Endogenous Risk Aversion and Ockham’s Razor

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Closely Related Publications


2. Pham, "A Large Deviations Approach to Optimal Long Term Investment", Finance and Stochastics, 2003
   | Derived Optimal Dynamic Portfolios


5. Dembo, Deuschel, and Dürr, "Large Portfolio Losses", Fin.& Sto., 2004
Samuelson's (1963) Friendly Wager:

**BET DISPLAYED HERE**

\[ E(W_1) = W_0 + 50 \quad \sqrt{Var[W_1]} = 150 \]
\[ Prob[W_1 < W_0] = 50\% \]

Samuelson's *Friend Declined This, But Was Willing to Accept* \( T = 100 \) *Such Bets:*

\[ E(W_{100}) = W_0 + 5000 \quad \sqrt{Var[W_{100}]} = 1500 \]
\[ Prob[W_{100} < W_0] = 0.04\% \]

Samuelson Challenged the Logic of *This* Reply:

(i) Shortfall Probability-Based Decision Rules Might Be Intransitive.
(ii) When \( T \) is Large, Losses Are Indeed Improbable, But Could Still Be Large.

(iii) \( E[U(W_1)] < U(W_0) \quad \forall W_0 \Rightarrow \]
\[ E[U(W_T)] < U(W_0) \]
Rabin Calibration Theorem: (Econometrica, 2000)

If Friend Had Used $E[U(W_1)]$, He Would Also Have Had to Reject:

\begin{equation}
E[W_1] = W_0 + 9900 \quad \sqrt{Var[W_1]} = 10100
\end{equation}

\begin{equation}
Prob[W_1 < W_0 - 200] = 0 !
\end{equation}

Rabin and Thaler's (2001) Conclusions:

(i) Friend Probably Would Have Accepted This.
(ii) No Repeated Bet; Large Loss is Impossible.
(iii) Thus, Friend Didn't Use any $E[U(W_1)]$.

Data sets dominated by smaller-scale investment opportunities are likely to yield much higher estimates of risk aversion than data sets dominated by larger-scale investment opportunities. Indeed, the correct conclusion for economists to draw, both from thought experiments and from actual data, is that people do not display a consistent coefficient of risk aversion, so it is a waste of time to try to measure it."

I Will Now Derive a Repeated Betting/Investment Criterion, That Is Not Subject to This Critique.
Specifically, I Will Argue That:

(i) Theoretical and empirical researchers' typical assumption of Power (CRRA) Utility implies even more paradoxical repeated betting/investment behavior.

(ii) Contrary to Samuelson's opinion, there is a useful probabilistic alternative to expected utility. It is based on maximizing the probability of outperforming a benchmark that the agent wants to exceed.

(iii) The Gärtner-Ellis Large Deviations Theorem is used to show that this is equivalent to using endogenous, policy-dependent coefficients of risk aversion, in accord with at least part of Rabin and Thaler's quoted claim.
Consider Thorp/Ziemba "Blackjack" Wager:

**BET DISPLAYED HERE**

**Expected CRRA Utility** \( U(W_T) = -W_T^{-\theta} \):

\[
E[U(W_T)] = E\left[U(W_0 \prod_{t=1}^{T} R_{pt})\right] \\
= E\left[-(W_0 \prod_{t=1}^{T} R_{pt})^{-\theta}\right] \\
= -W_0^{-\theta} \prod_{t=1}^{T} E\left[R_{pt}^{-\theta}\right] \\
= -W_0^{-\theta} \left( \pi (1 + p)^{-\theta} + (1 - \pi)(1 - p)^{-\theta}\right)^T
\]

**Maximum acceptable CRRA bet** \( \bar{p} \) solves:

\[\pi (1 + \bar{p})^{-\theta} + (1 - \pi)(1 - \bar{p})^{-\theta} = 1\]

while **FONC for optimal CRRA bet** \( p_\theta \) solves:

\[\log\left(\frac{1+p_\theta}{1-p_\theta}\right) = \frac{1}{1+\theta} \log\left(\frac{\pi}{1-\pi}\right)\]

**INDEPENDENT OF** \( T \) !!!
Numerical Example:

Prob[Win] ≡ π = 60%

<table>
<thead>
<tr>
<th>CRRA 1 + θ</th>
<th>$\bar{p}$ %</th>
<th>$p_\theta$ %</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>2.0</td>
<td>1.0</td>
</tr>
<tr>
<td>10</td>
<td>4.0</td>
<td>2.0</td>
</tr>
<tr>
<td>5</td>
<td>8.0</td>
<td>4.1</td>
</tr>
<tr>
<td>3</td>
<td>13.4</td>
<td>6.7</td>
</tr>
<tr>
<td>2</td>
<td>19.7</td>
<td>10.1</td>
</tr>
<tr>
<td>1 (Log)</td>
<td>38.9</td>
<td>20</td>
</tr>
</tbody>
</table>

Table 1: CRRA maximum acceptable ($\bar{p}$) and expected utility maximizing ($p_\theta$) bets when the probability of winning is π = 60%, i.e. the betting edge is 20% and the odds are 3:2.

