

Option pricing and hedging for stock prices with discrete jumps and stochastic volatility

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Background & Existing Literature

- Geometric Brownian motion *Black and Scholes(1973), Merton(1973)*
- Stochastic Volatility *Hull and White(1987), Heston(1993), Duffie et. al.(2000)*
- Jump diffusion *Merton(1976), Bakshi and Chen(1997), Ait-Sahalia(2002)*
- Pure jump processes *Eberlein and Jacod(1997), Madan(1999)*
- Discreteness *Harris(1991), Brown et. al. (1991), Gottlieb and Kalay(1985), Ball(1988)* – Discrete time or Rounding of Continuous path
- Birth Death model *Korn et. al. (1998)*

Birth & Death Model

Perrakis(1988), Korn et. al.(1998)

Jump size $\pm c$.

$N_t = c \times S_t$ is a birth and death process.

Probability that jump size $Y_t = 1$ is p_t .

Such a process can be considered as a discretized version of the Black-Scholes model if the intensity of jumps is proportional to N_t^2 .

Consider processes with intensity $\lambda_t N_t^2$.

- λ_t is a constant.
- λ_t a stochastic process and N_t is a birth and death process conditional on the λ_t process.

Let the measure associated with the process N_t be \mathcal{P} .

Let $\xi(t)$ be the underlying process of event times.

So $d\xi(t) = 1$ if there is a jump at time t .

$$dS(t) = cY(t)d\xi(t)$$

Convergence

$$X_t^{(n)} = \ln\left(\frac{N_t^{(n)}}{n}\right)$$

$$X_t^{*(n)} = X_t^{(n)} - \int_0^t E\left[\ln\left(1 + \frac{Y_u}{N_u}\right)\right] N_u^2 \lambda_u du - X_0^{(n)}$$

[1] Assume $\forall \eta > 0 \forall t > 0$

$$\lim_{\alpha \rightarrow \infty} \limsup_n P^{*(n)}(|x| I_{|x| > \alpha} * \nu_t^{*(n)} > \eta) = 0$$

Proof:

Step(1): $\forall u > 0$

$$E[\ln(1 + \frac{Y_u}{N_u})]N_u^2\lambda_u = (2p_u - 1)N_u\lambda_u + O(\frac{1}{n})$$

$$\xrightarrow{P} r_u - \frac{\lambda_u}{2} \implies p_u = \frac{1}{2}(1 + \frac{r_u - \frac{\lambda_u}{2}}{N_u\lambda_u})$$

Step(2): $\forall t > 0$

$$[X_t^{*(n)}, X_t^{*(n)}]_t \xrightarrow{P} \int_0^t \lambda_u du$$

Step(3): $X_t^{*(n)}$ local martingale

(2),(3),[1] and Thm VIII.3.12 imply

$$X^{*(n)} \xrightarrow{d} BM(0, \int_0^t \lambda_u du)$$

This and (1) imply $X^{(n)} \xrightarrow{d} X$

$$X_t = BM(X_0 + \int_0^t (r_u - \frac{1}{2}\lambda_u)du, \int_0^t \lambda_u du)$$

$$d(X_t) = (r_t - \frac{1}{2}\lambda_t)dt + \sqrt{\lambda_t}dW_t$$

$$X^{(n)} \xrightarrow{d} X, S^{(n)} = \exp(X^{(n)}) \xrightarrow{d} \exp(X)$$

since \exp is continuous function.

By Ito's formula,

$$\begin{aligned} d(e^{X_t}) &= S_t[(r_t - \frac{1}{2}\lambda_t)dt + \sqrt{\lambda_t}dW_t] \\ &\quad + \frac{1}{2}S_t\lambda_t dt \\ &= S_t r_t dt + S_t \sqrt{\lambda_t} dW_t \end{aligned}$$

Hedging

The market is complete when we add a market traded derivative security.

We can hedge an option by trading the stock, the bond and another option.

Let $F_1(x, t)$, $F_2(x, t)$ be the prices of two options at time t when the price of the stock is cx . Let $F_0(x, t)$ be the price of the stock.

Assume F_i are continuous in both arguments.

