Connecting Smiles and Basket Dynamics

F.Rapisarda
Product and Business Development Group

in collaboration with L.Bisesti, D.Brigo, A.Castagna, G.Mauri, F.Mercurio
E-mail: `francesco.rapisarda@bancaimi.it`

Summary

- Overview of the smile problem
- Local volatility models: mixture of densities' model
- A practical choice: an uncertain volatility model
- The link: a projection result
- Properties, precautions, niceties; explanatory power
- Mixtures of densities in the multivariate context:
  - brute force approach (Simply Correlated Mixture Dynamics)
  - subtler approach: MultiVariate Mixture Dynamics
  - consistency issues
- A practical example + conclusions

Motivations

Many financial derivatives are based on the values of many underlying assets:

European options in equity/FX

$$X_T = \left[\sum_{i=1}^{n} w_i S_T^{(i)} - K \right]^+$$

European swaptions

$$\begin{cases} X_T = D_{T_n} \left[S_{\alpha,\beta}(T_\alpha) - K \right]^+ \sum_{i=\alpha+1}^{\beta} \tau_i \, P(T_\alpha, T_i) \\ \\ S_{\alpha,\beta}(t) = \dfrac{1 - \prod_{j=\alpha+1}^{\beta} \frac{1}{1+\tau_j F_j(t)}}{\sum_{i=\alpha+1}^{\beta} \tau_i \prod_{j=\alpha+q}^{i} \frac{1}{1+\tau_j F_j(t)}} \end{cases}$$

In the real world, however, not only do traders trade multiasset derivatives, they also trade the *volatility smile/skew* of the underlyings.

Motivations (cont'd)

Mostly quantitative approaches have concentrated on the two issues in a separate fashion, *i.e.* either

- computing prices of multiasset derivatives in an efficient fashion (computationally or through approximations)
  or
- building models for a single asset dynamics that keep into account the volatility smiles.

We aim at solving both problems in one shot, formulating a "multiasset smile model" that consistently performs a joint dynamics for the many assets, while accounting for their volatility smiles.

The smile problem

The Black–Scholes model does not price all European options quoted on a single market in a consistent way.

Definition. The implied volatility is the parameter σ to plug into the Black–Scholes formula for an option maturing at T with strike K to match the corresponding market price:

$$S_0 e^{-qT}\Phi\left(\frac{\ln S_0/K+(r-q+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)$$

$$-Ke^{-rT}\Phi\left(\frac{\ln S_0/K+(r-q-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) = C(K,T)$$

If the implied volatility depended only on $T \implies$ modify BS

$$\frac{dS_t}{S_t} = (r-q)dt + \sigma_t dW_t$$

Implied volatility: $V_T = \sqrt{\frac{1}{T}\int_0^T \sigma_s^2 ds}$.

However, dependence on strike $K \implies$ smile/skew, with complex volatility structures. Many efforts are made to write down alternative dynamics that “price the smile” for a single asset. We tackle this issue by assuming a suitable volatility coefficient (“local volatiility”).

Mixture of densities (Brigo–Mercurio (2000))

Falls within the local volatility models family. Leads to an explicit dynamics for the asset:

$$dS_t = \mu S_t dt + \sigma(S_t, t) S_t dW_t \tag{1}$$

The asset under consideration underlies a given options market (needs not be tradable). It could be a stock, an exchange rate, or a Libor rate.

The basic assumptions are the following:

- the T–forward risk–adjusted measure $\mathbb{Q}^T$ exists;
- the dynamics of Eq. (1) has μ and σ deterministic and well behaved;
- the marginal density of S under $\mathbb{Q}^T$ is equal to the weighted average of the known densities of some given diffusion processes.

Such approach leads to an asset price dynamics flexible enough to reproduce a large variety of market volatility smiles.

Mixture of densities (cont'd)

Idea: asset S_t, marginal density $p_t(S)$. Suppose that at all times t the density can be expanded

$$\begin{cases} p_t(S) = \sum_{\alpha=1}^{\nu} \lambda_\alpha \, p_t^{(\alpha)}(S), \\ \lambda_\alpha \geq 0 \ \forall \alpha, \ \sum_{\alpha=1}^{\nu} \lambda_\alpha = 1 \end{cases} \tag{2}$$

Intuitively, write down auxiliary *fictitious* processes

$$d\mathfrak{z}_t^{(\alpha)} = \mu \mathfrak{z}_t^{(\alpha)} dt + v_\alpha(\mathfrak{z}_t^{(\alpha)}, t)\mathfrak{z}_t^{(\alpha)} dW_t$$

whose density is the $p_t^{(\alpha)}$ above. Their initial condition is $\mathfrak{z}_0^{(\alpha)} = S_0$, functions v_α sufficiently regular to ensure existence and uniqueness of the SDE.

Question. Is there a local volatility $\sigma(S,t)$ such that the $\mathbb{Q}^T$–density of S satisfies Eq. (2)?

Note. $p_t(S)$ is a proper density function:

$$\mathbb{E}\left\{S_t\right\} = \int_0^\infty x p_t(x) dx = \sum_{\alpha=1}^{\nu} \lambda_\alpha \int_0^\infty x p_t^{(\alpha)}(x) dx = \sum_{\alpha=1}^{\nu} \lambda_\alpha S_0 e^{\mu t}.$$

Mixture of densities (cont'd)

Solution. Apply the Fokker–Planck (Kolmogorov forward) equation

$$\frac{\partial p_t}{\partial t}+\frac{\partial}{\partial x}(\mu x p_t)-\frac{1}{2}\frac{\partial^2}{\partial x^2}(\sigma^2(x,t)x^2 p_t)=0$$

and

$$\frac{\partial p_t^{(\alpha)}}{\partial t}+\frac{\partial p_t^{(\alpha)}}{\partial x}(\mu x p_t^{(\alpha)})-\frac{1}{2}\frac{\partial^2}{\partial x^2}(v_\alpha^2(x,t)x^2 p_t^{(\alpha)})=0$$

Recalling $p_t(S)=\sum_{\alpha=1}^{\nu}\lambda_\alpha\, p_t^{(\alpha)}(S)$, from purely algebraic manipulations we end up with

$$\sigma(x,t)^2=\frac{\sum_{\alpha=1}^{\nu}\lambda_\alpha v_\alpha^2(x,t)p_t^{(\alpha)}}{\sum_{\alpha=1}^{\nu}\lambda_\alpha p_t^{(\alpha)}}$$

(square) local volatility function, hence the sought dynamics

$$dS_t=\mu S_t dt+\sigma(S_t,t)S_t dW_t.$$

This, however, only defines some *candidate* dynamics leading to the marginal density p_t. The next steps are: choose sensibly the "base" densities $p_t^{(\alpha)}$ and show existence and uniqueness of the solution to the SDE.

Properties

Suppose the existence and uniqueness condition is met. What are the consequences?

Option prices. The price at $t = 0$ of any European call option with maturity T, strike K and written on the asset is

$$X_0 = P(0,T)\mathbb{E}^T\left\{[S_T - K]^+\right\}$$

$$= P(0,T)\int_0^\infty [x-K]^+ p_T(x)dx$$

$$= P(0,T)\int_0^\infty [x-K]^+ \sum_{\alpha=1}^{\nu} \lambda_\alpha p_T^{(\alpha)}(x)dx$$

$$= \sum_{\alpha=1}^{\nu} \lambda_\alpha P(0,T)\int_0^\infty [x-K]^+ p_T^{(\alpha)}(x)dx$$

Greeks. The same applies for Greeks.

Why mixing densities? The use of analytical “base” densities $p_t^{(\alpha)}$ immediately leads to closed–form formulæ for European options, thus easing the calibration of the model. Moreover, the (virtually) unlimited number of parameters adds flexibility to the model. Still, we recover a full analytical dynamics that can be exploited for pricing exotics.

A convenient choice of base densities

Natural choice:

$$\begin{cases} \mathfrak{z}_0^{(\alpha)} = S_0 \\ v_\alpha(x,t) = x\sigma_\alpha(t) \implies d\mathfrak{z}_t^{(\alpha)} = \mu\mathfrak{z}_t^{(\alpha)}dt + \mathfrak{z}_t^{(\alpha)}\sigma_\alpha(t)dW_t \\ V_\alpha(t) = \sqrt{\int_0^t \sigma_\alpha(s)^2 ds} \\ p_t^{(\alpha)}(x) = \frac{1}{\sqrt{2\pi}xV_\alpha(t)} \exp\left[-\frac{\left(\ln\left(\frac{x}{S_0}\right) - \mu t + \frac{1}{2}V_\alpha^2(t)\right)^2}{2V_\alpha^2(t)} \right] \end{cases}$$

with σ_k deterministic (*mixture of lognormal densities*).

Everything fine (existence and uniqueness of the SDE) provided that $\sigma_\alpha(t)$ are well behaved (bounded from above and below by positive constants).

Why mixing lognormals?

- analytically tractable;
- easy to grasp (link to BS);
- log–returns $\ln(S_t/S_0)$, $t > 0$ are more leptokurtic than in the Gaussian case;
- work well in many practical situations: Ritchey (1990), Melick and Thomas (1997), Bhupinder (1998) and Guo (1998) found a good fitting quality to market options data.

The LMLV model: advantages and drawbacks

Advantages:

- Explicit marginal density
- Explicit option prices (mixtures of Black-Scholes prices)
- Nice fitting to smile-shaped implied volatility curves and surfaces
- Market completeness

Drawbacks:

- Unknown transition density
- Future implied volatilities must be calculated numerically (Monte Carlo)

Uncertain Volatility Models

- A *good* alternative model
  - Has explicit dynamics, possibly with known marginal density.
  - Implies analytical formulas for European options.
  - Implies a good fitting of market data.
  - Implies a nice evolution of the volatility structures in the future.
- It would be *great* if the alternative model also
  - Had explicit transition densities.
  - Implied closed form formulas for a number of path-dependent derivatives (barriers, lookbacks,...). Why? because a trading book may count up to thousands of exotics (see $e.g.$ the FX case)

A lognormal-mixture uncertain-volatility (UV) model [1]

We assume that the asset price dynamics under the risk neutral measure is

$$dS(t) = \begin{cases} S(t)[\mu(t)\,dt + \sigma_0\,dW(t)] & t \in [0, \varepsilon] \\ S(t)[\mu(t)\,dt + \eta(t)\,dW(t)] & t > \varepsilon \end{cases}$$

where η is a random variable that is independent of W and takes values in a set of N (given) deterministic functions:

$$\eta(t) = \begin{cases} \sigma_1(t) & \text{with probability } \lambda_1 \\ \sigma_2(t) & \text{with probability } \lambda_2 \\ \vdots & \qquad \vdots \\ \sigma_N(t) & \text{with probability } \lambda_N \end{cases}$$

The random value of η is drawn at time $t = \varepsilon$.

Model similar in spirit to (but derived independently from) those of Alexander, Brintalos, and Nogueira (2003) and Gatarek (2003).

Assuming a number of different possible scenarios for the asset forward volatility akin to Avellaneda, Levy and Parás (1995): vol varying continuously within a band $[\sigma_{min}, \sigma_{max}]$, thus mapping the pricing problem onto the solution of a nonlinear PDE. Here, finite number of possible forward vol states, so same degree of analytical tractability of the BS model.

---

[1] D.Brigo *et al.*, *Risk* magazine, May 2004

The LMUV model: advantages

A clear interpretation: the LMUV model is a Black-Scholes model where the asset volatility is unknown and one assumes different scenarios for it.

The LMUV model has the same advantages as the LMLV model:

- Explicit marginal density (mixture of lognormal densities)
- Explicit option prices (mixtures of Black-Scholes prices)
- Nice fitting to smile-shaped implied volatility curves and surfaces
- It allows for a natural extension to the lognomal LIBOR market model.

In addition, the LMUV model is analytically tractable also after time 0, since, for $t > \varepsilon$, S follows a geometric Brownian motion. We thus have:

- Explicit transition densities
- Explicit prices for a number of path-dependent payoffs

LMUV model vs LMLV model

Assume that the functions σ_i satisfy, for $t \geq \varepsilon$, the same assumptions as in the LMLV model. In particular, $\sigma_i(\varepsilon) = \sigma_0$, for $i = 1, \ldots, N$.

The marginal density of S at time t then coincides with that of the LMLV model, i.e.

$$p_t(y) = \sum_{i=1}^{N} \lambda_i \frac{1}{yV_i(t)\sqrt{2\pi}} \exp\left\{ -\frac{1}{2V_i^2(t)} \left[\ln \frac{y}{S_0} - M(t) + \tfrac{1}{2}V_i^2(t) \right]^2 \right\}$$

Under the LMUV model, the market is incomplete since the asset volatility is “stochastic”.

However, the dynamics of S is directly given under the pricing measure: LMUV and LMLV European option prices coincide.

LMUV model vs LMLV model (cont'd)

Proposition. *Under the previous assumptions on the functions σ_i, the LMLV model is the projection of the LMUV model onto the class of local volatility models, in that (Derman and Kani, 1998)*

$$\nu^2(T, K) = E[\eta^2(T)|S(T) = K]$$

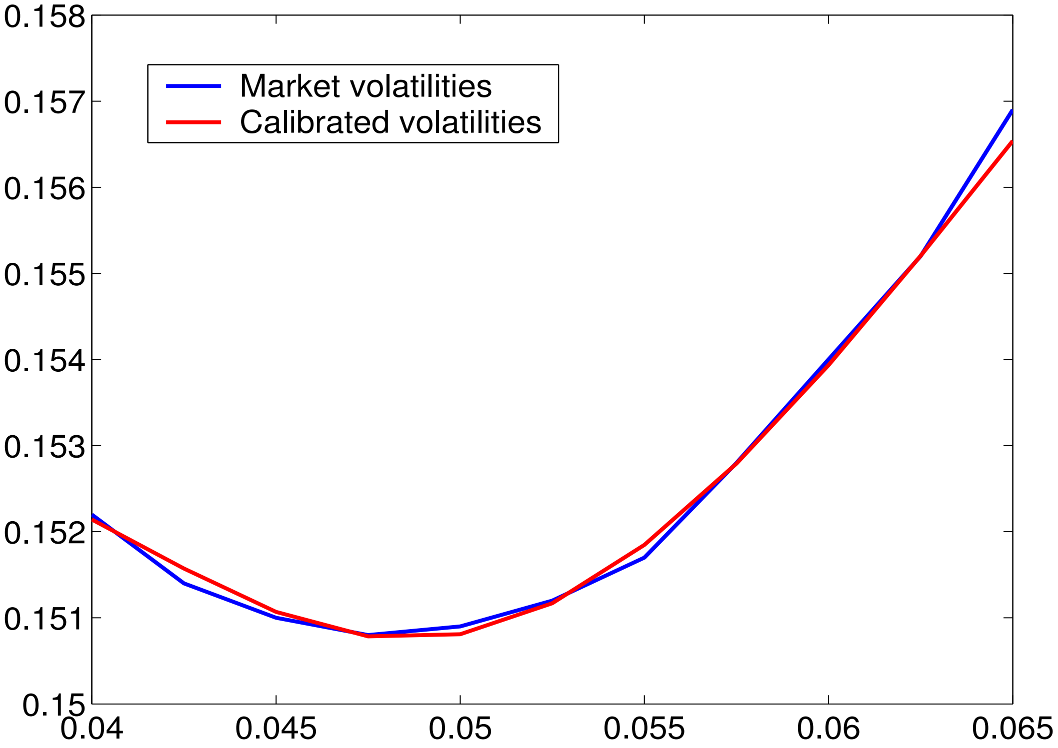
Proof. The equality follows from the definitions of $\eta(t)$ and $\nu(t, y)$ and a simple application of the Bayes rule.