{ I will argue that an agent wanting wealth to grow by at least 1% per bet should bet $p = 14.1\%$. After $T = 1000$ bets, there is 97% probability that cumulative ROR > 2, 200, 000%.

{ But it would be rejected by a CRRA bettor whose CRRA > 2.87, as would even better $T > 1000$ bets.
Risk Aversion Parameter Measure

Barsky, et.al. (QJE, 1997) Questionnaire:

Suppose that you are the only income earner in the family, and you have a good job guaranteed to give you your current (family) income for life. You are given the opportunity to take a new job, with a 50-50 chance it will double your current income, and a 50-50 chance that it will cut your income by 20 percent. Would you take the new job?

\{ If \textit{\text NO"}, CRRA $1 + \theta > 3.76$.

\{ 2 out of 3 Answer \textit{\text No"}.

Hence Range is $[3.76, +\infty]$

Barsky, et.al. Guessed Average is Over 10 !!

\{ Player Wouldn't Bet Any $p > 4\%$!
Underperformance Probabilities

\[ R_{pt} = 1 + r_{pt}; \text{ the (random) gross return at } t. \]
\[ W_T = W_0 \prod_{t=1}^{T} R_{pt} \equiv W_0 \left[ e^{\log R_p} \right]^T, \text{ where } \log R_p = \frac{1}{T} \sum_{t=1}^{T} \log R_{pt} \]
is the cumulative growth rate up to \( T \).

A comparison benchmark wealth path is:
\[ W_0 [e^{\log r}]^T = W_0 r^T, \text{ growing at constant gross rate } r, \]
e.g. \( \log r = 1\% \). Could also be a benchmark portfolio.

Here is \( \text{Prob} \left[ W_T \leq W_0 r^T \right] \equiv \text{Prob} \left[ \log R_p \leq 1\% \right]: \)

FIGURE DISPLAYED HERE
Maximum Outperformance Probability

(i) Only reject \( p \) that don't make:
\[
Prob \left[ W_T \leq W_{0rT} \right] \to 0 \text{ as } T \to \infty.
\]

(ii) Rank each \( p \) by size of its probability curve's Decay Rate to 0, denoted \( I_p(\log r) \):

\[
I_p(\log r) \equiv \max_{\theta} - \lim_{T \to \infty} \frac{1}{T} \log E \left[ e^{-\theta \sum_{t=1}^{T}(\log R_{pt} - \log r)} \right]
\]

\[
\arg \max_{p} I_p(\log r) \equiv \arg \max_{p} \max_{\theta > 0} \lim_{T \to \infty} - \frac{1}{T} \log E \left[ \left( \frac{W_T}{W_{0rT}} \right)^{-\theta} \right]
\]

\[
\equiv \arg \max_{p} \lim_{T \to \infty} - \frac{1}{T} \log E \left[ \left( \frac{W_T}{W_{0rT}} \right)^{-\theta(p)} \right] \quad (1)
\]

\[
\text{IID} \quad \arg \max_{p} \max_{\theta > 0} - \log E \left[ \left( \frac{R_p}{r} \right)^{-\theta} \right]
\]

\[
\equiv \arg \max_{p} \max_{\theta > 0} E \left[ - \left( \frac{R_p}{r} \right)^{-\theta} \right] \quad (2)
\]

\[
\equiv \arg \max_{p} E \left[ - \left( \frac{R_p}{r} \right)^{-\theta(p)} \right] \quad (3)
\]

In the example:

\[
E \left[ - \left( \frac{R_p}{r} \right)^{-\theta} \right] = - \left[ 0.6 \left( \frac{1 + p}{e^{0.01}} \right)^{-\theta} + 0.4 \left( \frac{1 - p}{e^{0.01}} \right)^{-\theta} \right].
\]
Example Calculations

\[ E \left[ - \left( \frac{R_p}{r} \right)^{-\theta} \right] = - \left[ 0.6 \left( \frac{1 + p}{e^{.01}} \right)^{-\theta} + 0.4 \left( \frac{1 - p}{e^{.01}} \right)^{-\theta} \right]. \]

For each \( p \), maximize this over \( \theta \), producing the policy-dependent coefficients of risk aversion \( 1 + \theta(p) \) used to evaluate the expected habit-formation power utility. Then, find the optimal \( p \) that maximizes it:

