. If there were no dependence on strike one could extend the model in a straightforward fashion by allowing a deterministic dependence of the underlying's instantaneous volatility on time, so that the dynamics could be represented by the following stochastic differential equation (SDE):

$$dS_t = \mu S_t dt + \sigma_t S_t dW_t, \tag{1}$$

σ_t being the deterministic instantaneous volatility referred to above. In that case, reconstruction of the time dependence of σ_t would follow by considering that, if $v(T_i)$ denotes the implied volatility for options maturing at time T_i, then

$$v(T_i)^2 T_i = \int_0^{T_i} \sigma_s^2 ds. \tag{2}$$

Implied volatility however does indeed show a strike dependence; in the common jargon, this behaviour is described with the term *smile* whenever volatility has a minimum at the forward asset price level, or *skew*

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when low–strike implied volatilities are higher than high–strike ones. In the following we will loosely speak of both effects as "volatility smile".

The aim of this paper is to incorporate the effect of the volatility smile observed on the market when pricing and hedging *multiasset* securities. A whole lot of such structured securities is nowadays offered to institutional and retail investors, in the form of options on baskets of stocks/FX rates and on combinations of forward interest rates such as e.g. European/Bermudan swaptions. In our approach we remain within a local volatility model for the individual assets composing the underlying of the option (be it a basket of stocks or a swap rate) that has proved to be quite effective in accounting for the observed single assets' smiles, but we move one step beyond the approach used on the street when writing the joint multiasset dynamics: instead of discretizing the multidimensional SDE through, say, Euler or higher order numerical schemes [22] under the assumption of a given instantaneous correlation structure, we incorporate the correlation in a new scheme that enjoys analytic multivariate densities and a fully analytic dynamics. In so doing we are able to sample a new manifold of instantaneous covariance structures (and a new manifold of dynamics) which ensures full compatibility with the individual volatility smiles.

The outline of the paper is the following: first, a brief (and by no means exhaustive) review of the main approaches to single asset smile modelling will be presented, along with the one that has been developed by this group [5, 6, 7]. Then, examples of typical securities that need a multivariate setting for proper pricing. The extension of the single asset model to the multivariate framework will follow, with a thorough discussion of the implications for the dynamics stemming from a naïve approach and from ours. In the end, the conclusions and proposals for further research.

Smiles in recent literature

Trying to incorporate the smile effect into a consistent theory has been the subject of a lot of research in recent years. Generally speaking four main streams of investigation can be identified:

- postulating an *alternative explicit dynamics* for the asset–price process that by construction ensures the existence of volatility smiles or skews. An example is the CEV process proposed by Cox [11] and Cox and Ross [12]. A general class of problems was presented by Carr *et al.* [10]. In general this approach does not reproduce accurately enough the market volatility structures.
- postulating a *continuum of traded strikes* [3]. This was extended yielding an explicit expression for the Black–Scholes implied volatility as a function of strike and maturity [15, 16, 13, 14]. This approach however needs a smooth interpolation of option prices between consecutive traded strikes and maturities. Explicit expressions for the risk–neutral stock price dynamics were also derived by minimizing the relative entropy to a prior distribution [1] and by assuming an analytical function describing the volatility surface [9].
- Another approach consists of finding the risk–neutral distribution on a lattice model for the underlying that leads to a best fit of the market option prices subject to a smoothness criterion [20, 8]. This approach has the drawback of being fully numerical.
- The last approach is an incomplete market approach, and includes stochastic volatility models [19, 18, 25] and jump–diffusion models [24].

A possible solution to the problem of finding a risk–neutral distribution that consistently prices all quoted options is given by assuming a particular parametric risk–neutral distribution dependent on several parameters and use the latter in conjunction with a calibration procedure to the market option prices. The model earlier proposed [5, 6, 7] finds a dynamics leading to a parametric risk–neutral distribution flexible enough for practical purposes.

The mixture of densities (MD) model

It is based on the hypothesis that, fixing a time T and denoting by P_{0T} the price at time 0 of the zero–coupon bond maturing at T, the dynamics of the asset underlying a given option market takes the form

$$dS_t = \mu S_t dt + \sigma(S_t, t) S_t dW_t \tag{3}$$

under the T–forward measure Q^T. Here, μ is a constant, W is a standard Q^T Brownian motion and σ (the "local volatility") is a well behaved deterministic function. In order to guarantee the existence of a unique strong solution to the above SDE, σ is assumed to satisfy the linear growth condition

$$\sigma^2(x,t)x^2 \leq L(1+x^2) \ \text{ uniformly in } t \tag{4}$$

for a suitable positive constant L.