A further analogy between the LMUV and LMLV models is that:

$$\mathrm{Corr}\big(\nu^2(t, S(t)), S(t)\big) = \mathrm{Corr}\big(\eta^2(t), S(t)\big) = 0$$

Single asset mixtures: the explanatory power

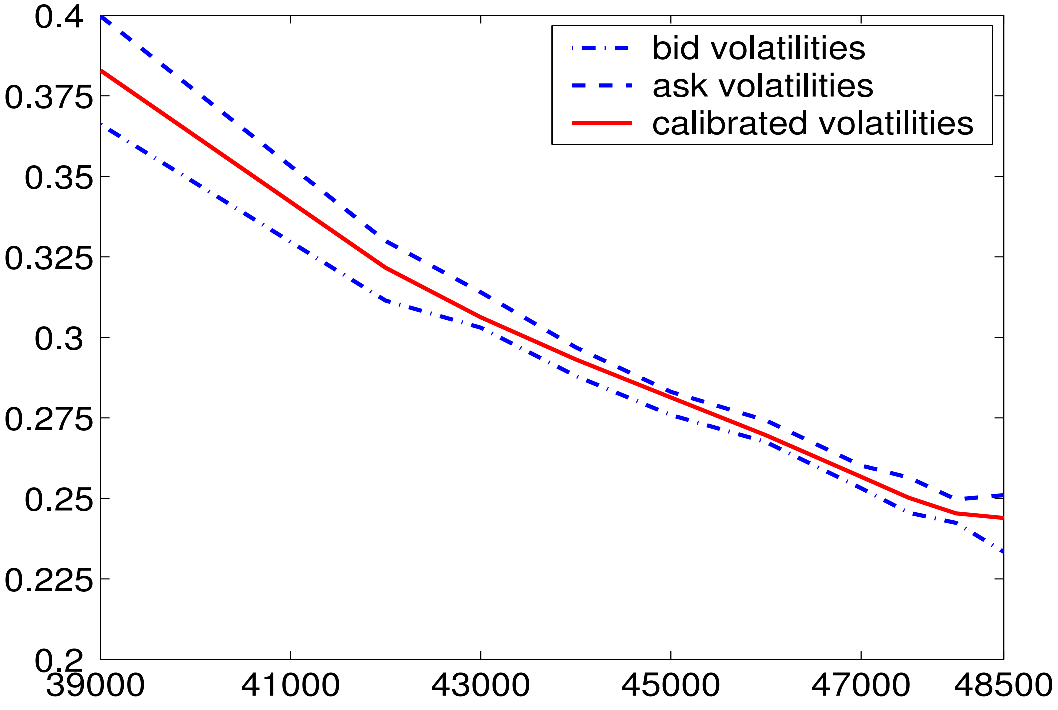
Two–year Euro caplet volatilities as of November 14th, 2000 (Libor resetting at 1.5 years. We set: $N = 2$, $v_i := \eta_i(1.5)$, $i = 1, 2$, $\lambda_2 = 1 - \lambda_1$. We minimize the squared percentage difference between model and market (mid) prices: $\lambda_1 = 0.241$, $\lambda_2 = 0.759$, $v_1 = 0.125$, $v_2 = 0.194$, $\alpha = 0.147$.



Single asset mixtures: the explanatory power (cont'd)

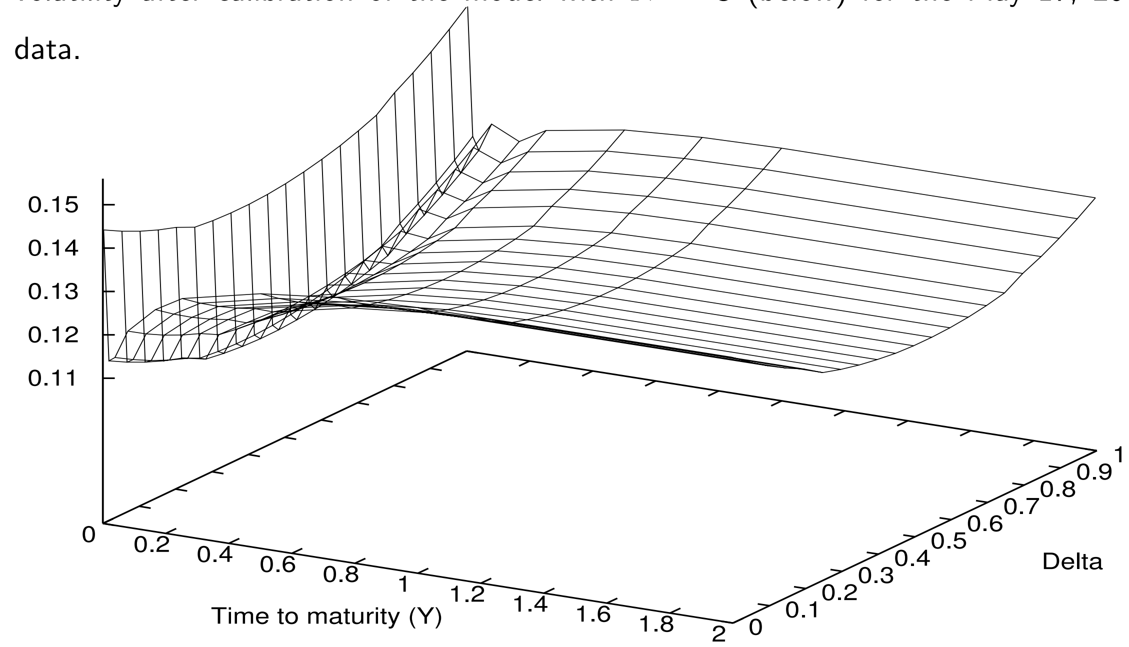
Italian MIB30 equity index on March 29, 2000, at 3,21pm (most liquid puts with the shortest maturity).

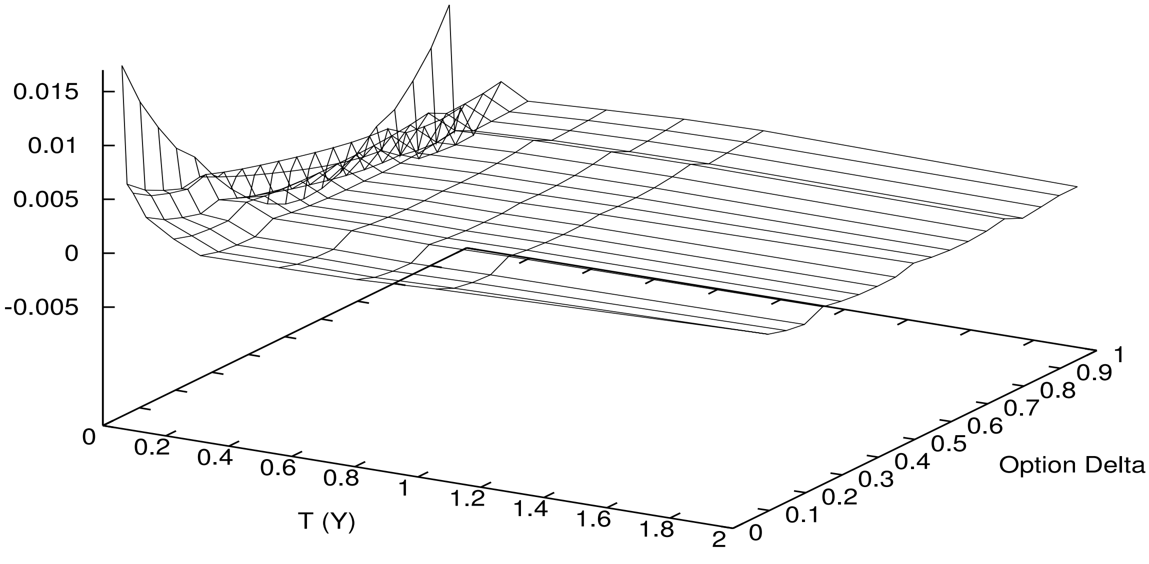
We set $N=3$, $v_i := \eta_i(T)$ $(i=1,2,3)$, $\lambda_3 = 1-\lambda_1-\lambda_2$. We minimize the squared percentage difference between model and market mid prices. We get: $\lambda_1 = 0.201$, $\lambda_2 = 0.757$, $v_1 = 0.019$, $v_2 = 0.095$, $v_3 = 0.229$, $\alpha = -1.852$.



Single asset mixtures: the explanatory power (cont'd)

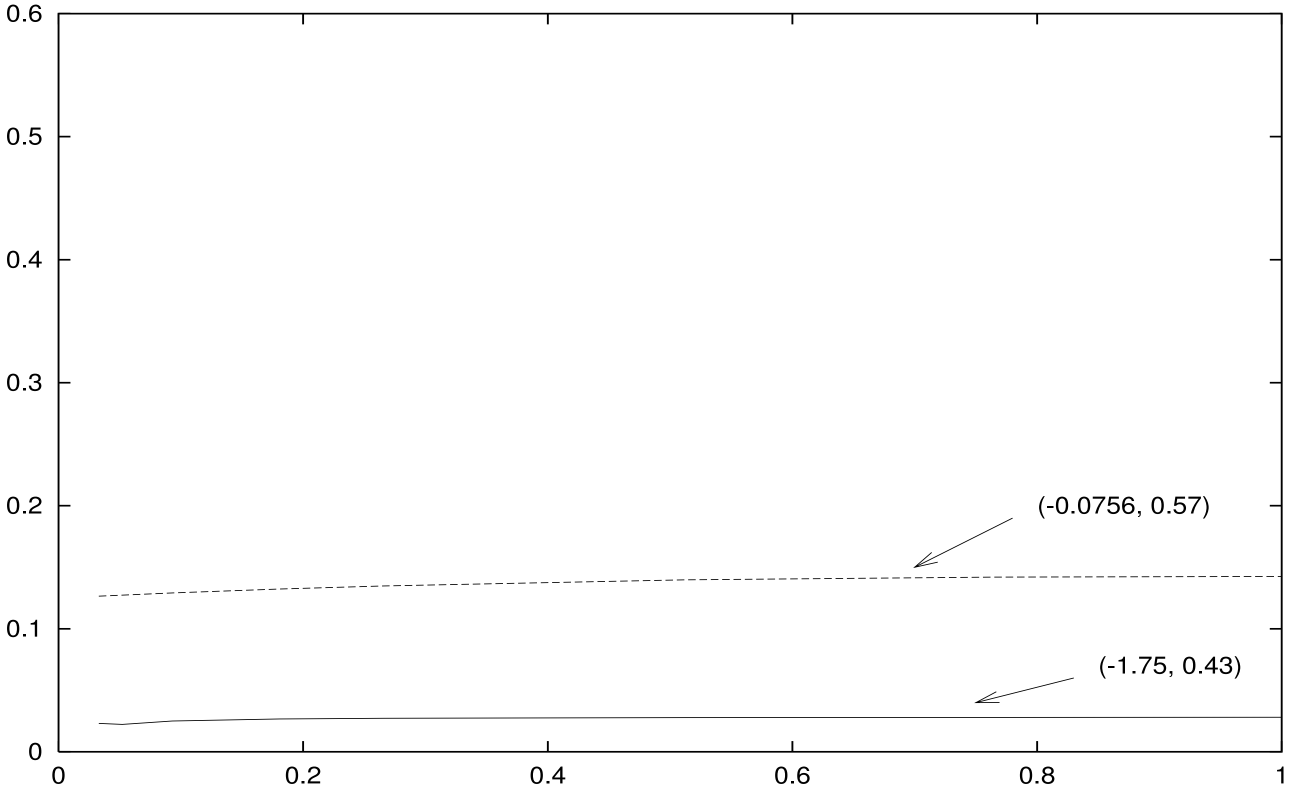
The EUR/USD market implied volatility surface (above) and absolute difference in implied volatility after calibration of the model with $N = 3$ (below) for the May 17, 2001 market data.





