Endogenous Risk Aversion and Ockham's Razor

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Michael Stutzer

Professor of Finance and Director, Burridge Center for Securities Analysis and Valuation University of Colorado

Closely Related Publications

1. Stutzer, \Asset Allocation Without Unobservable Parameters" Financial Analysts Journal, forthcoming 2004

2. Pham, \A Large Deviations Approach to Optimal Long Term Investment",Finance and Stochastics, 2003
 | Used My 2000 (Working Paper) Framework
 | Derived Optimal Dynamic Portfolios

3. Cont, et.al. \A Large Deviation Approach to Portfolio Management", IJTAF, 2002

4. Rabin (Econometrica, 2000) ; Rabin and Thaler (J. of Economic Perspectives, 2001).

5. Dembo, Deuschel, and Du±e, \Large Portfolio Losses", Fin.&Sto., 2004

Samuelson's (1963) Friendly Wager:

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$$E(W_1) = W_0 + 50 \quad \sqrt{Var[W_1]} = 150$$

$$Prob[W_1 < W_0] = 50\%$$

Samuelson's Friend Declined This, But Was Willing to Accept T = 100 Such Bets:

$$E(W_{100}) = W_0 + 5000 \quad \sqrt{Var[W_{100}]} = 1500$$

$$Prob[W_{100} < W_0] = 0.04\%$$

Samuelson Challenged the Logic of This Reply:

(i) Shortfall Probability-Based Decision Rules Might Be Intransitive.

(ii) When T is Large, Losses Are Indeed Improbable, But Could Still Be Large.

(iii) $E[U(W_1)] < U(W_0) \;\; \forall W_0 \Rightarrow$
$E[U(W_T)] < U(W_0)$

Rabin Calibration Theorem:
(Econometrica, 2000)

If Friend Had Used $E[U(W_1)]$, He Would Also Have Had to Reject:

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$$E[W_1] = W_0 + 9900 \quad \sqrt{Var[W_1]} = 10100$$

$$Prob[W_1 < W_0 - 200] = 0 \text{ !!}$$

Rabin and Thaler's (2001) Conclusions:

(i) Friend Probably Would Have Accepted This.
(ii) No Repeated Bet;Large Loss is Impossible.

(iii) Thus, Friend Didn't Use any $E[U(W_1)]$.

\Data sets dominated by smaller-scale investment opportunities are likely to yield much higher estimates of risk aversion than data sets dominated by larger-scale investment opportunities. Indeed, the correct conclusion for economists to draw, both from thought experiments and from actual data, is that people do not display a consistent coe ± cient of risk aversion, so it is a waste of time to try to measure it."

I Will Now Derive a Repeated Betting/Investment Criterion, That Is Not Subject to This Critique.

Specifically, I Will Argue That:

(i) Theoretical and empirical researchers' typical assumption of Power (CRRA) Utility implies even more paradoxical repeated betting/ investment behavior.

(ii) Contrary to Samuelson's opinion, there is a useful probabilistic alternative to expected utility. It is based on maximizing the probability of outperforming a benchmark that the agent wants to exceed.

(iii) The Gärtner-Ellis Large Deviations Theorem is used to show that this is equivalent to using endogenous, policy-dependent coefficients of risk aversion, in accord with at least part of Rabin and Thaler's quoted claim.

Consider Thorp/Ziemba "Blackjack" Wager:

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Expected CRRA Utility $U(W_T) = -W_T^{-\theta}$**:**

$$
\begin{aligned}
E[U(W_T)] &= E\left[U\left(W_0 \prod_{t=1}^{T} R_{pt}\right)\right] \\
&= E\left[-\left(W_0 \prod_{t=1}^{T} R_{pt}\right)^{-\theta}\right] \\
&= -W_0^{-\theta} \prod_{t=1}^{T} E\left[R_{pt}^{-\theta}\right] \\
&= -W_0^{-\theta} E\left[R_p^{-\theta}\right]^T \\
&= -W_0^{-\theta} \left(\pi(1+p)^{-\theta} + (1-\pi)(1-p)^{-\theta}\right)^T
\end{aligned}
$$

Maximum acceptable CRRA bet $\overline{p}$ **solves:**

$$\pi(1+\overline{p})^{-\theta} + (1-\pi)(1-\overline{p})^{-\theta} = 1$$

while FONC for optimal CRRA bet p_θ **solves:**

$$\log\left(\frac{1+p_\theta}{1-p_\theta}\right) = \frac{1}{1+\theta}\log\left(\frac{\pi}{1-\pi}\right)$$