We shall construct a self financing risk-less portfolio

$$V(t) = \sum_{i=0}^2 \phi^{(i)}(t) F_i(x, t)$$

Let $u^{(i)}(t) = \frac{\phi^{(i)}(t) F_i(x, t)}{V(t)}$ be the proportion of wealth invested in asset i .

$$\sum u^{(i)} = 1$$

Since V_t is self financing,

$$\begin{aligned} \frac{dV(t)}{V(t)} &= \sum_{i=0}^2 u^{(i)}(t) \frac{dF(x, t)}{F(x, t)} \\ &= u^{(0)}(t) \frac{1}{x} (dN_{1t} - dN_{2t}) \\ &\quad + \sum_{i=1}^2 u^{(i)}(t) (\alpha_{F_i}(x, t) dt \\ &\quad + \beta_{F_i}(x, t) dN_{1t} + \gamma_{F_i}(x, t) dN_{2t}) \end{aligned}$$

V_t is risk-less \implies

$$u^{(0)}(t)\frac{1}{x} + \sum_{i=1}^2 u^{(i)}(t)\beta_{F_i}(x, t) = 0,$$

$$-u^{(0)}(t)\frac{1}{x} + \sum_{i=1}^2 u^{(i)}(t)\gamma_{F_i}(x, t) = 0$$

No arbitrage $\implies \sum_{i=1}^2 u^{(i)}(t)\alpha_{F_i}(x, t) = \rho_t$ The hedge ratios are:

$$u^{(1)} = \frac{\rho(\beta_{F_2} + \gamma_{F_2})}{\alpha_{F_2}(\beta_{F_1} + \gamma_{F_1}) - \alpha_{F_1}\beta_{F_2} + \gamma_{F_2}}$$

$$u^{(2)} = -\frac{\rho(\beta_{F_1} + \gamma_{F_1})}{\alpha_{F_2}(\beta_{F_1} + \gamma_{F_1}) - \alpha_{F_1}\beta_{F_2} + \gamma_{F_2}}$$

$$u^{(0)} = 1 - u^{(1)} - u^{(2)}$$

Stochastic Intensity

Now we consider the case where the unobserved intensity λ_t is a stochastic process.

We first assume a two state Markov model for λ_t . *Naik(1993)*

Later we describe how we can have similar results for other models on λ_t . *Hull and White(1987)*

Suppose there is an unobserved state process θ_t which takes 2 values, say 0 and 1.

The transition matrix is \mathbf{Q} .

When $\theta_t = i$, $\lambda_t = \lambda_i$.

Counting process associated with θ_t is ζ_t .

Let us denote by $\{\mathcal{G}_t\}$ the complete filtration $\sigma(S_u, \lambda_u, 0 \leq u \leq t)$ and by P the probability measure on $\{\mathcal{G}_t\}$ associated with the process (S_t, λ_t) .

For hedging with stochastic intensity, same results as in the fixed λ case holds with

α, β, γ replaced by $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$

$$\begin{aligned}\tilde{\alpha} &= \left(\pi_0 \frac{\partial F_0}{\partial t} + \pi_1 \frac{\partial F_1}{\partial t} \right) \\ \tilde{\beta} &= \left(\pi_0 \beta_{F_0} F_0 + \pi_1 \beta_{F_1} F_1 \right) \\ \tilde{\gamma} &= \left(\pi_0 \gamma_{F_0} F_0 + \pi_1 \gamma_{F_1} F_1 \right)\end{aligned}$$

In this setting we need one option and the stock to hedge an option and do not need to invert at all time points as would be case if we did not use the posterior.

Let us assume that the risk-neutral measure is the measure associated with a birth-death process with event rate $\lambda_t^* N_t^2$ and probability of birth $p_t^* = \frac{1}{2}(1 + \frac{\rho_t}{\lambda_t^* N_t})$ where λ_t^* is a Markov process with state space $\{\lambda_1, \lambda_2\}$.

We get two different values of the expected price under the two values of $\theta(0)$.

The θ process is unobserved.

We cannot invert an option to get $\theta(0)$ because it takes two discrete values.

Need to introduce $\pi_i(t) = P(\theta_t = i | \mathcal{F}_t)$ where $\mathcal{F}_t = \sigma(S_u, 0 \leq u \leq t)$

As shown in Snyder(1973), under any $\hat{P} \in \mathcal{P}$ the π_{it} process evolves as:

$d\pi_{1t} = a(t)dt + b(t, 1)dN_{1t} + b(t, 2)dN_{2t}$ where $a(t)$ and $b(t, i)$ are \mathcal{F}_t adapted processes.