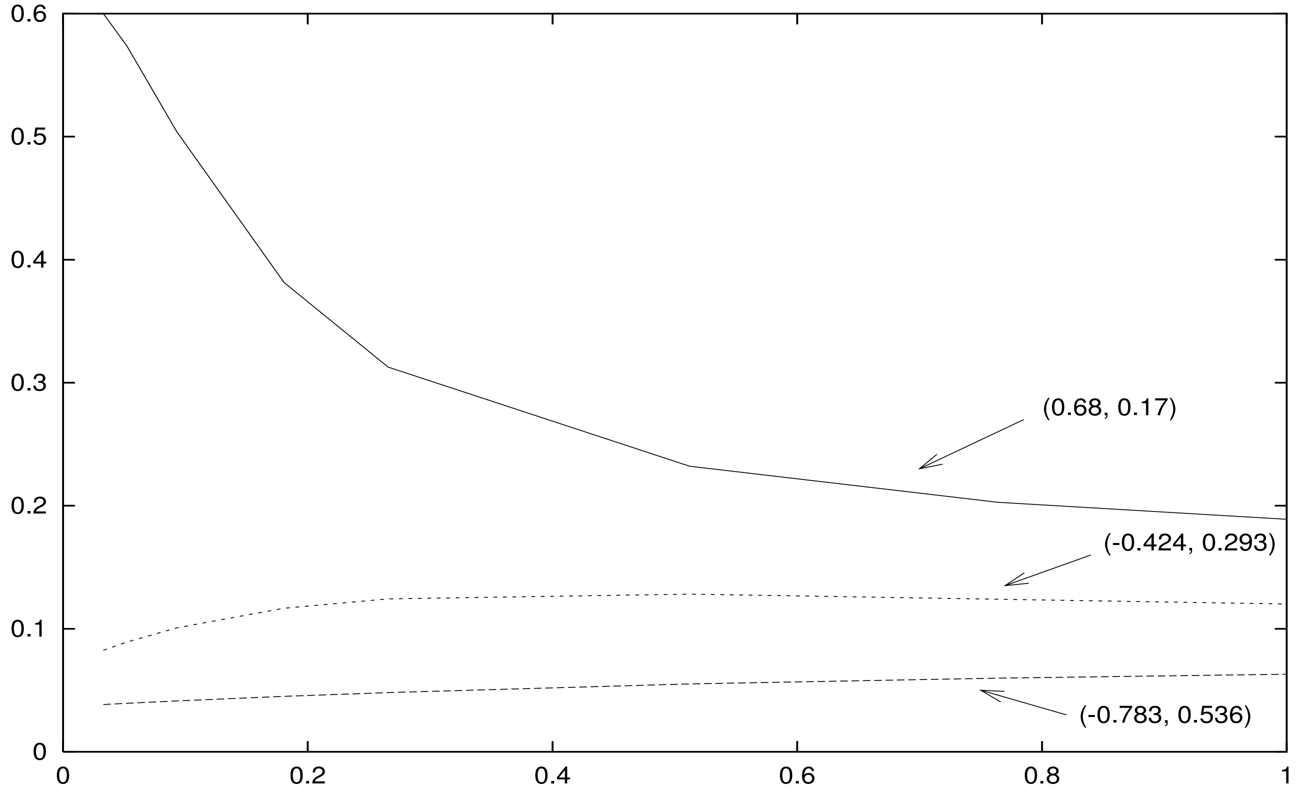
Single asset mixtures: the explanatory power (cont'd)

EUR/USD market: the integrated volatilities $\sqrt{\frac{1}{t}\int_0^t \sigma_s^{(k)2} ds}$ after calibration of the model with $N = 2$ on the May 17, 2001 market data. T is measured in years. Brackets enclose the shift and weight parameter: (β_i, λ_i) for each component.



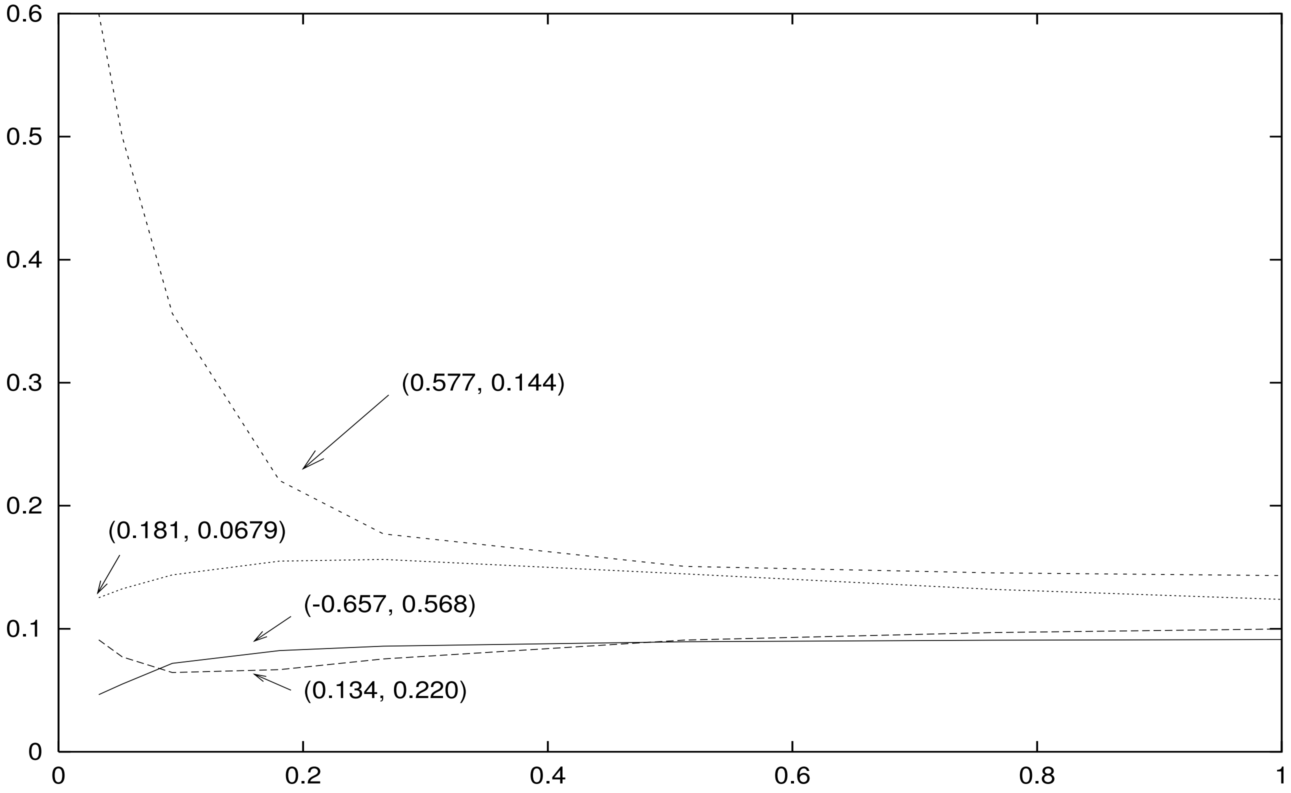
Single asset mixtures: the explanatory power (cont'd)

EUR/USD market: the integrated volatilities $\sqrt{\frac{1}{t}\int_0^t \sigma_s^{(k)2} ds}$ after calibration of the model with $N = 3$ on the May 17, 2001 market data. T is measured in years. Brackets enclose the shift and weight parameter: (β_i, λ_i) for each component.



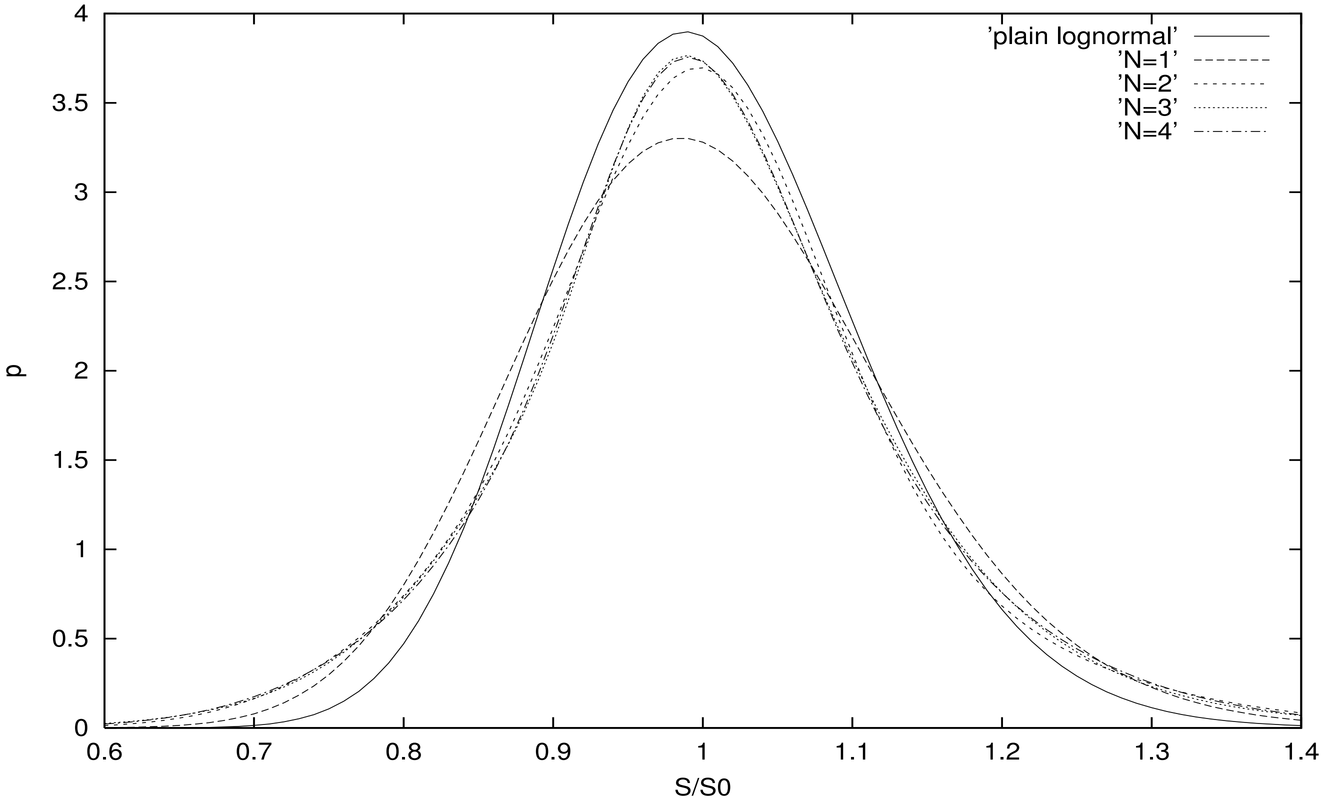
Single asset mixtures: the explanatory power (cont'd)

EUR/USD market: the integrated volatilities $\sqrt{\frac{1}{t}\int_0^t \sigma_s^{(k)2} ds}$ after calibration of the model with $N = 4$ on the May 17, 2001 market data. T is measured in years. Brackets enclose the shift and weight parameter: (β_i, λ_i) for each component.



Single asset mixtures: the explanatory power (cont'd)

EUR/USD: the probability densities resulting from the mixture after calibration of the model on the May 17, 2001 market data, for varying numbers N of basis densities (maturity T =1Y). The solid line represents a single (non shifted) lognormal process density calibrated to the whole implied volatility surface, the other lines refer to the shifted basis densities for $N = 1, \cdots 4$, as described in the legenda.



Calibration of the LMLV and LMUV models to FX volatility data: the market volatility matrix

	25Δ	50Δ	75Δ
1W	9.83%	9.45%	9.63%
2W	9.76%	9.40%	9.61%
1M	9.66%	9.25%	9.41%
2M	9.76%	9.40%	9.61%
3M	10.16%	9.85%	10.11%
6M	10.66%	10.40%	10.71%
9M	10.90%	10.65%	11.98%
1Y	10.99%	10.75%	11.09%
2Y	11.12%	10.85%	11.17%

Table 1: EUR/USD implied volatilities on 12 April 2002.

Calibration of the LMLV and LMUV models to FX volatility data: the calibrated volatility matrix

	25Δ	50Δ	75Δ
1W	9.55%	9.09%	9.55%
2W	9.66%	9.20%	9.67%
1M	9.76%	9.30%	9.76%
2M	10.06%	9.59%	10.07%
3M	10.32%	9.82%	10.35%
6M	10.84%	10.31%	10.89%
9M	11.12%	10.58%	11.19%
1Y	11.27%	10.73%	11.35%
2Y	11.39%	10.90%	11.53%

Table 2: Calibrated volatilities obtained through a suitable parametrization of the functions σ_i.

Calibration of the LMLV and LMUV models to FX volatility data: absolute errors

	25Δ	50Δ	75Δ
1W	-0.28%	-0.36%	-0.08%
2W	-0.10%	-0.20%	0.06%
1M	0.10%	0.05%	0.35%
2M	0.30%	0.19%	0.46%
3M	0.16%	-0.03%	0.24%
6M	0.18%	-0.09%	0.18%
9M	0.22%	-0.07%	0.21%
1Y	0.28%	-0.02%	0.26%
2Y	0.27%	0.05%	0.36%

Table 3: Differences between calibrated volatilities and market volatilities.

The pricing of barrier options under the LMUV model

Typical FX exotic products: *e.g.* up–and–out call (UOC) option, maturity T, strike K, barrier level H:

payoff $X_T = [Q_T - K]^+ \chi_{\{Q_\tau < H, \forall \tau \in [0,T]\}}$.

(Actually, a plethora of them liquidly traded, DOC/DOP/UOC/digitals/rebates/doubles . . .)

Assume $\mu(t) = r(t) - q(t)$. The price at time 0 of an up-and-out call with barrier $H > S_0$, strike K and maturity T is approximately (Lo-Lee, 2001)

$$\begin{aligned}
&\chi_{\{K<H\}} \sum_{i=1}^{N} \lambda_i \Bigg\{ S_0 e^{c_1+c_2+c_3} \left[\Phi\left(\frac{\ln\frac{S_0}{K}+c_1+2c_2}{\sqrt{2c_2}}\right) - \Phi\left(\frac{\ln\frac{S_0}{H}+c_1+2c_2}{\sqrt{2c_2}}\right)\right] \\
&- Ke^{c_3}\left[\Phi\left(\frac{\ln\frac{S_0}{K}+c_1}{\sqrt{2c_2}}\right) - \Phi\left(\frac{\ln\frac{S_0}{H}+c_1}{\sqrt{2c_2}}\right)\right] - He^{c_3+(\beta-1)(\ln\frac{S_0}{H}+c_1)+(\beta-1)^2 c_2} \\
&\cdot\left[\Phi\left(\frac{\ln\frac{S_0}{H}+c_1+2(\beta-1)c_2}{\sqrt{2c_2}}\right) - \Phi\left(\frac{\ln\frac{S_0K}{H^2}+c_1+2(\beta-1)c_2}{\sqrt{2c_2}}\right)\right] \\
&+ Ke^{c_3+\beta(\ln\frac{S_0}{H}+c_1)+\beta^2 c_2}\left[\Phi\left(\frac{\ln\frac{S_0}{H}+c_1+2\beta c_2}{\sqrt{2c_2}}\right) - \Phi\left(\frac{\ln\frac{S_0K}{H^2}+c_1+2\beta c_2}{\sqrt{2c_2}}\right)\right]\Bigg\}
\end{aligned}$$

where c_1, c_2, c_3 and β are integrals depending on $r(t)$, $q(t)$ and $\sigma_i(t)$.

Examples of FX barrier option prices under the LMLV and LMUV models

Note:

- LMUV analytic (approximate due to term structure of rates and vols across scenarios)
- LMLV requires MC simulation even for single underlying

Type	LMLV	LMUV	BS
UOC(T=3M,K=1,H=1.05)	29	28	36
UOC(T=6M,K=1,H=1.08)	48	47	57
UOC(T=9M,K=1,H=1.10)	57	55	68
DOC(T=3M,K=1.02,H=0.95)	96	98	99
DOC(T=6M,K=1.07,H=0.98)	65	67	59
DOC(T=9M,K=1.10,H=0.90)	68	70	59

Table 4: Barrier option prices in basis points ($S_0 = 1$).