Considering N diffusion processes $\mathfrak{z}_t^{(k)}$ with dynamics

$$d\mathfrak{z}_t^{(k)} = \mu \mathfrak{z}_t^{(k)} dt + v_k(\mathfrak{z}_t^{(k)}, t) dW_t \tag{5}$$

with marginal densities $p_t^{(k)}$ and v_k satisfying linear growth conditions, the marginal density p_t of S_t is assumed to be representable as the superposition of the $p_t^{(k)}$ [5, 6, 7]:

$$p_t(S) = \sum_k \lambda_k p_t^{(k)} \ \text{ with } \ \lambda_k \geq 0, \forall k \ \text{ and } \ \sum_k \lambda_k = 1. \tag{6}$$

The problem can then be cast in the following form: is there a local volatility σ for Eq. (3) such that Eq. (6) holds? Purely formal manipulation of the related Kolmogorov forward equation

$$\frac{\partial p_t}{\partial t} + \frac{\partial p_t}{\partial x}(\mu x p_t) - \frac{1}{2}\frac{\partial^2}{\partial x^2}(\sigma^2(x,t)x^2 p_t) = 0 \tag{7}$$

and of analogous equations for the $p_t^{(k)}$'s shows that a candidate σ is

$$\sigma(x,t) = \sqrt{\frac{\sum_{k=1}^N \lambda_k v_k^2(x,t) p_t^{(k)}}{\sum_{k=1}^N \lambda_k p_t^{(k)}}}. \tag{8}$$

A digression is in order now: suppose that everything in the above approach works fine, and with a judicious choice of the v_k's the unique strong solution for the dynamics of S exists. Then the pricing of European options on S would be immediate: given any call option, say, with strike K and maturity T, the option value at present time would be

$$O = P_{0T}\mathbb{E}^T\{[S_T - K]^+\} = \sum_{k=1}^N \lambda_k P_{0T} \int_0^{+\infty} [x-K]^+ p_T^{(k)}(x) dx = \sum_{k=1}^N \lambda_k O_k. \tag{9}$$

In plain words, the option price would simply be the weighted average of prices of call options written on the auxiliary processes $\mathfrak{z}$. Hedging would be just as easy. As a consequence, if the auxiliary densities $p_t^{(k)}$ are chosen so that the option prices O_k are computed analytically and allow for sufficient flexibility, a fast, stable and accurate calibration to the whole implied volatility structure over a wide range of option strikes and maturities follows [7].

The most natural choice for the $(\mathfrak{z}^{(k)}, v_k, p_t^{(k)})$ triplet is

$$
\begin{cases}
\mathfrak{z}_0^{(k)} = S_0 \\
v_k(x,t) = x\sigma_k(t) \\
V_k(t) = \sqrt{\int_0^t \sigma_k(s)^2 ds} \\
p_t^{(k)}(x) = \frac{1}{\sqrt{2\pi} x V_k(t)} \exp\left[-\frac{1}{2V_k^2(t)}\left(\ln\left(\frac{x}{S_0}\right) - \mu t + \frac{1}{2}V_k^2(t)\right)^2\right]
\end{cases}
\tag{10}
$$

with σ_k deterministic (*mixture of lognormal densities*).

It has been proven that with the above choice and additional nonstringent assumptions on the v_k the corresponding dynamics for S_t admits indeed a unique strong solution. A greater flexibility can also be achieved by shifting the auxiliary processes' density by a carefully chosen deterministic function of time (still preserving risk–neutrality). This is the so–called *shifted lognormal density* model.

We are not going in any more detail on the single asset case; the above description gives a sufficient base for presenting our generalization of the mixture of lognormal densities to the multivariate setting (as before, at first on the basis of pure formal manipulations, then with full rigour) with the specific aim of finding a method to infer the "implied volatility" of a basket of securities from the individual components and/or an explicit dynamics for the multiasset system. Later in the paper, formal proofs of the general consistency of the model and of the existence and uniqueness of the solution to the multivariate version of Eq. (3) will be provided.

Formulation of the problem

One of the aims of this approach is to be able to compute the smile effect on the implied volatilities for basket options, *i.e.* options on the average

$$
B_t = \sum_k w_k S_t^{(k)}, \tag{11}
$$

where $S_t^{(k)}$ is the k–th component of the basket, and may represent a stock price, to fix ideas. (It must be noted in passing that $S^{(k)}$ could also represent a forward rate process $F_k(t)$ in the Libor Market Model (LMM) and instead of (11) we could have a more complicated expression representing a swap rate.)

Such options have the most varied nature: from the plain European call/put options on the value of the basket at maturity T,

$$
\Pi_T = [\omega(B_T - K)]^+ \tag{12}
$$

($\omega = \pm 1$ for a call/put respectively), to options somewhat more complicated, such as Asian options on the basket, Himalaya options, rainbow options and so on.

In the sequel, and for the ultimate purpose of this application, we will be concerned only with European options on a basket. Even the pricing of a plain call on a basket made of five stocks can be a formidable problem in its own right, in both the analytic and computational sense.

It is customary to seek for analytic (and by all means approximate, dealing with multisecurities) formulæ for the prices of such options. Some of these formulæ have been examined elsewhere [4] along with their regimes of validity in terms of volatility and correlation structures of the individual stocks composing the basket. They typically rely on dictating a specific dynamics for the "basket price" B_t, and solving the pricing problem through the formulæ implied by such dynamics. And, surprise surprise!, the geometric Brownian motion (GBM) postulate for the dynamics of B_t is often used on the market.

In other words, the appeal of the GBM model and of Black–Scholes' theory is still so wide that, even if the problem at hand is intrinsically multidimensional, we are still led to think of the basket value in (11) as having a lognormal dynamics, thus neglecting its internal composition in terms of stocks having each its own dynamics and character. The approximation goes even so far in our minds that we naturally start to wonder, at some point, if we can somehow keep into consideration possible smile effects on the basket. Because the main limitation of the formulæ (apart from being approximate, and yielding grossly wrong answers in specific cases [4]) is that they structurally cannot keep into account any smile effect on the individual stocks' volatility, and therefore on the "basket volatility".