INDEPENDENT OF T **!!!**

Numerical Example:

Prob[Win]$\equiv \pi$ = 60%

Agents Who Use Expected CRRA Utility		
CRRA 1 + θ	**$\overline{p}$ %**	**p_θ%**
20	**2.0**	**1.0**
10	**4.0**	**2.0**
5	**8.0**	**4.1**
3	13.4	**6.7**
2	**19.7**	**10.1**
1 (Log)	**38.9**	**20**

Table 1: **CRRA maximum acceptable ($\overline{p}$) and expected utility maximizing (p_θ) bets when the probability of winning is π = 60%, i.e. the betting edge is 20% and the odds are 3:2.**

{ I will argue that an agent wanting wealth to grow by at least 1% per bet should bet p = 14.1%. After T = 1000 bets, there is 97% probability that cumulative ROR $> 2,200,000$%.

{ But it would be rejected by a CRRA bettor whose CRRA > 2.87, as would even better $T > 1000$ bets.

Risk Aversion Parameter Measure

Barsky, et.al. (QJE, 1997) Questionnaire:

Suppose that you are the only income earner in the family, and you have a good job guaranteed to give you your current (family) income for life. You are given the opportunity to take a new job, with a 50-50 chance it will double your current income, and a 50-50 chance that it will cut your income by 20 percent. Would you take the new job?

{ If \NO", CRRA 1 + $\theta > 3.76$.

{ 2 out of 3 Answer \No".

Hence Range is [3.76, +∞]

Barsky, et.al. Guessed Average is Over 10 !!

{ Player Wouldn't Bet Any $p >$ 4%!

Underperformance Probabilities

R_{pt} = 1 + r_{pt}; the (random) gross return at t.
$W_T = W_0 \prod_{t=1}^{T} R_{pt} \equiv W_0 \left[e^{\overline{\log R_p}} \right]^T$, where $\overline{\log R_p} = \frac{1}{T} \sum_{t=1}^{T} \log R_{pt}$ is the cumulative growth rate up to T.

A comparison benchmark wealth path is:
$W_0 \left[e^{\log r} \right]^T = W_0 r^T$, growing at constant gross rate r, e.g. $\log r$ = 1%. Could also be a benchmark portfolio.

Here is $Prob\left[W_T \leq W_0 r^T\right] \equiv Prob\left[\overline{\log R_p} \leq 1\%\right]$:

FIGURE DISPLAYED HERE

Maximum Outperformance Probability

(i) Only reject p that don't make:
$Prob\left[W_T \leq W_0 r^T\right] \to 0$ **as** $T \to \infty$.

(ii) Rank each p by size of its probability curve's Decay Rate to 0, denoted $I_p(\log r)$:

$$I_p(\log r) \equiv \max_{\theta} - \lim_{T\to\infty} \frac{1}{T} \log E\left[e^{-\theta \sum_{t=1}^{T}(\log R_{pt} - \log r)}\right]$$

$$\begin{aligned}
\arg\max_p I_p(\log r) &\equiv \arg\max_p \max_{\theta>0} \lim_{T\to\infty} -\frac{1}{T}\log E\left[\left(\frac{W_T}{W_0 r^T}\right)^{-\theta}\right] \\
&\equiv \arg\max_p \lim_{T\to\infty} -\frac{1}{T}\log E\left[\left(\frac{W_T}{W_0 r^T}\right)^{-\theta(p)}\right] \quad (1) \\
&\overset{IID}{\equiv} \arg\max_p \max_{\theta>0} -\log E\left[\left(\frac{R_p}{r}\right)^{-\theta}\right] \\
&\equiv \arg\max_p \max_{\theta>0} E\left[-\left(\frac{R_p}{r}\right)^{-\theta}\right] \quad (2) \\
&\equiv \arg\max_p E\left[-\left(\frac{R_p}{r}\right)^{-\theta(p)}\right] \quad (3)
\end{aligned}$$

In the example:

$$E\left[-\left(\frac{R_p}{r}\right)^{-\theta}\right] = -\left[.6\left(\frac{1+p}{e^{.01}}\right)^{-\theta} + .4\left(\frac{1-p}{e^{.01}}\right)^{-\theta}\right].$$