A multivariate context

We want to price a European call option on a basket

$$B_t = \sum_k^n w_k S_t^{(k)},$$

$$X_T = [B_T - K]^+$$

(typical in equities, FX and, *mutatis mutandis*, in interest rates), where each $S_t^{(k)}$ has a smile $\Longrightarrow$ local volatility $\sigma^{(k)}(S^{(k)}, t)$.

We are also given an exogenous instantaneous correlation structure

$$R = (\rho_{ij})_{i,j=1\cdots n}$$

Brute force scheme: use Euler Monte Carlo (or Milstein, or higher order) scheme with instantaneous covariance

$$\tilde{C}_{ij}(S * (i), S^{(j)}, t) = \sigma^{(i)}(S^{(i)}, t)\sigma^{(j)}(S^{(j)}, t)\rho_{ij}$$

Need for a suitably fine discretization time grid $\tau_1 = 0, \cdots \tau_N = T$ and costly simulations.

A multivariate context (cont'd)

To fix ideas, consider a basket of two entities $S_t^{(1)}$, $S_t^{(2)}$ and an option

$$X_T = [w_1 S_T^{(1)} + w_2 S_T^{(2)} - K]^+.$$

We have calibrated a mixture model, based on two lognormal densities for each asset, so we have

$$\sigma^{(1)}(S,t)^2 = \frac{\lambda_1 \sigma_t^{(1)} \pi_t^{(1,1)}(S) + \lambda_2 \sigma_t^{(2)} \pi_t^{(1,2)}(S)}{\lambda_1 \pi_t^{(1,1)}(S) + \lambda_2 \pi_t^{(1,2)}(S)}$$

and

$$\sigma^{(2)}(S,t)^2 = \frac{\xi_1 \eta_t^{(1)} \pi_t^{(2,1)}(S) + \xi_2 \eta_t^{(2)} \pi_t^{(2,2)}(S)}{\eta_1 \pi_t^{(2,1)}(S) + \eta_2 \pi_t^{(2,2)}(S)}$$

Each asset follows in our model

$$\frac{dS_t^{(1)}}{S_t^{(1)}} = \mu_1 dt + \sigma^{(1)}(S_t^{(1)}, t) dW_1$$

and

$$\frac{dS_t^{(2)}}{S_t^{(2)}} = \mu_2 dt + \sigma^{(2)}(S_t^{(2)}, t) dW_2$$

Simply Correlated Mixture Dynamics (SCMD)

In order to price the option, we could use a discretized scheme

$$d\begin{pmatrix} S_t^{(1)} \\ S_t^{(2)} \end{pmatrix} = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} \begin{pmatrix} S_t^{(1)} \\ S_t^{(2)} \end{pmatrix} dt$$

$$+\tilde{c}(S_t^{(1)}, S_t^{(2)}, t) \begin{pmatrix} S_t^{(1)} \\ S_t^{(2)} \end{pmatrix} \begin{pmatrix} dW_1 \\ dW_2 \end{pmatrix}$$

with $\tilde{c}\tilde{c}' = \tilde{C}$

$$\begin{cases} \tilde{C}_{11}(S^{(1)}, S^{(2)}, t) = \sigma^{(1)}(S^{(1)}, t)^2 \\ \tilde{C}_{12}(S^{(1)}, S^{(2)}, t) = \sigma^{(1)}(S^{(1)}, t)\,\sigma^{(2)}(S^{(2)}, t)\,\rho_{12} \\ \tilde{C}_{22}(S^{(1)}, S^{(2)}, t) = \sigma^{(2)}(S^{(2)}, t)^2 \end{cases}$$

instantaneous covariance matrix.

An alternative consists in building up a mixture of lognormal densities already at the bivariate level.

A multivariate context (cont'd)

Try to build up a mixture of lognormal *bivariate* densities:

$$[S_t^{(1)}, \sigma^{(1)}(S,t), p_t^{(1)}(S)] \leftrightarrow [(\lambda_1, \sigma_t^{(1)}, \pi_t^{(1,1)}), (\lambda_2, \sigma_t^{(2)}, \pi_t^{(1,2)})]$$

$$[S_t^{(2)}, \sigma^{(2)}(S,t), p_t^{(2)}(S)] \leftrightarrow [(\xi_1, \eta_t^{(1)}, \pi_t^{(2,1)}), (\xi_2, \eta_t^{(2)}, \pi_t^{(2,2)})]$$

Build four time–dependent covariance matrices

$$\Xi_{11}(t), \Xi_{12}(t), \Xi_{21}(t), \Xi_{22}(t)$$

from $(\alpha, \beta = 1, 2)$

$$\Xi_{\alpha\beta}(t) = \begin{pmatrix} \int_0^t \sigma_s^{(\alpha)2} ds & \int_0^t \sigma_s^{(\alpha)} \eta_s^{(\beta)} \rho_{12} ds \\ \int_0^t \sigma_s^{(\alpha)} \eta_s^{(\beta)} \rho_{12} ds & \int_0^t \eta_s^{(\beta)2} ds \end{pmatrix}$$

Build also four bivariate lognormal densities

$$\begin{cases} p_t^{(\alpha\beta)}(S^{(1)}, S^{(2)}) = \frac{1}{2\pi\sqrt{\det \Xi_{\alpha\beta}(t)} S^{(1)} S^{(2)}} \exp\left[-\frac{\tilde{\mathrm{x}}(\Xi_{\alpha\beta}(t))^{-1}\tilde{\mathrm{x}}}{2} \right] \\ \tilde{x}_i = \ln\left(\frac{S^{(i)}}{S_0^{(i)}}\right) - \int_0^t \left(\mu_s^{(i)} - \frac{{\sigma_i^{(\alpha)}}^2(s)}{2}\right) ds \end{cases}$$

A multivariate context (cont'd)

Finally, combine all four densities into a single bivariate density:

$$\begin{aligned} p(S^{(1)}, S^{(2)}, t) = \quad & \lambda_1 \xi_1 p^{(11)}(S^{(1)}, S^{(2)}, t) \\ & + \lambda_1 \xi_2 p^{(12)}(S^{(1)}, S^{(2)}, t) \\ & + \lambda_2 \xi_1 p^{(21)}(S^{(1)}, S^{(2)}, t) \\ & + \lambda_2 \xi_2 p^{(22)}(S^{(1)}, S^{(2)}, t) \end{aligned}$$

Remarks

- Is this a density? Yes: it is everywhere positive and unit normalized;
- is this compatible with the univariate densities? That is, if I integrate out e.g. $S^{(2)}$, do I get the starting mixture density for the first asset $S^{(1)}$? Yes: try it.

The second point above implies the consistency of the multivariate model with the univariate smiles (any single–asset property is reproduced).

Multivariate mixture of densities (MVMD)

n–dimensional stochastic process $\mathbf{x}(t)$

$$\frac{dx_i(t)}{x_i(t)} = \mu_i(t)dt + \mathbf{C}_i(\mathbf{x}, t) \cdot d\mathbf{W}_t$$

$\mathbf{W}_t$ standard d–dimensional Brownian motion; $\mu_i(t), \mathbf{C}_i(\mathbf{x}, t)$ deterministic functions.

Denote $C_{ij}(\mathbf{x}, t) = \mathbf{C}_i(\mathbf{x}, t) \cdot \mathbf{C}_j(\mathbf{x}, t)$.

Multidimensional Fokker–Planck equation

$$\frac{\partial p_t}{\partial t} + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}[\mu_t^{(i)} x_i p_t] - \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}[C_{ij} x_i x_j p_t] = 0$$

$p_t(\mathbf{x})$ multivariate marginal density.

Assume again

$$p_t(\mathbf{x}) = \sum_{k=1}^{N} \lambda_k p_t^{(k)}(\mathbf{x}), \qquad \lambda_k \geq 0 \ \forall k, \qquad \sum_{k=1}^{N} \lambda_k = 1$$

then, again formally

$$\frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}\left[\left(C_{ij}(\mathbf{x}, t)p_t - \sum_{k=1}^{N} \lambda_k \sigma_{ij}^{(k)}(\mathbf{x}, t) p_t^{(k)}\right) x_i x_j\right] = 0$$

solved by

$$C_{ij}(\mathbf{x}, t) = \frac{\sum_{k=1}^{N} \lambda_k \sigma_{ij}^{(k)}(\mathbf{x}, t) p_t^{(k)}}{\sum_{k=1}^{n} \lambda_k p_t^{(k)}}$$

Multivariate extension of the MD model (cont'd)

It can be proven that *under the same assumptions made for the univariate model*, existence and uniqueness of the solution to the vector SDE is ensured.

Practical choice:

$$\begin{cases} p_t^{(\alpha)}(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det \Xi^{(\alpha)}(t)\Pi_{i=1}^n x_i}} \exp\left[-\frac{\tilde{\mathbf{x}}(\Xi^{(\alpha)}(t))^{-1}\tilde{\mathbf{x}}}{2}\right] \\ \Xi_{ij}^{(\alpha)}(t) = \int_0^t \sigma_i^{(\alpha)}(s)\cdot\sigma_j^{(\alpha)}(s)ds \\ \tilde{x}_i = \ln\left(\frac{x_i}{S_0^{(i)}}\right) - \int_0^t \left(\mu_s^{(i)} - \frac{{\sigma_i^{(\alpha)}}^2(s)}{2}\right) ds \end{cases}$$

Rendering the multivariate MD consistent with the univariate version:

Calibrate the univariate MD for each asset $S_t^{(i)}$:

$$p_t^{(i)}(x) = \sum_{\alpha=1}^{\nu} \lambda_{i\alpha}\pi_t^{(i\alpha)}(x), \qquad \lambda_{i\alpha} \geq 0, \forall\alpha \qquad \sum_{\alpha} \lambda_{i\alpha} = 1$$

$\pi_t^{(i\alpha)}$ univariate lognormal.

$$\begin{cases} p_t(\mathbf{x}) = \sum_{i_1,i_2,\cdots i_n=1}^{\nu} \lambda_{1,i_1}\cdots\lambda_{n,i_n} p_t^{(i_1\cdots i_n)}(\mathbf{x}) \\ p_t^{(i_1\cdots i_n)}(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det\Xi^{(i_1\cdots i_n)}(t)}\Pi_{i=1}^{n} x_i} \\ \qquad\qquad \exp\left[-\frac{\tilde{\mathbf{x}}^{(i_1\cdots i_n)}(\Xi^{(i_1\cdots i_n)}(t))^{-1}\tilde{\mathbf{x}}^{(i_1\cdots i_n)}}{2}\right] \\ \Xi_{lm}^{(i_1\cdots i_n)}(t) = \int_0^t \sigma_{l,i_l}(s)\sigma_{m,i_m}(s)\rho_{lm}\,ds \\ \tilde{x}_l^{(i_1\cdots i_n)} = \ln\left(\frac{x_l}{S_0^{(l)}}\right) - \int_0^t \left(\mu_s^{(li_l)} - \frac{{\sigma_s^{(li_l)}}^2}{2}\right) ds \end{cases}$$

MVMD vs. SCMD joint dynamics

Recall Euler MC

$$\tilde{C}_{11}(S_1,t)=\frac{\sum_{\alpha=1}^{\nu}\lambda_\alpha\sigma_{\tau_l}^{(\alpha)2}\pi_{\tau_l}^{(1\alpha)}(S_1)}{\sum_{\alpha=1}^{\nu}\lambda_\alpha\pi_{\tau_l}^{(1\alpha)}(S_1)}$$

and

$$\tilde{C}_{12}(S_1,S_2,t)=\sqrt{\frac{\sum_{\alpha=1}^{\nu}\lambda_\alpha\sigma_{\tau_l}^{(\alpha)}\pi_{\tau_l}^{(1\alpha)}(S_1)}{\sum_{k=1}^{\nu}\lambda_\alpha\pi_{\tau_l}^{(1\alpha)}(S_1)}}\sqrt{\frac{\sum_{\beta=1}^{\nu}\xi_\beta\eta_{\tau_l}^{(\beta)}\pi_{\tau_l}^{(2\beta)}(S_2)}{\sum_{\beta=1}^{\nu}\xi_\beta\pi_{\tau_l}^{(2\beta)}(S_2)}}\rho_{12}$$

to be compared with their analytic counterparts

$$C_{11}(S_1,S_2,t)=\frac{\sum_{\alpha,\beta=1}^{\nu}\lambda_\alpha\xi_\beta\sigma_{\tau_l}^{(\alpha)2}p_{\tau_l}^{(\alpha\beta)}(S_1,S_2)}{\sum_{\alpha\beta=1}^{\nu}\lambda_\alpha\xi_\beta p_{\tau_l}^{(\alpha\beta)}(S_1,S_2)}$$

and

$$C_{12}(S_1,S_2,t)=\frac{\sum_{\alpha,\beta=1}^{\nu}\lambda_\alpha\xi_\beta\sigma_{\tau_l}^{(\alpha)}\eta_{\tau_l}^{(\beta)}\rho_{12}p_{\tau_l}^{(\alpha\beta)}(S_1,S_2)}{\sum_{\alpha\beta=1}^{\nu}\lambda_\alpha\xi_\beta p_{\tau_l}^{(1\alpha\beta)}(S_1,S_2)}$$

In SCMD $\tilde{C}_{11}(S_1,t)$, in MVMD $C_{11}(s_1,S_2,t)$.

Identity happens only when $\rho_{12}=0$, otherwise in general

$$\begin{cases} C_{11}(S_1,S_2,t)=\tilde{C}_{11}(S_1,t)+O(\rho_{12}^2) \\ \\ C_{12}(S_1,S_2,t)=\tilde{C}_{12}(S_1,S_2,t)+O(\rho_{12}^2) \end{cases}$$

General consistency of the MVMD model

Proposition. *Given any* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *and any* $t \geq 0$*, under the MVMD the expectation of* $f(x_l(t))$ *is*

$$\mathbb{E}_0\{f(x_l(t))\} = \int dx_l f(x_l) \pi_t^{(l)}$$

Proof: Trivial. Simply integrate away all variables but x_l. Single asset properties (option prices, Greeks, . . .) are exactly reproduced.

Consequences for pricing

The MVMD model allows to sample *directly* (through e.g. MC) the terminal density, whereas EMC needs discretization.