In the following we will tackle the problem in a rigorous way, through the generalization of a dynamical model that has proven to perform quite well on some markets [5, 6, 7] and that is under extension to the equity markets case.

An extension of the MD model to handle multivariate problems

To fix ideas, suppose we are faced with the following problem: we want to price an option maturing at T on the basket of securities of Eq. (11). Each of these securities will have a "smiley" volatility structure, and we strongly suspect that the basket will show a smile in its implied volatility, too.

Through Eqs. (3–10) we now have a piece of machinery that allows us to calibrate an MD to the implied volatility smile structure of the individual component $S^{(k)}$ of the basket. Suppose we have already calibrated the individual MDs to such smile surfaces, thus finding the local volatilities governing the dynamics of each $S^{(k)}$. In order to price a plain European option on the basket the traditional approach suggests to use a Monte Carlo simulation that samples suitably discretized paths according to the drift rate of each component (risk–free minus dividend yield) and to the diffusion matrix given by the local volatility function deriving from the mixture of densities model. Therefore, assuming to have an exogenously computed structure of instantaneous correlations ρ_{ij} (computed e.g. through historical analysis and supposed constant over time) among the stocks' returns, we could apply a naïve Euler Monte Carlo scheme and simulate the joint evolution of the stocks through a suitably discretized time grid $\tau_1 = 0 \cdots \tau_N = T$ with a covariance matrix whose (i, j) component over the (τ_l, τ_{l+1}) propagation interval (constant in the Euler scheme) is

$$\tilde{C}_{ij}(S_i, S_j, t) = \sqrt{\frac{\sum_{k=1}^{\nu} \lambda_{ik}\sigma_{ik}(\tau_l) p_{\tau_l}^{(ik)}(S_i)}{\sum_{k=1}^{\nu} \lambda_{ik} p_{\tau_l}^{(ik)}(S_i)}} \sqrt{\frac{\sum_{k=1}^{\nu} \lambda_{jk}\sigma_{jk}(\tau_l) p_{\tau_l}^{(jk)}(S_j)}{\sum_{k=1}^{\nu} \lambda_{jk} p_{\tau_l}^{(jk)}(S_j)}} \rho_{ij}. \tag{13}$$

(It must be mentioned that quite recently an interesting extension of this approach was suggested [17] based on the assumption that basket option prices be available at all strikes and maturities, *á la* Dupire [15, 16]. This is however not generally the case in real markets). It can easily be shown that the approach of Eq. (13) is consistent with both the individual dynamics induced by a MD model for each stock and with the instantaneous correlation structure ρ_{ij} imposed. However, besides the practical and heartwarming possibility of controlling the instantaneous correlation, one must be aware of its main limitations:

Remark. *By imposing that the instantaneous covariance of the multidimensional process be of the form (13) one is moving only within a given manifold of the possible local volatility structures for the multidimensional process. Moreover, this scheme imposes costly discretization procedures on the* $\{\tau_i\}$ *grid for pricing even the simplest derivatives.*

One could try to do something different and approach the problem so that, under suitable assumptions, the individual MD models (one for each stock, separately calibrated each on its volatility surface) could be merged so as to provide a coherent multiasset model that allows for a degree of (semi)analytic tractability comparable to the one typical of the univariate case.

For this purpose we need to briefly revise some properties of the multidimensional Itô processes, and extend in a trivial fashion the existing MD model.

The multidimensional Kolmogorov equation

Consider an n–dimensional stochastic process $\mathbf{x}(t)$ whose generic i–th component follows the SDE

$$\frac{dx_i(t)}{x_i(t)} = \mu_i(t)dt + \mathbf{C}_i(\mathbf{x}, t) \cdot d\mathbf{W}_t \tag{14}$$

where μ_t and $\mathbf{C}_i(\mathbf{x}, t)$ are deterministic functions of time and of the state of the process. Bold letters will denote vector/matrix quantities in the following. $\mathbf{W}_t$ is a standard d–dimensional Brownian motion, $d \leq n$.

Denoting $C_{ij}(\mathbf{x}, t) = \mathbf{C}_i(\mathbf{x}, t) \cdot \mathbf{C}_j(\mathbf{x}, t)$, the associated Kolmogorov forward PDE to be satisfied by the corresponding probability density is

$$\frac{\partial p_t}{\partial t} + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}[\mu_t^{(i)} x_i p_t] - \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}[C_{ij} x_i x_j p_t] = 0 \tag{15}$$

where all functions are evaluated at $(\mathbf{x}, t)$.

The existence and uniqueness theorem for strong solutions still holds for a vector SDE, provided that the generalized linear growth conditions hold: that there exists a positive K such that

$$\|\mathbf{C}(\mathbf{x}, t)\|^2 \|\mathbf{x}\|^2 \leq K(1 + \|\mathbf{x}\|^2) \tag{16}$$

and similarly for μ_t [21]. The symbol $\|\|$ denotes here vector and matrix norms.