We still need one market traded option and the stock to hedge an option. But to get the hedge ratios, we need $\pi^*(t), a(t), b(t)$.

- The hedge ratios involve $a(t), b(t)$. So we need them to be predictable. But if we have to invert an option to get them, then we need to observe the price at time t to infer π_t and from there to get a_t and b_t . So they are no more predictable.
- Given a value of $\pi(t_0)$, the whole process $\pi(t), t > t_0$ is determined by the conditional distribution of the

θ process given the observed process S_t . $\pi(t)$ is completely determined by historical data. We do not have the freedom of determining $\pi(t)$ by inverting options. Thus, inferring $\pi(t)$ at each time point t independently will give incorrect prices and lead to arbitrage.

Bayesian Framework

As shown in Yashin(1970) and Elliott et. al.(1995), the posterior of $\theta_j(t)$ is given by:

$$\begin{aligned} \pi_j(t) = & \pi_j(0) + \int_0^t \sum_i q_{ij} \pi_i(u) du \\ & + \int_0^t \pi_j(u) (\bar{\lambda}(u) - \lambda_j) N_u^2 du \\ & + \sum_{0 < u < t} b_j(u) \end{aligned}$$

where $\bar{\lambda}(t) = \sum_i \pi_i(t) \lambda_i$ and

$$b_j(u) = \pi_j(u-) \left(\frac{\lambda_j p_{\lambda_j}(S_{u-} \rightarrow S_u)}{\sum_i \pi_i(u) \lambda_i p_{\lambda_i}(S_{u-} \rightarrow S_u)} - 1 \right)$$

Thus, $a_j(u) = \sum_i q_{ij} \pi_i(u) + \int_0^t \pi_j(u) (\bar{\lambda}(u) - \lambda_j) N_u^2$

Now we can hedge as in the constant intensity case with modified hedge ratios.

Pricing

How to get $E(e^{-\int_0^T \rho_s ds} X | \mathcal{F}_0)$ under fixed values of

$$\mathbf{Q}, \lambda_0, \lambda_1, \pi_0(0)?$$

Fix $\theta_0 = i$

Generate the θ process.

Generate the ξ waiting times as non-homogeneous Poisson process.

At each event time S_t jumps by $\pm c$ with probability p_t and $1 - p_t$.

Get the expectation under $\theta_0 = i$ for $i = 0, 1$.

Now take the average of these with respect to π_0 .

Generating ξ :

Generate T_0 from $Exp(\lambda_{\theta_0} N_0^2)$

Let $\tau_0 = \inf\{t : \theta_t \neq \theta_0\}$

If $T_0 < \tau_0$, jump at time T_0 .

Otherwise, generate T_1 from $Exp(\lambda_{\theta_{\tau_0}} N_{\tau_0}^2)$

$\tau_1 = \inf\{t : \theta_t \neq \theta_{\tau_0}\}$

If $T_1 < \tau_1$, jump at time $\tau_0 + T_1$

Continue.

This is justified by memorylessness.

Description of data

- The data was obtained from the optionmetrics database on 3 stocks: Ford (Dec 2002), IBM (June 2002) and ABMD(Feb 2003). The stock data is transaction by transaction. The option data is daily best bid and ask prices for all options traded on that day.
- The data is filtered for after hour and international market trading. The data now is on tradings in NASDAQ regular hours.
- The tick size is $\frac{1}{16}$ for Ford and $\frac{1}{100}$ for IBM and ABMD.
- We shall use the data for the first day of the month as training sample and for the rest of the days as test sample. Estimating risk-neutral parameters by inverting option prices in training sample.

Reducing number of options required

- We need to infer 5 parameters $q_{01}, q_{10}, \lambda_0, \lambda_1, \pi_0(0)$
 \Rightarrow Invert the prices of 5 options with different maturities at time 0.
- Some stocks do not have so many options. Using the filtering equations it is possible to infer the required quantities by using one option at 5 points of time close to 0.