A long jump up to T instead of a sequence of small steps $\Longrightarrow$ CPU time generally saved.

$$X_T = [\sum_{i=1}^{n} w_i S_T^{(i)} - K]^+$$

$$\begin{aligned} X_0 &= P(0,T)\mathbb{E}^T\{X_T\} \\ &= P(0,T) \int dx_1 dx_2 \cdots dx_n X_T p_T(x_1, x_2, \cdots x_n) \\ &= P(0,T) \int dx_1 dx_2 \cdots dx_n X_T p_T^{(\alpha_1, \alpha_2, \cdots \alpha_n)}(x_1, x_2, \cdots x_n) \end{aligned}$$

MVMD *vs* SCMD

$$d\underline{S}(t) = \mathrm{diag}(\mu)\underline{S}(t)dt + \mathrm{diag}(\underline{S}(t))C(t, \underline{S}(t))d[W_1, .., W_n]'$$

$$p_{\underline{S}(t)} = \sum_{\alpha_1,\ldots,\alpha_n=1}^{\nu} \lambda_1^{\alpha_1} \cdots \lambda_n^{\alpha_n} \quad p_{[S_1^{\alpha_1}(t),\ldots,S_n^{\alpha_n}(t)]'} \qquad \textbf{VS}$$

$$d\underline{S} = \mathrm{diag}(\mu)\underline{S}dt + \mathrm{diag}((\nu_{\mathrm{mix}}^{\alpha}(t, S_\alpha))_\alpha)\mathrm{diag}(\underline{S})d[W_1, .., W_n]'$$

$$dW_i dW_j = \rho_{i,j}dt, \qquad p_{\underline{S}(t)} = ?!?,$$

Pros MVMD:

- Terminal distribution of $\underline{S}_T$ (and of basket $A_T = \underline{a}'\underline{S}_T$) can be simulated one-shot, no time discretization.
- Explicit multivariate distribution that is the most natural non-trivial generalization of the scalar case, i.e. a multivariate mixture

Cons MVMD:

- “Combinatorial explosion”: A possibly large number of densities to mix in the multivariate mixture (typically ν^n, e.g. $3^{10} = 59049$). BUT... Typically $\nu = 2, 3$, and there is a hierarchy in the base densities: $\lambda_\alpha^1 > \lambda_\alpha^2 >> \lambda_\alpha^3$, so that with weights given by $\lambda_\alpha^1\lambda_\beta^2\lambda_\gamma^3\cdots$ only few multivariate densities have appreciable weights, thus easing the simulations.
- No immediate statistical interpretation of ρ's if not as cross-sectional fitting parameters...

Pros SCMD: 1) A clear interpretation for ρ as instantaneous correlation among the single names. 2) Number of densities to mix does not increase with n but remains equal to ν.

Cons SCMD: 1) Need time discretization to price simple claims on $A_T = \underline{a}'\underline{S}_T$. As T increases, we need more time steps.

A misnomer: the basket smile

Ill–defined concept: the basket density is not lognormal even if its constituents' were!

However, traders use the Black–Scholes formulæ to price plain vanilla options, and widespread market practice is to approximate the basket density with a lognormal density with suitable rate and volatilities (moment–matching procedure): $\tilde{B}_t$ basket approximant with

$$\frac{d\tilde{B}_t}{\tilde{B}_t} = \tilde{\mu}_t dt + \tilde{\sigma}_t dW_t$$

Choose $\tilde{\mu}_t$ such that the first moment is matched at all times:

$$\mathbb{E}\{\tilde{B}_t\} = B_0 e^{\int_0^t \tilde{\mu}_s ds} = \mathbb{E}\{B_t\} = \sum_{i=1}^{n} w_i \mathbb{E}\{S_t^{(i)}\}$$

Def.: the basket implied volatility is the parameter $\Sigma(K,T)$ that, plugged into the Black–Scholes formula for a European option on the approximant $\tilde{B}_t$, allows to reproduce the market price of the corresponding option on the actual basket.

From the model prices of European options $\Pi(K,T)$, computed through the multivariate model, back out the "basket implied volatilities" $\Sigma(K,T)$ for the approximant such that Black–Scholes formulæ reproduce such prices.

Technicalities

The number of multivariate base densities scales like ν^n!

Combinatorial explosion

However, a few empirical facts on the mixture density model:

- Typically $\nu = 2, 3$;
- hierarchy in the base densities: $\lambda_1 > \lambda_2 >> \lambda_3$.

The multivariate densities bear weights given by $\lambda^{(1)}_{\alpha}\lambda^{(2)}_{\beta}\lambda^{(3)}_{\gamma}\cdots$ and therefore only few multivariate densities have appreciable weights, thus easing the simulations.

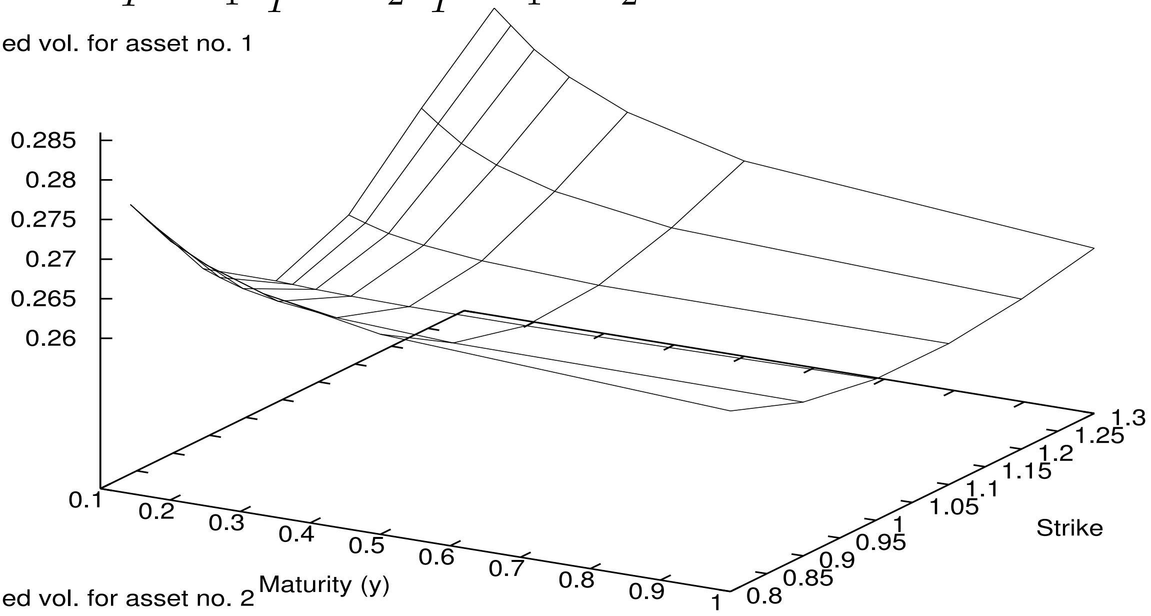
Example: 2×2 synthetic basket

$S_0^{(1)} = 1$, $\mu_1 = 5\%$, $\sigma_1(t)$, $\sigma_2(t)$, $\lambda_1 = 0.6$, $\lambda_2 = 1 - \lambda_1$

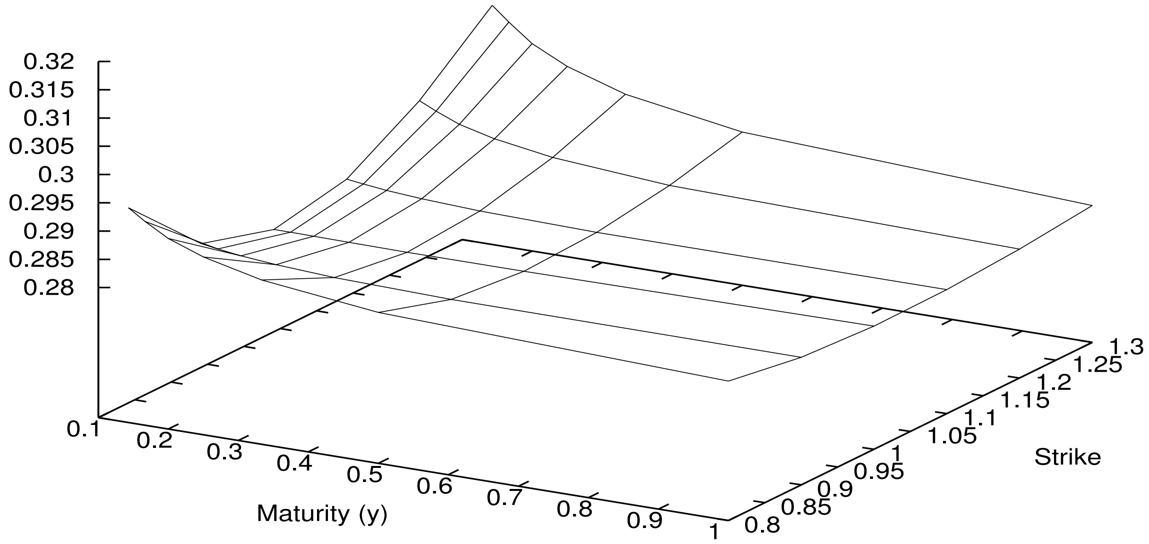
$S_0^{(2)} = 1$, $\mu_2 = 3\%$, $\eta_1(t)$, $\eta_2(t)$, $\xi_1 = 0.7$, $\xi_2 = 1 - \xi_1$

$B_T = w_1 S_T^{(1)} + w_2 S_T^{(2)}$, $w_1 = w_2 = 0.5$, $T =$1Y.

Implied vol. for asset no. 1



Implied vol. for asset no. 2



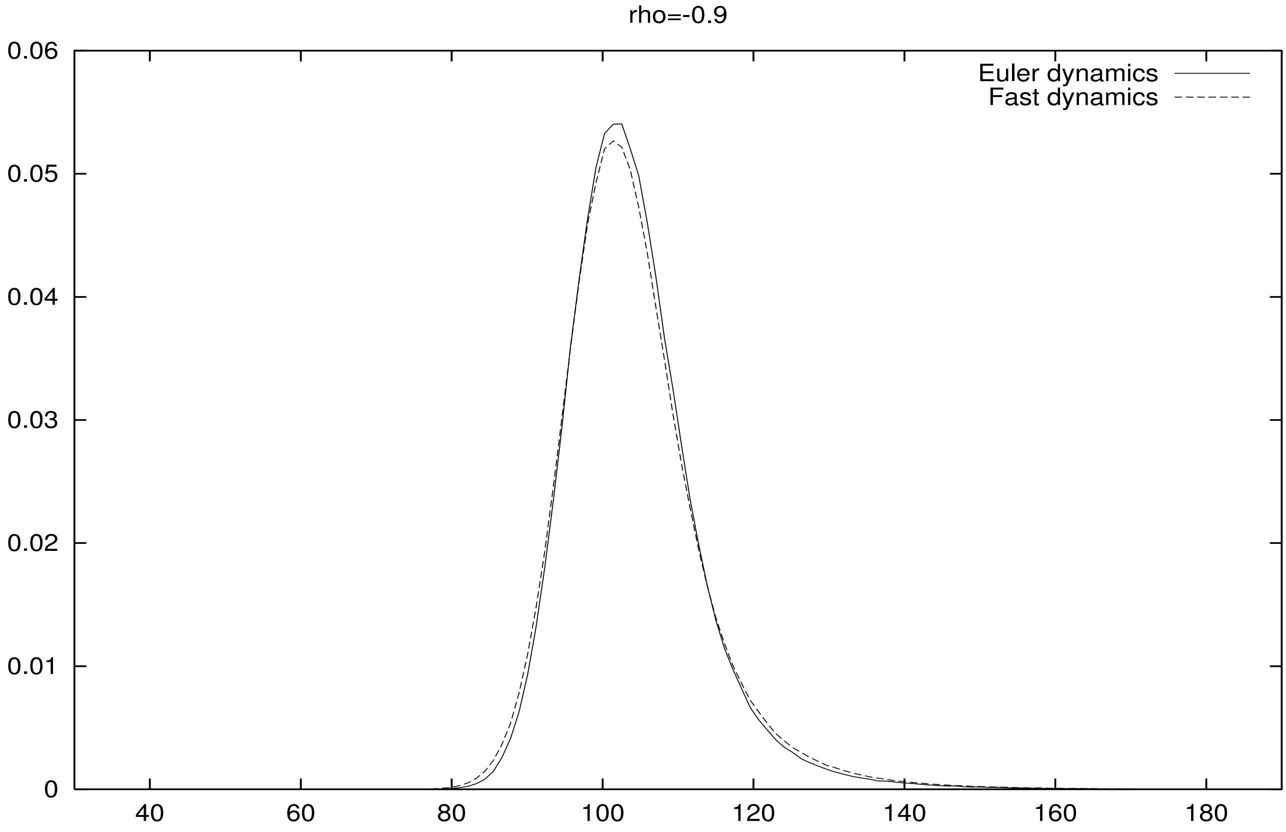
Example: 2×2 synthetic basket

$S_0^{(1)} = 1$, $\mu_1 = 5\%$, $\sigma_1(t)$, $\sigma_2(t)$, $\lambda_1 = 0.6$, $\lambda_2 = 1 - \lambda_1$

$S_0^{(2)} = 1$, $\mu_2 = 3\%$, $\eta_1(t)$, $\eta_2(t)$, $\xi_1 = 0.7$, $\xi_2 = 1 - \xi_1$

$B_T = w_1 S_T^{(1)} + w_2 S_T^{(2)}$, $w_1 = w_2 = 0.5$, $T =$1Y.

$\rho = -90\%$; the basket density $p_T(B, \rho)$ under the EMC scheme, continuous line; $\Pi_T(B, \rho)$ under the MVMD scheme, dashed line.