Inspired by the univariate approach which gave rise to the MD model, let us postulate that the density at any time t of the multivariate process $\mathbf{x}$ be equal to a weighted average of densities $p_t^{(k)}$

$$p_t(\mathbf{x}) = \sum_{k=1}^{N} \lambda_k p_t^{(k)}(\mathbf{x}), \qquad \lambda_k \geq 0 \ \forall k, \qquad \sum_{k=1}^{N} \lambda_k = 1 \tag{17}$$

each corresponding to a given dynamics corresponding to the Kolmogorov equation

$$\frac{\partial p_t^{(k)}}{\partial t} + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}[\mu_t^{(i)} x_i p_t^{(k)}] - \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}[\sigma_{ij}^{(k)}(\mathbf{x}, t) x_i x_j p_t^{(k)}] = 0, \tag{18}$$

The condition that p_t satisfy Eq. (15) and that each $p_t^{(k)}$ satisfy the corresponding equation leads through standard algebra to the PDE

$$\frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} \left[\left(C_{ij}(\mathbf{x}, t) p_t - \sum_{k=1}^{N} \lambda_k \sigma_{ij}^{(k)}(\mathbf{x}, t) p_t^{(k)} \right) x_i x_j \right] = 0. \tag{19}$$

This is a rather complex PDE, that we solve by setting, for all i, j [1]

$$C_{ij}(\mathbf{x}, t) = \frac{\sum_{k=1}^{N} \lambda_k \sigma_{ij}^{(k)}(\mathbf{x}, t) p_t^{(k)}}{\sum_{k=1}^{n} \lambda_k p_t^{(k)}}. \tag{20}$$

[1] A heuristic argument suggests that this is the only possible solution with physical meaning: it can be easily proven that the most general solution of the equation $\sum_{ij} \frac{\partial^2}{\partial x_i \partial x_j} f_{ij}(\mathbf{x}) = 0$ has a Fourier transform satisfying $(\mathbf{q}, f(\mathbf{q})\mathbf{q}) = 0$. The only matrix function $f(\mathbf{q})$ satisfying it and infinitely differentiable with respect to $\mathbf{q}$ is constant. This constant must be zero in order to have finite first and second moments of the multivariate density p_t.

The connection between univariate and multivariate MD

Let us make a strong assumption, namely that the volatility coefficient for the k–th "base" density $p_t^{(k)}$ of Eq. (18) is a deterministic function of time, independent of the state, and of the particular form $\sigma_{ij}^{(k)}(\mathbf{x},t) = \sigma_i^{(k)}(t) \cdot \sigma_j^{(k)}(t)$.

Under this hypothesis we already know the dynamics corresponding to Eq. (18), and we can explicitly write the density $p_t^{(k)}$ statisfying it at all times:

$$p_t^{(k)}(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det \Xi^{(k)}(t)}\Pi_{i=1}^n x_i} \exp\left[-\frac{\tilde{\mathbf{x}}(\Xi^{(k)}(t))^{-1}\tilde{\mathbf{x}}}{2}\right], \tag{21}$$

where $\Xi^{(k)}(t)$ is the $n \times n$ integrated covariance matrix of returns for the many components of the process $\mathbf{x}$:

$$\Xi_{ij}^{(k)}(t) = \int_0^t \sigma_i^{(k)}(s) \cdot \sigma_j^{(k)}(s) ds \tag{22}$$

($\Xi^{(k)}$ is assumed to be invertible at all times) and

$$\tilde{x}_i = \ln x_i - \ln x_i(0) - \int_0^t \left(\mu_s^{(i)} - \frac{{\sigma_i^{(k)}}^2(s)}{2}\right) ds. \tag{23}$$

Life is not simple unless one is ready to make the further assumption $\sigma_i^{(k)}(t) \cdot \sigma_j^{(k)}(t) = \sigma_i^{(k)}(t)\sigma_j^{(k)}(t)\rho_{ij}$. Note that this is similar to what done in (13), the main difference being that this imposition is now made at the level of the diffusion processes corresponding to the constituent densities, rather than of the actual, "physical" diffusion. At first sight this might appear innocuous: after all, we would be doing something pretty standard *if we were dealing with only one density.* But the very fact that the densities will get mixed up through Eq. (17) will have dramatic consequences on the actual structure of correlations, both instantaneous and average. But first, let us prove that under a further assumption we can be fully consistent with the dynamics specified by the MD model for the individual stocks.

Let's assume that we have calibrated an MD model for each $x_i(t)$: if $\pi_t^{(i)}$ is the density of S_i, we write

$$\pi_t^{(i)}(x) = \sum_{k=1}^{\nu_i} \lambda_{ik}\pi_t^{(ik)}(x), \quad \text{with} \quad \lambda_{ik} \geq 0, \forall k \quad \text{and} \quad \sum_k \lambda_{ik} = 1 \tag{24}$$

$\pi_t^{(ik)}$ being a suitably modified lognormal density (by suitably modified we mean any modification such as e.g. shifts in the process [5, 6, 7]; no need to go into details right now. The most general case of shifted lognormal densities is a trivial extension of the present approach.)

For notational simplicity we will assume that the number of base densities ν_i will be the same, ν, for all stocks. The exogenous correlation structure ρ_{ij} is given by the symmetric, positive–definite matrix R.