- Another possibility of reducing the number of parameters to be estimated is to assume that the θ process is in equilibrium when we start observing. Then we need to estimate (or invert for) 4 parameters ($q_{01}, q_{10}, \lambda_0, \lambda_1$). The chain is irreducible and $T_{ii} \sim \text{Exp}(q_{10} + q_{01})$ that is non-lattice and finite mean.

$$\pi_0(0) = \frac{q_{10}}{q_{10} + q_{01}} \quad (\text{e.g. Ross Pg214})$$

Computational Details

The plan is to find the parameter set that minimizes the root mean square error between the bid-ask-midpoint and the daily average of the predicted option price, for all options in the training sample. I followed a diagonally scaled steepest descent algorithm with central difference approximation to the differential. The starting values of $\lambda_0, \lambda_1, q_{01}, q_{10}$ are obtained by an iterative method that has 2 steps. Let the underlying state process be η_t , that is the intensity is λ_i when η_t is i . In one step, the MLE of the parameters is obtained given the η_t process. In the next step, for each t when there is a jump in the stock price, we assign η_t to that i which maximizes the probability of an event. When this method converges, we do a finite search on the parameter π .

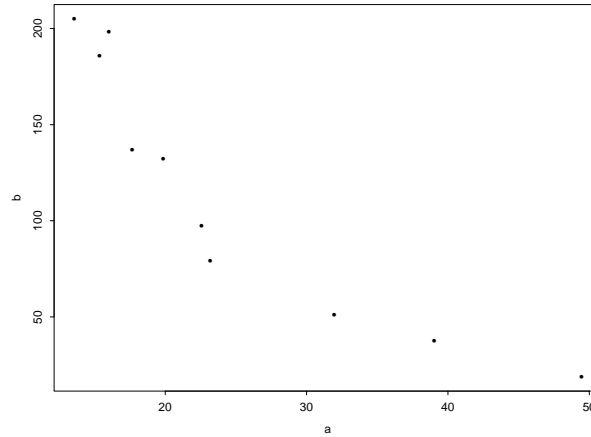


Figure 1: Estimation: ABMD Data, general model

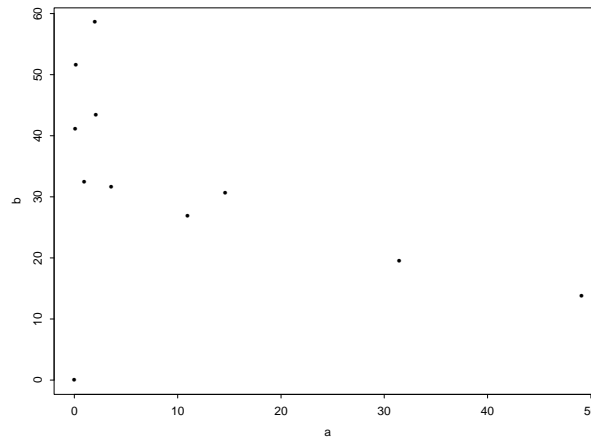
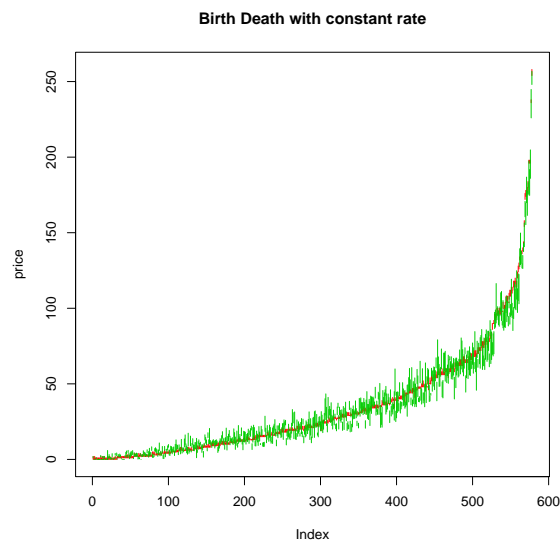
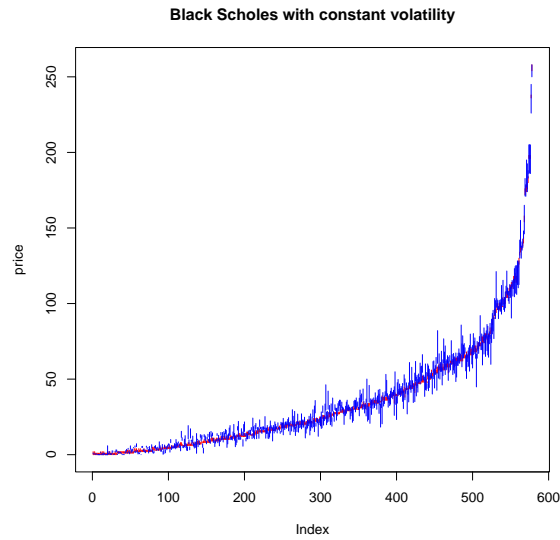


Figure 2: Estimation: ABMD Data, Birth Death model with Stochastic volatility

Date:Dec 2,2000. Tick= $\frac{1}{100}$. n=12. Range of option prices \$0-\$1.60. Avg bid ask spread=50.