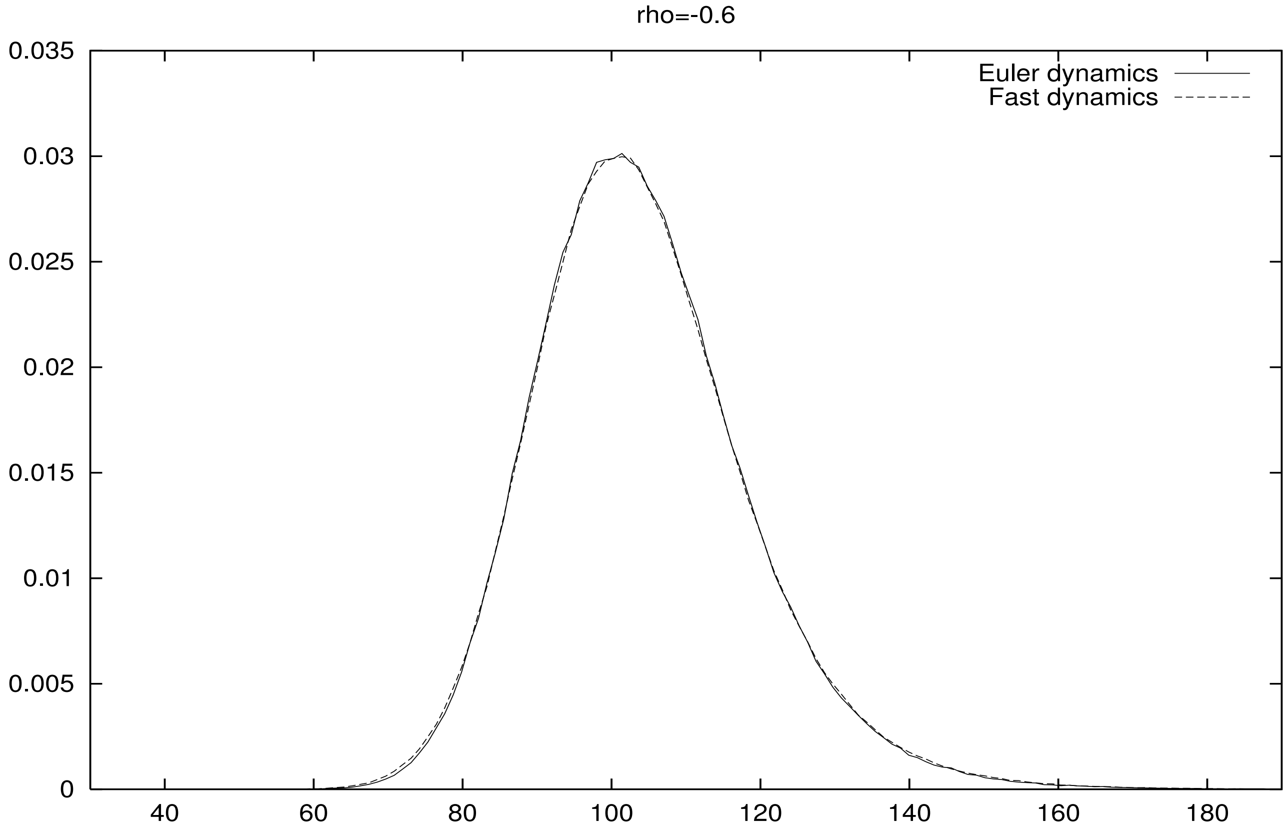
Example: 2×2 synthetic basket

$S_0^{(1)} = 1$, $\mu_1 = 5\%$, $\sigma_1(t)$, $\sigma_2(t)$, $\lambda_1 = 0.6$, $\lambda_2 = 1 - \lambda_1$

$S_0^{(2)} = 1$, $\mu_2 = 3\%$, $\eta_1(t)$, $\eta_2(t)$, $\xi_1 = 0.7$, $\xi_2 = 1 - \xi_1$

$B_T = w_1 S_T^{(1)} + w_2 S_T^{(2)}$, $w_1 = w_2 = 0.5$, $T =$1Y.

$\rho = -60\%$; the basket density $p_T(B, \rho)$ under the EMC scheme, continuous line; $\Pi_T(B, \rho)$ under the MVMD scheme, dashed line.



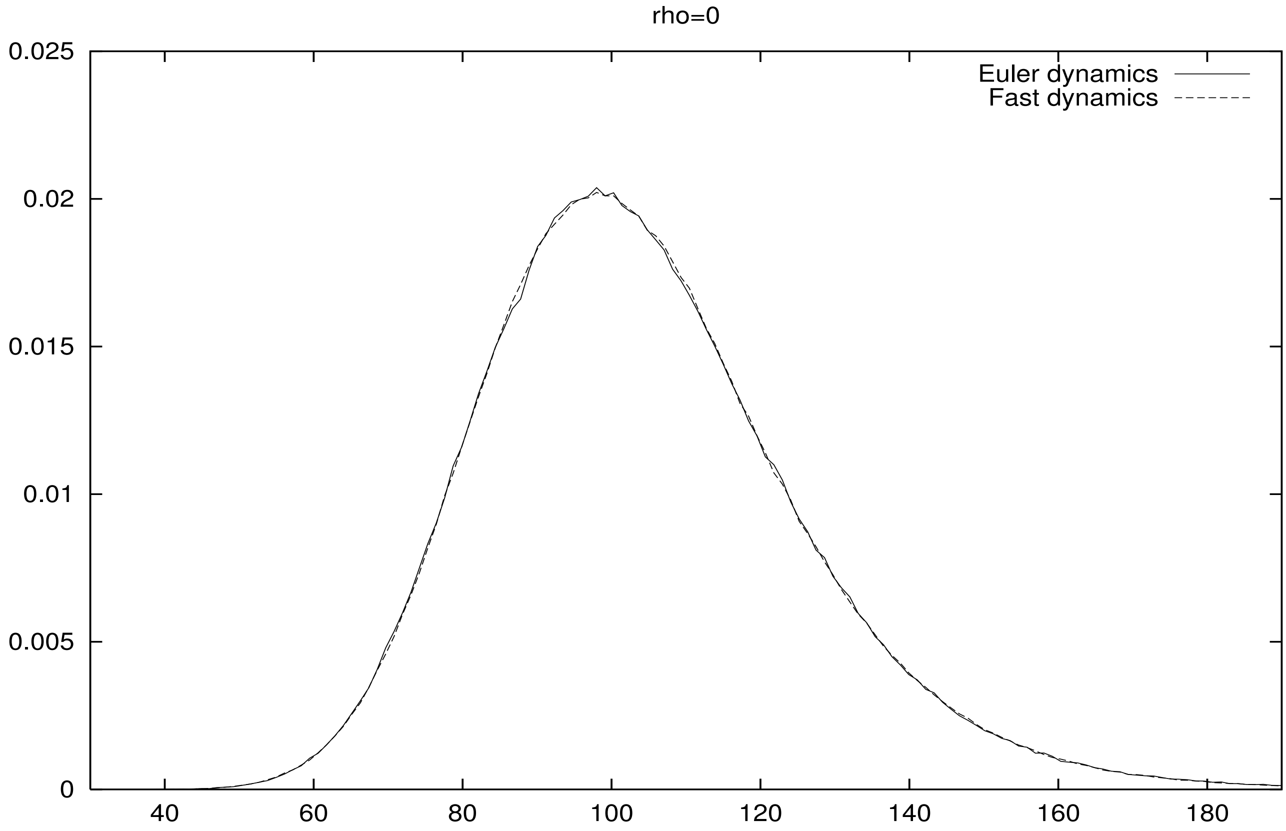
Example: 2×2 synthetic basket

$S_0^{(1)} = 1$, $\mu_1 = 5\%$, $\sigma_1(t)$, $\sigma_2(t)$, $\lambda_1 = 0.6$, $\lambda_2 = 1 - \lambda_1$

$S_0^{(2)} = 1$, $\mu_2 = 3\%$, $\eta_1(t)$, $\eta_2(t)$, $\xi_1 = 0.7$, $\xi_2 = 1 - \xi_1$

$B_T = w_1 S_T^{(1)} + w_2 S_T^{(2)}$, $w_1 = w_2 = 0.5$, $T =$1Y.

$\rho = 0$; the basket density $p_T(B, \rho)$ under the EMC scheme, continuous line; $\Pi_T(B, \rho)$ under the MVMD scheme, dashed line.



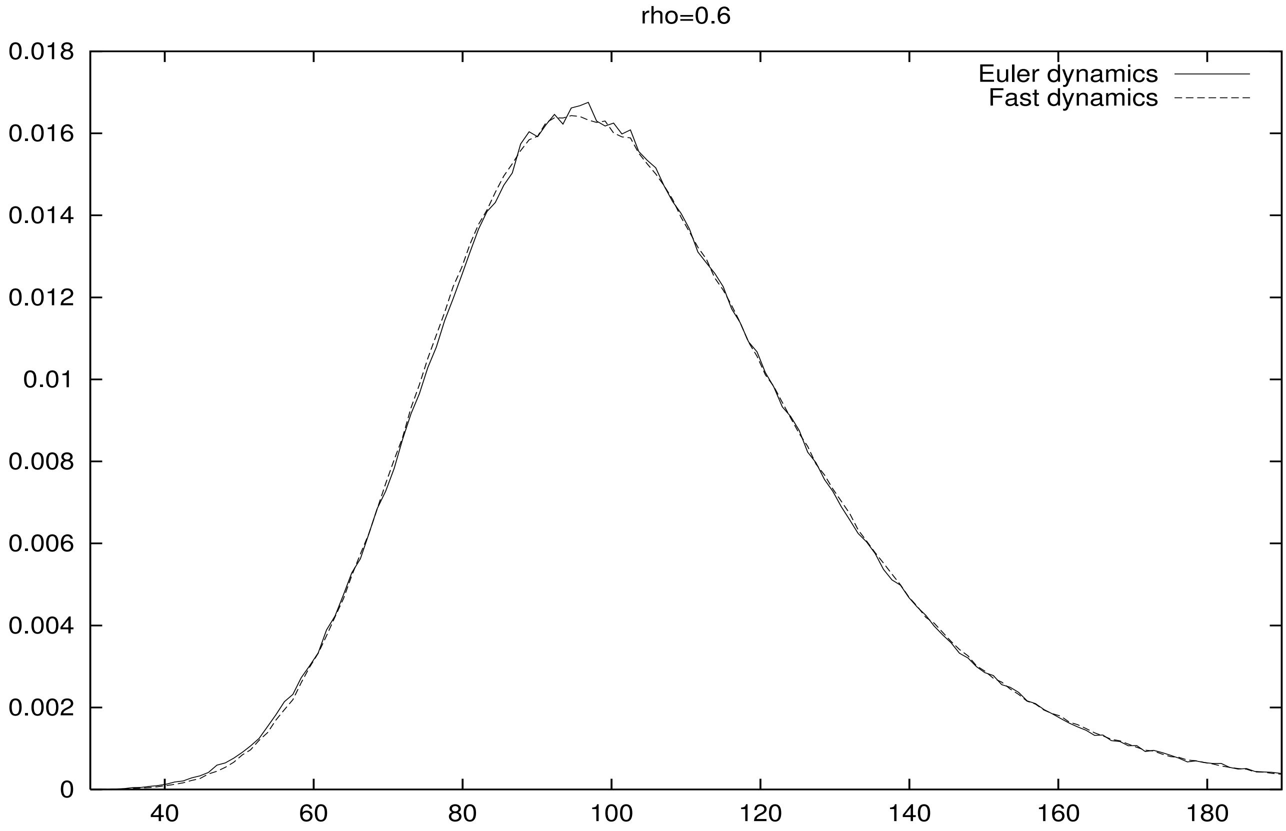
Example: 2×2 synthetic basket (cont'd)

$S_0^{(1)} = 1$, $\mu_1 = 5\%$, $\sigma_1(t)$, $\sigma_2(t)$, $\lambda_1 = 0.6$, $\lambda_2 = 1 - \lambda_1$

$S_0^{(2)} = 1$, $\mu_2 = 3\%$, $\eta_1(t)$, $\eta_2(t)$, $\xi_1 = 0.7$, $\xi_2 = 1 - \xi_1$

$B_T = w_1 S_T^{(1)} + w_2 S_T^{(2)}$, $w_1 = w_2 = 0.5$, $T =$1Y.

$\rho = 60\%$; the basket density $p_T(B, \rho)$ under the EMC scheme, continuous line; $\Pi_T(B, \rho)$ under the MVMD scheme, dashed line.



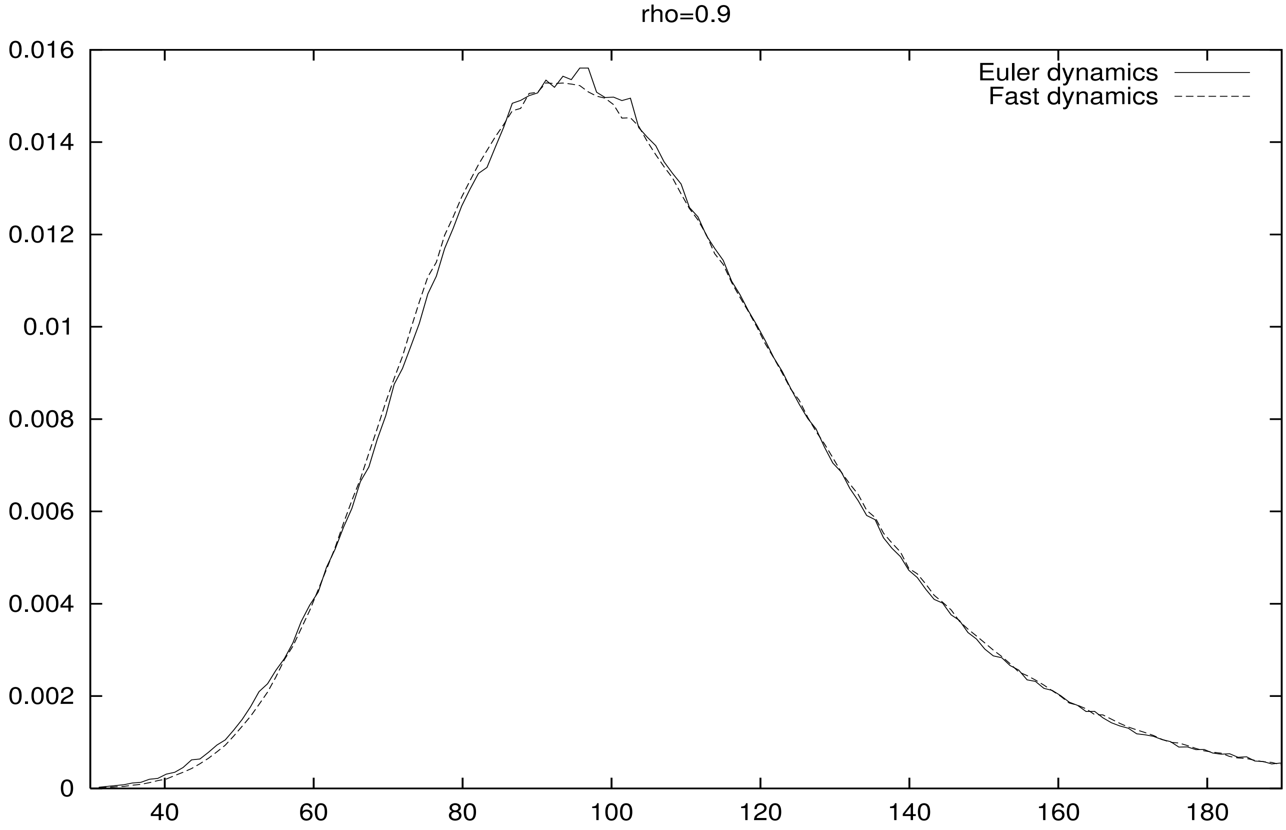
Example: 2×2 synthetic basket (cont'd)

$$S_0^{(1)} = 1,\ \mu_1 = 5\%,\ \sigma_1(t),\ \sigma_2(t),\ \lambda_1 = 0.6,\ \lambda_2 = 1 - \lambda_1$$

$$S_0^{(2)} = 1,\ \mu_2 = 3\%,\ \eta_1(t),\ \eta_2(t),\ \xi_1 = 0.7,\ \xi_2 = 1 - \xi_1$$

$$B_T = w_1 S_T^{(1)} + w_2 S_T^{(2)},\ w_1 = w_2 = 0.5,\ T = \text{1Y}.$$

$\rho = 90\%$; the basket density $p_T(B,\rho)$ under the EMC scheme, continuous line; $\Pi_T(B,\rho)$ under the MVMD scheme, dashed line.