The most natural tentative choice for the base densities of Eq. (17) is

$$p_t(\mathbf{x}) = \sum_{i_1,i_2,\cdots i_n=1}^{\nu} \lambda_{1,i_1} \cdots \lambda_{n,i_n} p_t^{(i_1\cdots i_n)}(\mathbf{x}) \tag{25}$$

with

$$p_t^{(i_1\cdots i_n)}(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det \Xi^{(i_1\cdots i_n)}(t)}\Pi_{i=1}^n x_i} \exp\left[-\frac{\tilde{\mathbf{x}}^{(i_1\cdots i_n)}\Xi^{(i_1\cdots i_n)}(t)^{-1}\tilde{\mathbf{x}}^{(i_1\cdots i_n)}}{2}\right]. \tag{26}$$

Here, $\Xi^{(i_1\cdots i_n)}(t)$ is the integrated covariance matrix whose (l,m) element is

$$\Xi_{lm}^{(i_1\cdots i_n)}(t) = \int_0^t \sigma_{l,i_l}(s)\sigma_{m,i_m}(s)\rho_{lm} ds \tag{27}$$

and, generalizing Eq. (23)

$$\tilde{x}_l^{(i_1\cdots i_n)} = \ln x_l - \ln x_l(0) - \int_0^t \left(\mu_s^{(l i_l)} - \frac{\sigma_s^{(l i_l)^2}}{2} \right) ds. \tag{28}$$

Putting notational complexity aside, what we are ultimately doing is to mix in all possible ways the component densities for the individual stocks, still ensuring consistency with the starting models for the components stocks, and imposing the instantaneous correlation structure R at the level of the constituent densities.

This is indeed consistent with the mixture of densities models through which we have specified the dynamics of the single components of $\mathbf{x}$:

Proposition. *Given any $f : \mathbb{R} \longrightarrow \mathbb{R}$ and any $t \geq 0$, under the assumptions of Eqs. (25)–(27) the expectation of $f(x_l(t))$ is*

$$\mathbb{E}_0\{f(x_l(t))\} = \int dx_l f(x_l) \pi_t^{(l)} \tag{29}$$

$\pi_t^{(l)}$ being given by Eq. (24).

Proof. The proof is trivial: it is enough to compute the multiple integral

$$\begin{aligned} \mathbb{E}_0\{f(x_l(t)\} &= \int dx_1 \cdots \int dx_l \cdots \int dx_n f(x_l) p_t(\mathbf{x}) \\ &= \sum_{i_1,i_2,\cdots i_n=1}^{\nu} \lambda_{1,i_1} \cdots \lambda_{n,i_n} \int dx_1 \cdots \int dx_l \cdots \int dx_n f(x_l) p_t^{(i_1\cdots i_n)}(\mathbf{x}) \end{aligned} \tag{30}$$

Integrating out all variables but $x^{(l)}$ in each of the integrals in the right hand side we have

$$\begin{aligned} \mathbb{E}\{f(x_l(t)\} &= \sum_{i_1,i_2,\cdots i_n=1}^{\nu} \lambda_{1,i_1} \cdots \lambda_{n,i_n} \int dx_l f(x_l) \pi_t^{(l,i_l)}(x_l) \\ &= \sum_{i_l=1}^{\nu} \lambda_{l,i_l} \int dx_l f(x_l) \pi_t^{(l,i_l)}(x_l) = \int dx_l f(x_l) \pi_t^{(l)}(x_l) \end{aligned} \tag{31}$$

since by the condition that probability integrate up to one we know that $\sum_{k=1}^{\nu} \lambda_{ik} = 1$ for all i. □

An immediate consequence of this is the following

Corollary. *Under the assumptions of Eqs. (25)–(27) all moments, at any order, at any time, of the individual components are exactly reproduced.*

In other words the theory we have built is fully consistent from the statistical viewpoint with the MD model postulated at the beginning for each component at any time. This implies also dynamical consistency, which is also directly proven in the following

Proposition. *Under the assumptions of Eqs. (25)–(27) the unconditional dynamics of each component of* $\mathbf{x}$ *is exactly the same as in the corresponding MD model.*

Proof. Take for instance the first component: using the shorthand $\mathbf{i}$ for $(i_1, \cdots i_n)$ integrate Eq. (18) with respect to $x_2, \cdots x_n$:

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\int dx_2 \cdots \int dx_n p_t^{(\mathbf{i})}\right) + \frac{\partial}{\partial x_1}\left[\mu_t^{(1)} x_1 \int dx_2 \cdots \int dx_n p_t^{(\mathbf{i})}\right] \\
&+ \int dx_2 \frac{\partial}{\partial x_2}\left[\mu_t^{(2)} x_2 \int dx_3 \cdots \int dx_n p_t^{(\mathbf{i})}\right] \\
&+ \cdots + \int dx_n \frac{\partial}{\partial x_n}\left[\mu_t^{(n)} x_n \int dx_2 \cdots \int dx_{n-1} p_t^{(\mathbf{i})}\right] \\
&- \frac{1}{2}\frac{\partial^2}{\partial x_1^2}\left[\sigma^{(1i_1)}(t)^2 x_1^2 \int dx_2 \cdots \int dx_n p_t^{(\mathbf{i})}\right] \\
&- \int dx_2 \frac{\partial^2}{\partial x_1 \partial x_2}\left[\sigma^{(1i_1)}(t)\cdot\sigma^{(2i_2)}(t) x_1 x_2 \int dx_3 \cdots \int dx_n p_t^{(\mathbf{i})}\right] \\
&- \int dx_2 \int dx_3 \frac{\partial^2}{\partial x_2 \partial x_3}\left[\sigma^{(2i_2)}(t)\cdot\sigma^{(3i_3)}(t) x_2 x_3 \int dx_4 \cdots \int dx_n p_t^{(\mathbf{i})}\right] + \cdots = 0
\end{aligned}
\tag{32}
$$