Example Calculations

$$E\left[-\left(\frac{Rp}{r}\right)^{-\theta}\right] = -\left[.6\left(\frac{1+p}{e^{.01}}\right)^{-\theta} + .4\left(\frac{1-p}{e^{.01}}\right)^{-\theta}\right].$$

For each p, maximize this over θ, producing the policy-dependent coefficients of risk aversion $1+\theta(p)$ used to evaluate the expected habit-formation power utility. Then, find the optimal p **that maximizes it:**

Agent Who Wants to Beat Benchmark $\log r = 1.0\%$ Per Bet			
p%	Value of E	$I_p(1\%)$ %	Risk Aversion $1+\theta(p)$
6.0	≈ -1	≈ 0	**1.06**
8.0	**-.9994**	**0.06**	**3.03**
14.1	-.9982	0.18	1.43
20.0	**-.9987**	**0.13**	**1.25**
33.0	≈ -1	≈ 0	**1.008**

A less risk tolerant agent would only try to beat the easier benchmark $\log r = 0.1\%$:

Agent Who Wants to Beat Benchmark $\log r = 0.1\%$ Per Bet			
p%	Value of E	$I_p(1\%)$ %	Risk Aversion $1+\theta(p)$
1.0	**-.9954**	**0.46**	**10.73**
4.5	-.9877	1.23	4.53
14.1	**-.9924**	**0.77**	**1.88**
20.0	**-.9954**	**0.46**	**1.48**
33.0	**-.9996**	**0.04**	**1.09**

"Indeed, the correct conclusion for economists to draw, both from thought experiments and from actual data, is that people do not display a consistent coefficient of risk aversion, so it is waste of time to try to measure it." (Rabin and Thaler (2001), p.225)

Summary of Behavioral Implications

Agents Who Want to Beat Di®erent Benchmark Growth Rates $\log r$			
$\log r$%	p_{opt} **%**	$I_{p_{opt}}(\log r)$**%**	**Risk Aversion $1 + \theta(p_{opt})$**
2.01	**20.0**	**0.0**	1
1.0	**14.1**	**0.18**	1.43
0.5	**10.0**	**0.51**	2.02
0.1	**4.5**	**1.23**	4.53

Samuelson's Fears Aren't Realized

- Intransitivities Can't Happen
  - Samuelson Didn't Envision a Benchmark

- Size of Potential Loss Isn't Ignored
  - A Benchmark Helps Fix That, Too.

- Why Not Give Larger Losses Higher Marginal Impact, And Then Weight Them By Their Respective Probabilities ?
  - That is Expected Concave Utility ! Absorb This Talk !

- Expected Concave Utility is Tractable
  - So is Benchmark Outperformance Prob.

Other Alternatives

- **External Habit Formation**

$$U = -(R_p/X)^{-\theta}$$

- **\Prospect" Loss Aversion**

$$U = \begin{cases} -R^{-\theta_1} & if \;\; R \geq 1 \\ -\lambda R^{-\theta_2} & if \;\; R < 1 \end{cases}$$

- **Epstein-Zin Recursive**

$$U_t = \left\{(1-\delta)C_t^{1-1/\psi} + \delta\left(E_t[U_{t+1}^{1-\theta}]\right)^{\frac{1}{(1-\theta)/(1-1/\psi)}}\right\}^{\frac{1}{1-1/\psi}}$$

{ These Alternatives Have More Adjustable Parameters

Ockham's Razor Favors Mine

Bayesian analysis also shows that a hypothesis with fewer adjustable parameters automatically has an enhanced posterior probability, because the predictions it makes are sharp. (Berger and Je®erys, 1992)

$$\frac{PostProb[H_0]}{PostProb[H_A]} = \frac{L(p_{obs}|H_0)}{L(p_{obs}|H_A)} \frac{\mu(H_0)}{\mu(H_A)} \stackrel{Def.}{=} B*PriorOdds$$

Candidates for H_A:

Habit-Formation CRRA: 2 Adjustable Parameters.
Benartzi-Thaler Loss Aversion: 3 Parameters.
Epstein-Zin: 3 Parameters

{ Likelihood functions $L(p \mid H)$ Must Be Computed Before p_{obs} **is Observed !**

{ Extra Parameters Spread Out Likelihood Unless Directly Measured With Low Error.

{ When the Benchmark is Observable, H_0 has 0 Adjustable Parameters !