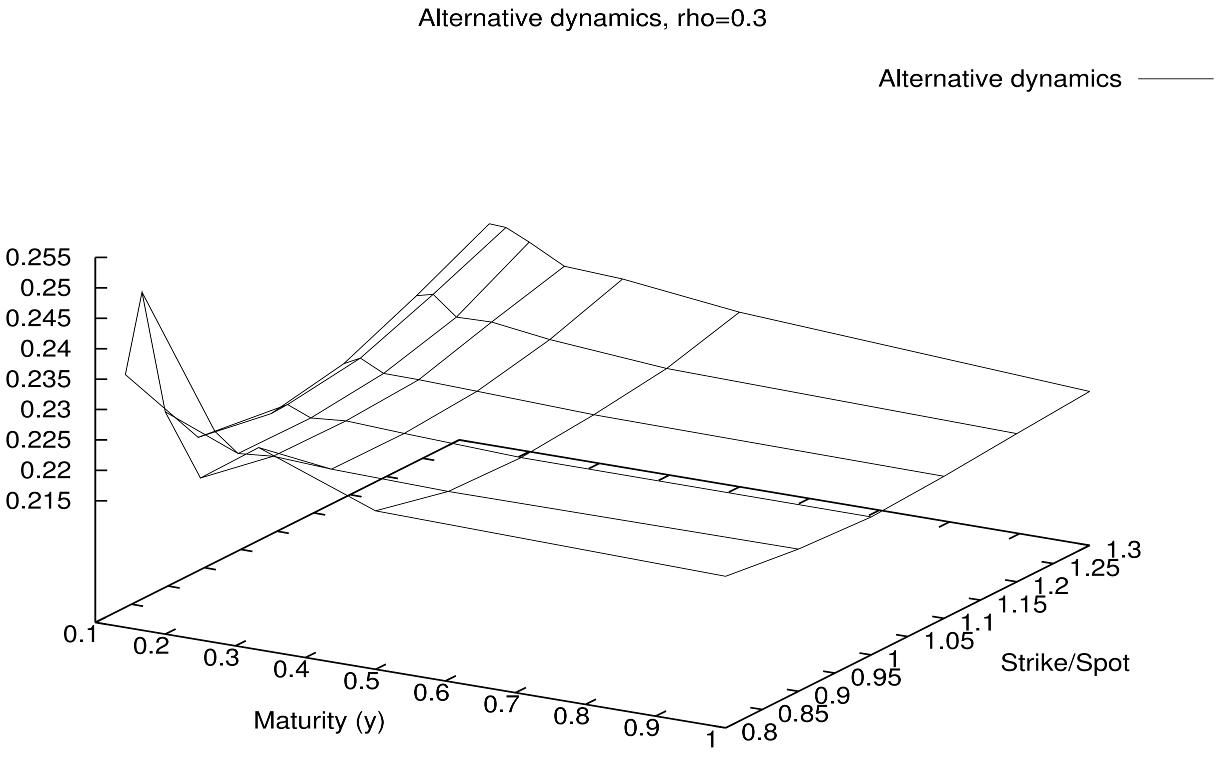
Example: 2×2 synthetic basket (cont'd)

$S_0^{(1)} = 1$, $\mu_1 = 5\%$, $\sigma_1(t)$, $\sigma_2(t)$, $\lambda_1 = 0.6$, $\lambda_2 = 1 - \lambda_1$

$S_0^{(2)} = 1$, $\mu_2 = 3\%$, $\eta_1(t)$, $\eta_2(t)$, $\xi_1 = 0.7$, $\xi_2 = 1 - \xi_1$

$B_T = w_1 S_T^{(1)} + w_2 S_T^{(2)}$, $w_1 = w_2 = 0.5$, T =1Y.

The basket smile for $\rho = 30\%$.



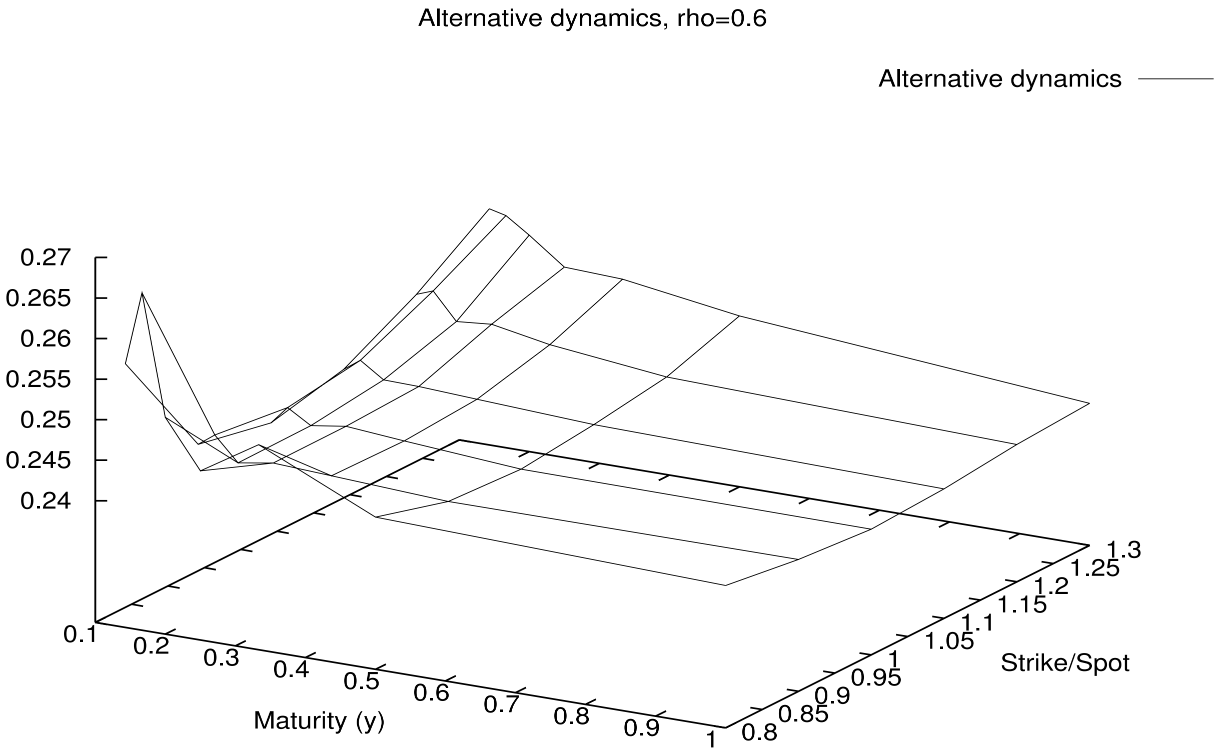
Example: 2×2 synthetic basket (cont'd)

$S_0^{(1)} = 1$, $\mu_1 = 5\%$, $\sigma_1(t)$, $\sigma_2(t)$, $\lambda_1 = 0.6$, $\lambda_2 = 1 - \lambda_1$

$S_0^{(2)} = 1$, $\mu_2 = 3\%$, $\eta_1(t)$, $\eta_2(t)$, $\xi_1 = 0.7$, $\xi_2 = 1 - \xi_1$

$B_T = w_1 S_T^{(1)} + w_2 S_T^{(2)}$, $w_1 = w_2 = 0.5$, T =1Y.

The basket smile for $\rho = 60\%$.



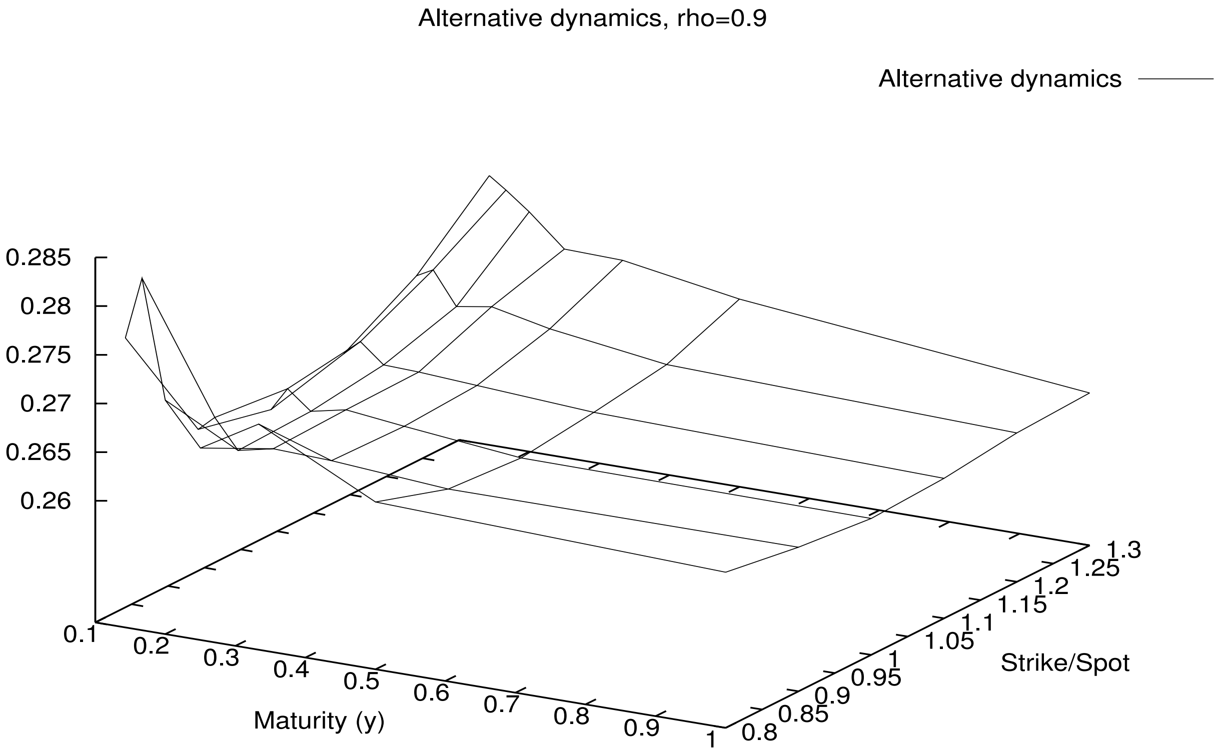
Example: 2×2 synthetic basket (cont'd)

$S_0^{(1)} = 1$, $\mu_1 = 5\%$, $\sigma_1(t)$, $\sigma_2(t)$, $\lambda_1 = 0.6$, $\lambda_2 = 1 - \lambda_1$

$S_0^{(2)} = 1$, $\mu_2 = 3\%$, $\eta_1(t)$, $\eta_2(t)$, $\xi_1 = 0.7$, $\xi_2 = 1 - \xi_1$

$B_T = w_1 S_T^{(1)} + w_2 S_T^{(2)}$, $w_1 = w_2 = 0.5$, T =1Y.

The basket smile for $\rho = 90\%$.



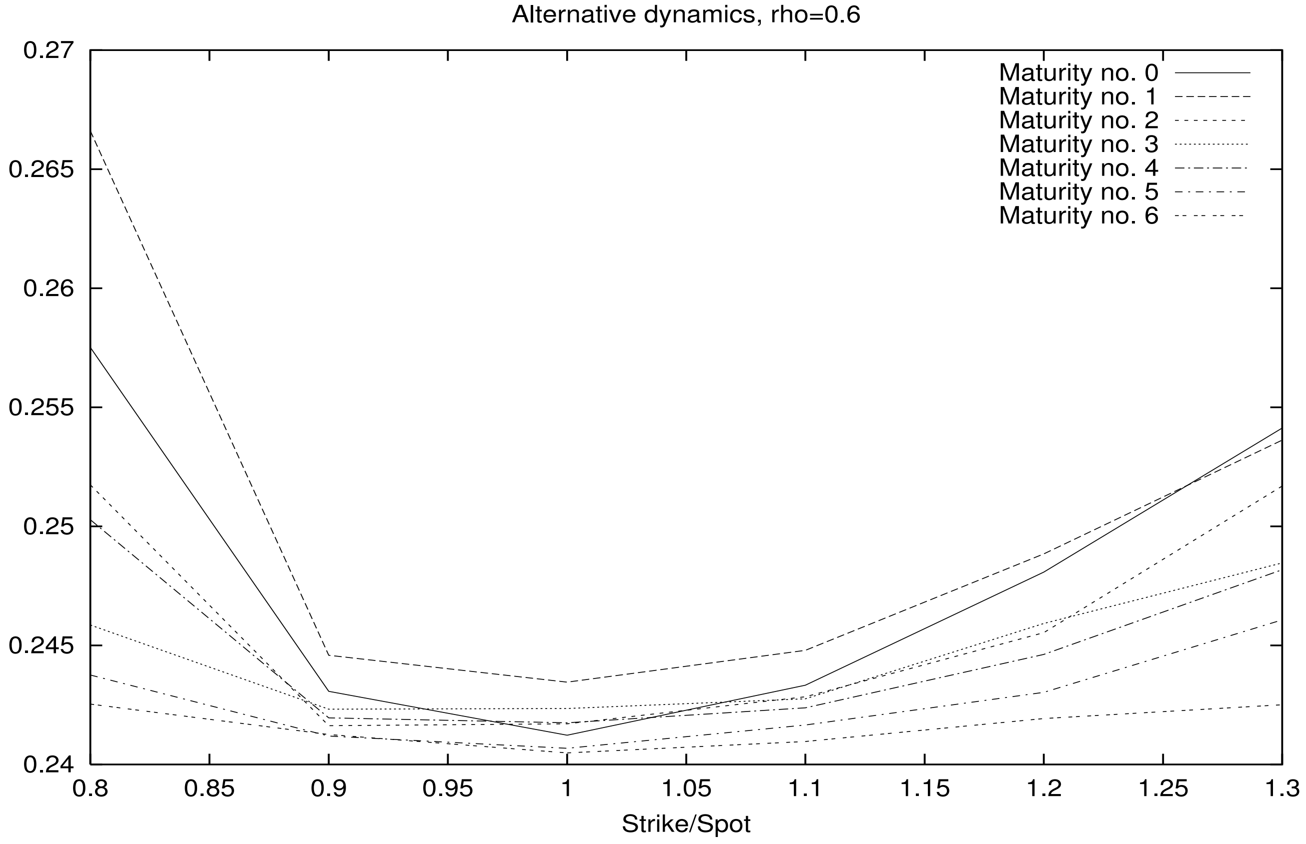
Example: 2×2 synthetic basket (cont'd)

$S_0^{(1)} = 1$, $\mu_1 = 5\%$, $\sigma_1(t)$, $\sigma_2(t)$, $\lambda_1 = 0.6$, $\lambda_2 = 1 - \lambda_1$

$S_0^{(2)} = 1$, $\mu_2 = 3\%$, $\eta_1(t)$, $\eta_2(t)$, $\xi_1 = 0.7$, $\xi_2 = 1 - \xi_1$

$B_T = w_1 S_T^{(1)} + w_2 S_T^{(2)}$, $w_1 = w_2 = 0.5$, $T =$1Y.