Noting that $\pi_t^{(1i_1)}(x_1) = \int dx_2 \cdots dx_n p_t^{(\mathbf{i})}(\mathbf{x})$ and denoting

$$
\pi_t^{(1i_1,li_l)}(x_1, x_l) = \int dx_2 \cdots \int dx_{l-1} \int dx_{l+1} \cdots dx_n p_t^{(\mathbf{i})}(\mathbf{x}) \tag{33}
$$

the above equation becomes

$$
\begin{aligned}
&\frac{\partial}{\partial t}\pi_t^{(1i_1)} + \frac{\partial}{\partial x_1}\left[\mu_t^{(1)} x_1 \pi_t^{(1i_1)}\right] \\
&+\mu_t^{(2)}\left[x_2 \pi_t^{(1i_1,2i_2)}\right]_{x_2=0}^{x_2=\infty} + \cdots + \mu_t^{(n)}\left[x_n \pi_t^{(1i_1,ni_n)}\right]_{x_n=0}^{x_n=\infty} \\
&-\frac{1}{2}\frac{\partial^2}{\partial x_1^2}\left[\sigma^{(1i_1)}(t)^2 x_1^2 \pi_t^{(1i_1)}\right] \\
&-\frac{\partial}{\partial x_1}\left[\sigma^{(1i_1)}(t)\cdot\sigma^{(2i_2)}(t) x_1 x_2 \pi_t^{(1i_1,2i_2)}\right]_{x_2=0}^{x_2=\infty} - \cdots \\
&-\frac{\partial}{\partial x_1}\left[\sigma^{(1i_1)}(t)\cdot\sigma^{(ni_n)}(t) x_1 x_n \pi_t^{(1i_1,ni_n)}\right]_{x_n=0}^{x_n=\infty} \\
&-\left[\sigma^{(2i_2)}(t)\cdot\sigma^{(3i_3)}(t) x_2 x_3 \pi_t^{(2i_2,3i_3)}\right]_{x_2=x_3=0}^{x_2=x_3=\infty} + \cdots = 0
\end{aligned}
\tag{34}
$$

All terms evaluated in zero and infinity in the above equation vanish because in order to have finite first and second moment with respect to all components, the following limits hold

$$
\begin{aligned}
&\lim_{x_i\to+\infty} x_i p_t(\mathbf{x}) = 0 \qquad &&\lim_{x_i\to 0} x_i p_t(\mathbf{x}) = 0 \\
&\lim_{x_i\to+\infty} x_i^2 p_t(\mathbf{x}) = 0 \qquad &&\lim_{x_i\to 0} x_i^2 p_t(\mathbf{x}) = 0
\end{aligned}
\tag{35}
$$

and similar conditions hold for the derivatives of p_t.

Each term in Eq. (18) for each base multivariate density therefore reduces to the corresponding term of the Kolmogorov equation for the one–body density of x_1. □

The consequences for pricing are immediate: suppose that $\mathbf{x}$ represents the vector $\mathbf{S}$ of stock prices composing the basket of Eq. (11). A conventional scheme for pricing a plain option on the basket in a way fully consistent with individual local volatilities would require, as already said, a sufficiently fine time discretization coupled with a Monte Carlo integration with instantaneous covariance given by Eq. (13) (or by more complicated discretization schemes for SDEs, see e.g. Milstein's [22]). The present choice allows instead to compute the option price (12) through a set of single–step Monte Carlo integrations (one integration for each combination $(i_1, \cdots i_n)$). But the actual consequences of this approach are wider, in that they affect the many–body dynamics in a deeper way.

So far we have only provided a candidate density and dynamics for the multivariate problem. We must still prove that our choice does indeed lead to a unique strong solution of the vector SDE for $\mathbf{x}_t$. Moreover a deep analysis of the consequences of this choice has to be performed.

Existence and uniqueness of a strong solution follows from the usual technical assumptions for the individual volatilities $\sigma_t^{(li_l)}$: supposing they are all bounded form above and below [5], standard algebra yields

$$\left\{ \begin{array}{l} n^2\tilde{\sigma}^2 \leq \|C\|^2 = \sum_{l,m=1}^{n} C_{lm}(\mathbf{x},t)^2 \leq n^2\hat{\sigma}^2, \\ \\ \tilde{\sigma} = \inf_{t\geq 0}\{\min_{l=1\cdots n, i_l=\ \cdots\nu}\}\sigma_t^{(l,i_l)}\} \\ \\ \hat{\sigma} = \sup_{t\geq 0}\{\max_{l=1\cdots n, i_l=\ \cdots\nu}\}\sigma_t^{(l,i_l)}\} \end{array} \right. \tag{36}$$

and the linear growth condition holds.

Choice and consequences of an exogenously given correlation structure

Multivariate mixtures: the instantaneous correlation structure

The link between the correlation structure R imposed on the constituent densities and the actual instantaneous correlation "felt" by the real process is not straightforward. An apparent inconsistency already shows up at the instantaneous variance level. Without loss of generality consider a two–dimensional process $\mathbf{x}_t = (x_t, y_t)$ modelled through a MD model. Denote by λ_k and ξ_k the coefficients of the individual MD for x and y, respectively, and by $\sigma_t^{(k)}$ and $\eta_t^{(k)}$ the corresponding instantaneous volatilities.