We obtain the intervals for option prices under various values of the intensity parameter. The figures plot the length of the predicted interval against the distance of the predicted interval from the observed interval. The intervals from the birth-death process are clearly much smaller than those of the general model with the same error levels.



These figures present the predicted intervals and the bid-ask midpoint for the Ford data.

Generalizations

According to Snyder(1973), the $\pi_i(t)$ process evolves as: $d\pi(t) = a(t)dt + b(t, 1)dN_{1t} + b(t, 2)dN_{2t}$ where $a(t)$ and $b(t, i)$ are \mathcal{F}_t adapted processes. Let us see the form of the posterior in the general case:

Let $c_t(v|N_{t_0,t})$ be the posterior characteristic function for λ_t given an observed path

$$dc_t(v|N_{t_0,t}) = \lambda_{N_t, N_{t+dt}}^{*-1} \times E(\exp(iv\lambda_t)(\lambda_{N_t, N_{t+dt}} - \lambda_{N_t, N_{t+dt}}^*)|N_{t_0,t})d\xi_t \\ + E(\exp(iv\lambda_t)\Psi_t(v|N_{t_0,t}, \lambda_t)|N_{t_0,t})dt - E(\exp(iv\lambda_t)g(N_t)(\lambda_t - \hat{\lambda}_t)|N_{t_0,t})dt$$

where

$$\lambda_{N_t, \zeta_t} = \begin{cases} g(N_t)\lambda_t p_{\lambda_t, N_t} & \text{if } \zeta_t = N_t + 1 \\ g(N_t)\lambda_t(1 - p_{\lambda_t, N_t}) & \text{if } \zeta_t = N_t - 1 \\ 0 & \text{o. w.} \end{cases}$$

$$\lambda_{N_t, \zeta_t}^* = E(\lambda_{N_t, \zeta_t}(t, N_{t_0,t}, \lambda_t)|N_{t_0,t})$$

$$\Psi_t(v|N_{t_0,t}, \lambda_t) = E(\exp(iv\Delta\lambda_t)|N_{t_0,t}, \lambda_t)$$

For example, let $d\lambda_t = f_t(\lambda_t)dt + G_t(\lambda_t)dW_t$, where W_t is Brownian motion. In this case

$$\Psi_t(v|N_{t_0,t}, \lambda_t) = ivf_t(\lambda_t) - \frac{1}{2}v^2G_t^2(\lambda_t).$$

As long as the observed process is Markov jump process, the second and third terms are same as in the two state Markov case and hence yield the same function on taking inverse Fourier transform. The first term, on taking inverse Fourier transform yields L, the Kolmogorov-Fokker-Plank differential operator associated with λ .

$$d\pi_t = L + \int_0^t \pi_j(u)(\bar{\lambda} - \lambda_j)g(S(u))du \\ + \sum_{0 < u < t} \pi_j(u-) \left(\frac{\lambda_j p_{\lambda_j}(S_{u-} \rightarrow S_u)}{\sum_i \pi_i(u)\lambda_i p_{\lambda_i}(S_{u-} \rightarrow S_u)} - 1 \right)$$

Another possible direction is to consider jumps of size > 1 . But then we no longer have the distribution of jump size from simple martingale considerations. We have to either assume the jump distribution, or estimate it, or impose some optimization criterion to get a unique price. Also, if the jump magnitude can take k values, we need $k - 1$ market traded options and the stock to hedge an option.

Conclusions

- Quadratic Variation is not observable
- Bounds are small as opposed to general jump models which produce huge intervals
- General jump models are not amenable to hedging, Birth-Death process is
- Stochastic intensity
- Combining risk neutral estimation with updating by historical data
- Not inverting option prices at all point of time as options are not as frequently traded as stocks and hence not very reliable