The basket smile for $\rho = 60\%$.



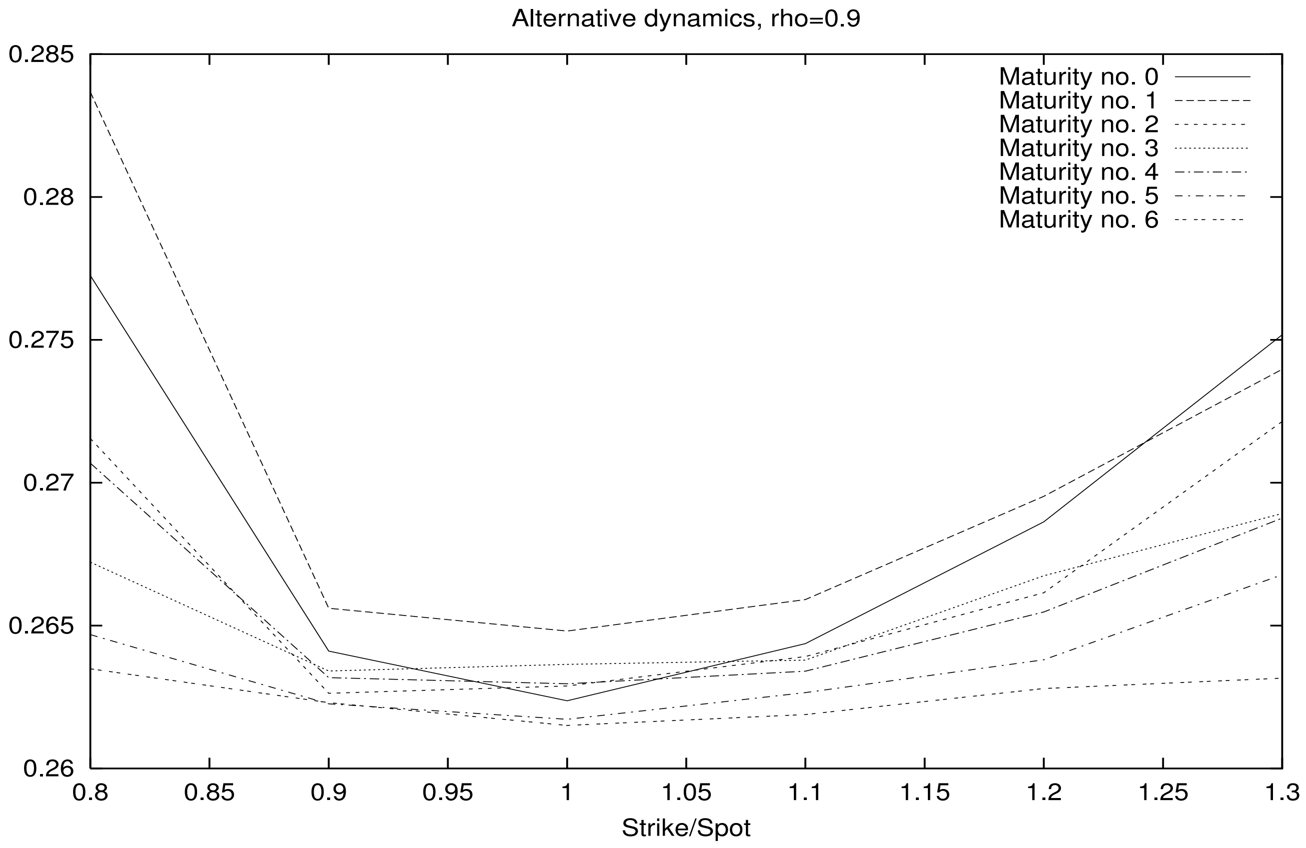
Example: 2×2 synthetic basket (cont'd)

$S_0^{(1)} = 1$, $\mu_1 = 5\%$, $\sigma_1(t)$, $\sigma_2(t)$, $\lambda_1 = 0.6$, $\lambda_2 = 1 - \lambda_1$

$S_0^{(2)} = 1$, $\mu_2 = 3\%$, $\eta_1(t)$, $\eta_2(t)$, $\xi_1 = 0.7$, $\xi_2 = 1 - \xi_1$

$B_T = w_1 S_T^{(1)} + w_2 S_T^{(2)}$, $w_1 = w_2 = 0.5$, $T =$1Y.

The basket smile for $\rho = 90\%$.



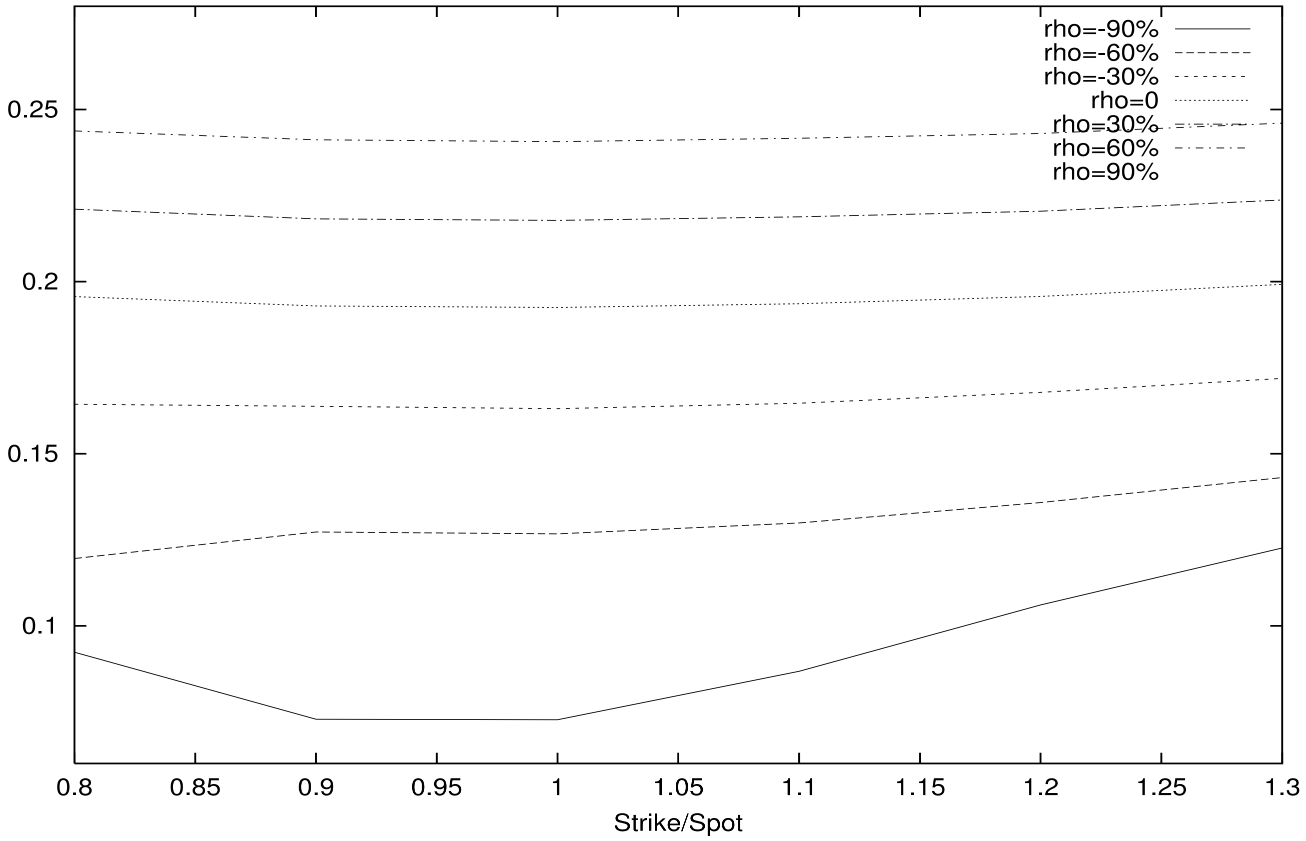
Example: 2×2 synthetic basket (cont'd)

$S_0^{(1)} = 1$, $\mu_1 = 5\%$, $\sigma_1(t)$, $\sigma_2(t)$, $\lambda_1 = 0.6$, $\lambda_2 = 1 - \lambda_1$

$S_0^{(2)} = 1$, $\mu_2 = 3\%$, $\eta_1(t)$, $\eta_2(t)$, $\xi_1 = 0.7$, $\xi_2 = 1 - \xi_1$

$B_T = w_1 S_T^{(1)} + w_2 S_T^{(2)}$, $w_1 = w_2 = 0.5$, T =6m.

The basket smile for T=6m for different ρ's.



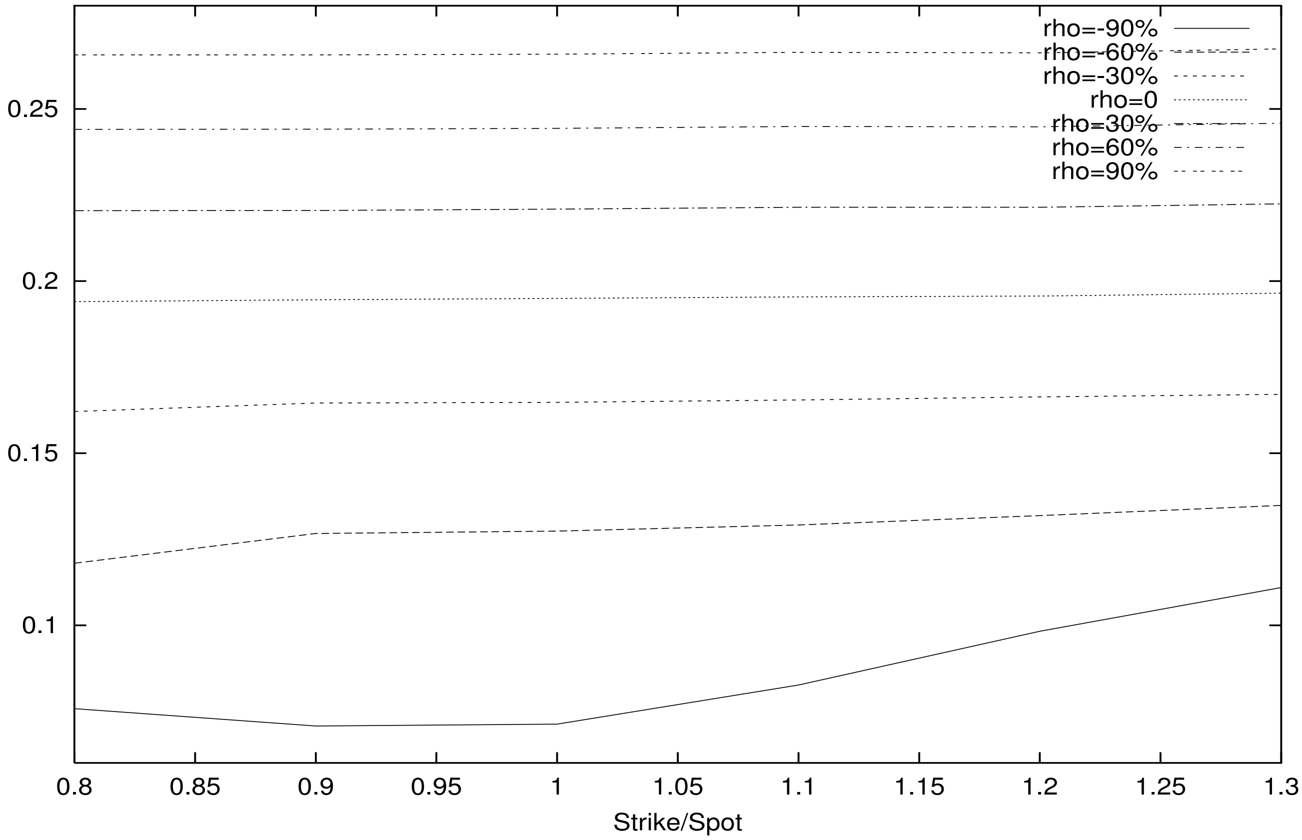
Flat vols:

$$S_0^{(1)} = 1,\ \mu_1 = 5\%,\ \sigma(t) = \sigma_1(t)$$

$$S_0^{(2)} = 1,\ \mu_2 = 3\%,\ \eta(t) = \eta_1(t)$$

$$B_T = w_1 S_T^{(1)} + w_2 S_T^{(2)},\ w_1 = w_2 = 0.5,\ T = \text{6m}.$$

The basket smile at flat assets' volatility for T=6m for different ρ's.



Conclusions and perspectives

Rather general uncertain volatility (and rate) model with special application to the FX options market. The model

- Has the tractability required (known marginal and transition densities, explicit European option prices);
- Prices analytically a number of exotic derivatives (barrier options, forward start options,...);
- Accommodates general implied volatility surfaces;
- Allows for Vega bucketing, i.e. for the calculation of the sensitivities with respect to each volatility quote.

  A further extension is based on Markov chains (volatility and rates are drawn on several fixed dates).
- Conceptually simple extension of an asset price model that works generally well for implied volatility smiles/skews on single assets
- Analytic expression for the *multivariate* local diffusion coefficient and for the multivariate marginal density
- General consistency of the model: individual smiles and dynamics are exactly reproduced
- Semianalytic pricing the European derivatives on many assets $\Longrightarrow$ CPU time generally shorter for small baskets
- Sampling from a different joint dynamics, *i.e.* from another manifold of the possible volatility structures for a multidimensional process
- Possible application to the LMM