Recall that in a standard Monte Carlo scheme the instantaneous variance for the x component, say, of the process over the interval (τ_l, τ_{l+1}) would be (see Eq. (13))

$$\tilde{C}_{11}(x,t) = \frac{\sum_{k=1}^{\nu} \lambda_k \sigma_{\tau_l}^{(k)2} \pi_{\tau_l}^{(1k)}(x)}{\sum_{k=1}^{\nu} \lambda_k \pi_{\tau_l}^{(1k)}(x)} \tag{37}$$

and the instantaneous covariance of returns between components one and two would be

$$\tilde{C}_{12}(x,y,t) = \sqrt{\frac{\sum_{k=1}^{\nu} \lambda_k \sigma_{\tau_l}^{(k)} \pi_{\tau_l}^{(1k)}(x)}{\sum_{k=1}^{\nu} \lambda_k \pi_{\tau_l}^{(1k)}(x)}} \sqrt{\frac{\sum_{k=1}^{\nu} \xi_k \eta_{\tau_l}^{(k')} \pi_{\tau_l}^{(2k)}(y)}{\sum_{k=1}^{\nu} \xi_k \pi_{\tau_l}^{(2k)}(y)}}\rho \tag{38}$$

to be compared with the expressions

$$C_{11}(x,y,t) = \frac{\sum_{k,k'=1}^{\nu} \lambda_k \xi_{k'} \sigma_{\tau_l}^{(k)2} p_{\tau_l}^{(kk')}(x,y)}{\sum_{k,k'=1}^{\nu} \lambda_k \xi_{k'} p_{\tau_l}^{(kk')}(x,y)} \tag{39}$$

and

$$C_{12}(\mathbf{x},t) = \frac{\sum_{k,k'=1}^{\nu} \lambda_k \xi_{k'} \sigma_{\tau_l}^{(k)} \eta_{\tau_l}^{(k')} \rho p_{\tau_l}^{(kk')}(x,y)}{\sum_{k,k'=1}^{\nu} \lambda_k \xi_{k'} p_{\tau_l}^{(kk')}(x,y)} \tag{40}$$

of our model.

An evident difference is that, while in Eq. (37) the instantaneous covariance of x depends only on x itself, and not on y, the opposite is true of Eq. (39). In other words, the diffusion matrix is now *fully* dependent on the components of the multidimensional process. Moreover, the two equations (38) and (40) are structurally different. However, there must be a link between the two, because we know that in the limit when the correlation ρ between the two components vanishes, they will in fact evolve ignoring one another.

By the choice we made at the beginning, $p_t^{(kk')}$ is a bivariate lognormal density, *i.e.* it has the expression

(41)
$$p_t^{(kk')}(x,y) = \frac{1}{2\pi\sqrt{a_{11}a_{22}-\rho^2 a_{12}^2}}\frac{1}{xy}\exp\left[-\frac{1}{2}\tilde{x}^2\frac{a_{22}}{a_{11}a_{22}-\rho^2 a_{12}^2} - 2\tilde{x}\tilde{y}\frac{a_{12}\rho}{a_{11}a_{22}-\rho^2 a_{12}^2} + \tilde{y}^2\frac{a_{11}}{a_{11}a_{22}-\rho^2 a_{12}^2}\right]$$

with $\tilde{x}$ and $\tilde{y}$ defined as in Eq. (23) and

$$\begin{cases} a_{11} = \int_0^t \sigma_s^{(k)^2} ds \\ a_{22} = \int_0^t \eta_s^{(k')^2} ds \\ a_{12} = \int_0^t \sigma_s^{(k)} \eta_s^{(k')} ds \end{cases} \tag{42}$$

The tetrachoric expansion for the bivariate normal density with correlation ρ reads [26]

$$n(x,y,\rho) = n(x)n(y)\sum_{k=0}^{\infty}\frac{\rho^k}{k!}He_k(x)He_k(y) \tag{43}$$

(He_k is the kth Hermite polynomial) which, applied to $p_t^{(kk')}$ yields

$$\begin{aligned} p_t^{(kk')}(x,y) \simeq\ & \frac{1}{\sqrt{2\pi a_{11}}}\frac{1}{x}\exp\left[-\frac{1}{2a_{11}}\tilde{x}^2\right]\frac{1}{\sqrt{2\pi a_{22}}}\frac{1}{y}\exp\left[-\frac{1}{2a_{22}}\tilde{y}^2\right] \\ & + \frac{1}{\sqrt{2\pi a_{11}}}\frac{1}{x}\exp\left[-\frac{1}{2a_{11}}\tilde{x}^2\right]\frac{1}{\sqrt{2\pi a_{22}}}\frac{1}{y}\exp\left[-\frac{1}{2a_{22}}\tilde{y}^2\right]\tilde{x}\tilde{y}\frac{a_{12}}{a_{11}a_{22}}\rho + O(\rho^2) \end{aligned} \tag{44}$$

and similarly expanding Eqs. (39) and (40) we get

$$\begin{cases} C_{11}(x,y,t) = \tilde{C}_{11}(x,t) + O(\rho^2) \\ C_{12}(x,y,t) = \tilde{C}_{12}(x,y,t) + O(\rho^2) \end{cases} \tag{45}$$

In other words the traditional multivariate simulation scheme can be seen an approximation of the present one, valid for weakly correlated systems. The instantaneous correlation structure induced by this model is

$$\rho_t = \frac{\rho\sum_{k,k'=1}^{\nu}\lambda_k\xi_{k'}\sigma_t^{(k)}\eta_t^{(k')}p_t^{(kk')}(x,y)}{\sqrt{\left(\sum_{k,k'=1}^{\nu}\lambda_k\xi_{k'}\sigma_t^{(k)2}p_t^{(kk')}(x,y)\right)\left(\sum_{k,k'=1}^{\nu}\lambda_k\xi_{k'}\eta_t^{(k')2}p_t^{(kk')}(x,y)\right)}} \leq \rho \tag{46}$$

where we made use of the Schwartz inequality.