<table>
<thead>
<tr>
<th>Agent Who Wants to Beat Benchmark log( r = 1.0% ) Per Bet</th>
<th>Value of ( E )</th>
<th>( I_p(1%) ) %</th>
<th>Risk Aversion ( 1 + \theta(p) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.0</td>
<td>( 0.9994 )</td>
<td>0.06</td>
<td>3.03</td>
</tr>
<tr>
<td>8.0</td>
<td>-0.9982</td>
<td>0.18</td>
<td>1.43</td>
</tr>
<tr>
<td>14.1</td>
<td>-0.9987</td>
<td>0.13</td>
<td>1.25</td>
</tr>
<tr>
<td>20.0</td>
<td>( 0.9987 )</td>
<td>0.13</td>
<td>1.008</td>
</tr>
<tr>
<td>33.0</td>
<td>( 0.9954 )</td>
<td>0.04</td>
<td>1.09</td>
</tr>
</tbody>
</table>

A less risk tolerant agent would only try to beat the easier benchmark log\( r = 0.1\% \):

<table>
<thead>
<tr>
<th>Agent Who Wants to Beat Benchmark log( r = 0.1% ) Per Bet</th>
<th>Value of ( E )</th>
<th>( I_p(1%) ) %</th>
<th>Risk Aversion ( 1 + \theta(p) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>-0.9954</td>
<td>0.46</td>
<td>10.73</td>
</tr>
<tr>
<td>4.5</td>
<td>-0.9877</td>
<td>1.23</td>
<td>4.53</td>
</tr>
<tr>
<td>14.1</td>
<td>-0.9924</td>
<td>0.77</td>
<td>1.88</td>
</tr>
<tr>
<td>20.0</td>
<td>-0.9954</td>
<td>0.46</td>
<td>1.48</td>
</tr>
<tr>
<td>33.0</td>
<td>-0.9996</td>
<td>0.04</td>
<td>1.09</td>
</tr>
</tbody>
</table>

Indeed, the correct conclusion for economists to draw, both from thought experiments and from actual data, is that people do not display a consistent coefficient of risk aversion, so it is waste of time to try to measure it." (Rabin and Thaler (2001), p.225)
## Summary of Behavioral Implications

<table>
<thead>
<tr>
<th>( \log r ) %</th>
<th>( p_{opt} ) %</th>
<th>( I_{p_{opt}}(\log r) ) %</th>
<th>Risk Aversion ( 1 + \theta(p_{opt}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.01</td>
<td>20.0</td>
<td>0.0</td>
<td>1</td>
</tr>
<tr>
<td>1.0</td>
<td>14.1</td>
<td>0.18</td>
<td>1.43</td>
</tr>
<tr>
<td>0.5</td>
<td>10.0</td>
<td>0.51</td>
<td>2.02</td>
</tr>
<tr>
<td>0.1</td>
<td>4.5</td>
<td>1.23</td>
<td>4.53</td>
</tr>
</tbody>
</table>
Samuelson's Fears Aren't Realized

- Intransitivities Can't Happen
  - Samuelson Didn't Envision a Benchmark

- Size of Potential Loss Isn't Ignored
  - A Benchmark Helps Fix That, Too.

- Why Not Give Larger Losses Higher Marginal Impact, And Then Weight Them By Their Respective Probabilities?
  - That is Expected Concave Utility! Absorb This Talk!

- Expected Concave Utility is Tractable
  - So is Benchmark Outperformance Prob.
Other Alternatives

- External Habit Formation
  \[ U = -(R_p/X)^{-\theta} \]

- Prospect“ Loss Aversion
  \[ U = \begin{cases} 
  -R^{-\theta_1} & \text{if } R \geq 1 \\
  -\lambda R^{-\theta_2} & \text{if } R < 1 
  \end{cases} \]

- Epstein-Zin Recursive
  \[ U_t = \left\{ (1 - \delta)C_t^{1-1/\psi} + \delta (E_t[U_{t+1}^{1-\theta}])\left(\frac{1}{\theta-\theta/1/\psi}\right) \right\}^{1-1/\psi} \]

{ These Alternatives Have More Adjustable Parameters
Ockham's Razor Favors Mine

Bayesian analysis also shows that a hypothesis with fewer adjustable parameters automatically has an enhanced posterior probability, because the predictions it makes are sharp. (Berger and Je®erys, 1992)

\[
\frac{\text{PostProb}[H_0]}{\text{PostProb}[H_A]} = \frac{L(p_{\text{obs}}|H_0) \mu(H_0)}{L(p_{\text{obs}}|H_A) \mu(H_A)} \overset{\text{Def.}}{=} B \cdot \text{PriorOdds}
\]

Candidates for \( H_A \):

- Habit-Formation CRRA: 2 Adjustable Parameters.
- Benartzi-Thaler Loss Aversion: 3 Parameters.
- Epstein-Zin: 3 Parameters

\{ Likelihood functions \( L(p \mid H) \) Must Be Computed Before \( p_{\text{obs}} \) is Observed ! \}

\{ Extra Parameters Spread Out Likelihood Unless Directly Measured With Low Error. \}

\{ When the Benchmark is Observable, \( H_0 \) has 0 Adjustable Parameters ! \}