The physical significance of these results, in particular with respect to the positive definiteness of the instantaneous covariance matrix [23] is enforced by the following

Proposition. *The matrix* $\mathbf{C}(\mathbf{x},t)$ *of Eqs. (20,25 and 27) is positive definite.*

Proof. Omitting the explicit dependence on $\mathbf{x}$ the generic element of $\mathbf{C}(\mathbf{x},t)$ can be written as

$$C_{kl} = \frac{1}{\sum_{i_1,i_2,\cdots i_n=1}^{\nu}\lambda_{1,i_1}\cdots\lambda_{n,i_n}p_t^{(i_1\cdots i_n)}}\sum_{i_1,i_2,\cdots i_n=1}^{\nu}\lambda_{1,i_1}\cdots\lambda_{n,i_n}\sigma_{k,i_k}(t)\sigma_{l,i_l}(t)\rho_{kl}p_t^{(i_1\cdots i_n)} \tag{47}$$

and therefore

$$\mathbf{C} = \frac{1}{\sum_{i_1,i_2,\cdots i_n=1}^{\nu} \lambda_{1,i_1}\cdots\lambda_{n,i_n} p_t^{(i_1\cdots i_n)}} \sum_{i_1,i_2,\cdots i_n=1}^{\nu} \lambda_{1,i_1}\cdots\lambda_{n,i_n} p_t^{(i_1\cdots i_n)}$$

$$\begin{pmatrix} \sigma_{1,i_1}(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_{n,i_n}(t) \end{pmatrix} \begin{pmatrix} \rho_{11} & \rho_{12} & \cdots \\ \vdots & \ddots & \vdots \\ \rho_{n1} & \rho_{n2} & \cdots \end{pmatrix} \begin{pmatrix} \sigma_{1,i_1}(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_{n,i_n}(t) \end{pmatrix} \tag{48}$$

The thesis is a trivial consequence of the fact that **C** is a linear combination of positive definite matrices with positive coefficients, thanks to the conditions of Eq. (6). □

Multivariate mixtures: the average ("term", "terminal") correlation structure

The average logreturn for component x is

$$\mathbb{E}_0\{\ln x - \ln x_0\} = \sum_k \lambda_k \int_0^t \left(\mu_s - \frac{\sigma_s^{(k)2}}{2}\right) ds \tag{49}$$

and its variance is

$$\mathrm{Var}_0\{\ln x - \ln x_0\} = \sum_k \lambda_k \int_0^t \sigma_s^{(k)2} ds \tag{50}$$

The average covariance of returns of x and y is instead

$$\mathrm{Cov}_0\{\ln x - \ln x_0, \ln y - \ln y_0\} = \sum_{kk'} \lambda_k \xi_{k'} \int_0^t \sigma_s^{(k)} \eta_s(k') \rho ds, \tag{51}$$

giving rise to an average correlation between returns up to time t

$$< \rho_t >_0 = \frac{\mathrm{Cov}_0\{\ln x - \ln x_0, \ln y - \ln y_0\}}{\sqrt{\mathrm{Var}_0\{\ln x - \ln x_0\}\mathrm{Var}_0\{\ln y - \ln y_0\}}} = \frac{\rho \sum_{kk'} \lambda_k \xi_{k'} \int_0^t \sigma_s^{(k)} \eta_s^{(k')} ds}{\sqrt{(\sum_k \lambda_k \int_0^t \sigma_s^{(k)2} ds)(\sum_{k'} \xi_{k'} \int_0^t \eta_s^{(k')2} ds)}}. \tag{52}$$

Conclusions and perspectives

Throughout this paper we have illustrated how to extend in a conceptually simple fashion an asset price model already shown to reproduce quite general implied volatility structures commonly observed on the market [5, 6, 7], lying within the manifold of the so–called local volatility models. The extension aims at inferring an analytic expression for the local volatility of a multivariate security (such as e.g. a basket of stocks) that is consistent with *i)* the individual dynamics of each component of the security, *ii)* a given instantaneous correlation structure (and the link with what is usually done in a standard multivariate setting is established in the limit of low correlations). The main practical advantage is that our approach allows for a semianalytic pricing of European style derivatives on the multivariate seturity in a way that keeps into account the smile structures of the individual component securities, thus reducing computational time. Finally, an issue not to be disregarded is that the dynamics we propose is alternative to the discretized multiasset dynamics commonly postulated when dealing with multisecurities.

Future extensions of this work include the testing of this approach in actual situations, two of which are particularly challenging for practitioners: baskets of many stocks with a smile structure of implied volatilities, and swap rates' derivatives within the Libor Forward Model (LFM).

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