Non-parametric calibration of jump-diffusion models

Peter Tankov

(joint work with Rama Cont)

Centre de Mathématiques Appliquées – Ecole Polytechnique

Peter.Tankov@polytechnique.org

http://www.cmap.polytechnique.fr/~tankov/

Exponential Lévy models

$$S_t = S_0 e^{rt+X_t},$$

where (X_t) is a Lévy process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$, $T < \infty$.

$$E[e^{iuX_t}] = e^{t\psi(u)}$$

$$\psi(u) = iu\gamma - \frac{\sigma^2 u^2}{2} + \int \nu(dx)(e^{iux} - 1 - iux1_{|x|\leq 1})$$

Jump-diffusion case: $\sigma > 0$, $\nu(\mathbb{R}) < \infty$.

No arbitrage opportunity $\Leftrightarrow \exists\, Q \sim P : e^{X_t}$ is a Q-local martingale.

$$\frac{\sigma^2}{2} + \gamma + \int (e^x - 1 - x1_{|x|\leq 1})\nu(dx) = 0.$$

Option prices:

$$C^Q(t, K) = e^{-rt} E^Q[(S_0 e^{rt+X_t} - K)^+]$$

Examples of parametric exp-Lévy models

- Merton's model (Merton (1976)):

$$\sigma > 0, \quad \nu(x) = \frac{\lambda}{\delta\sqrt{2\pi}} e^{\frac{(x-\theta)^2}{2\delta^2}}.$$

- Variance gamma (Madan, Milne (1991)):

$$\sigma = 0, \quad \nu(x) = \frac{Ae^{-\eta_+ x}}{x} 1_{x>0} + \frac{Ae^{-\eta_- |x|}}{|x|} 1_{x<0}.$$

- Tempered stable (Carr, Geman, Madan, Yor (2002)):

$$\sigma = 0, \quad \nu(x) = \frac{A_+ e^{-\eta_+ x}}{x^{1+\alpha}} 1_{x>0} + \frac{A_- e^{-\eta_- |x|}}{|x|^{1+\alpha}} 1_{x<0}.$$

The calibration problem

- Retrieve an exponential Lévy model from prices of traded options.

$\mathcal{M}_L = \{\text{risk-neutral probabilities } Q : (X, Q) \text{ is a Lévy process}\}$

- Calibration problem for exact data, consistent with the model: given market prices $C_M(T_i, K_i)$, $i \in I$, find $\mathcal{Q}^E \subseteq \mathcal{M}_L$:

$$\forall Q \in \mathcal{Q}^E,\ C_M(T_i, K_i) = C^Q(T_i, K_i) \quad \text{for all } i \in I. \qquad (E)$$

- If the market data are known exactly, but are not necessarily consistent, we want our model to reproduce them in the best possible way: find $\mathcal{Q}^{LS} \subseteq \mathcal{M}_L$:

$$\forall Q^* \in \mathcal{Q}^{LS},\ \|C_M - C^{Q^*}\|_w^2 = \inf_{Q \in \mathcal{M}_L} \|C_M - C^Q\|_w^2, \qquad (LS)$$

where

$$\|C_M - C^Q\|_w^2 := \sum_{i \in I} w_i (C_M(T_i, K_i) - C^Q(T_i, K_i))^2.$$

Ill-posedness of least squares calibration

- Lack of identification
- The problem (LS) need not have a solution
- Even if a solution does exist, it may not be continuous with respect to market data
- The solution is hard to find numerically due to non-convexity of the pricing error

Lack of identification in least squares calibration

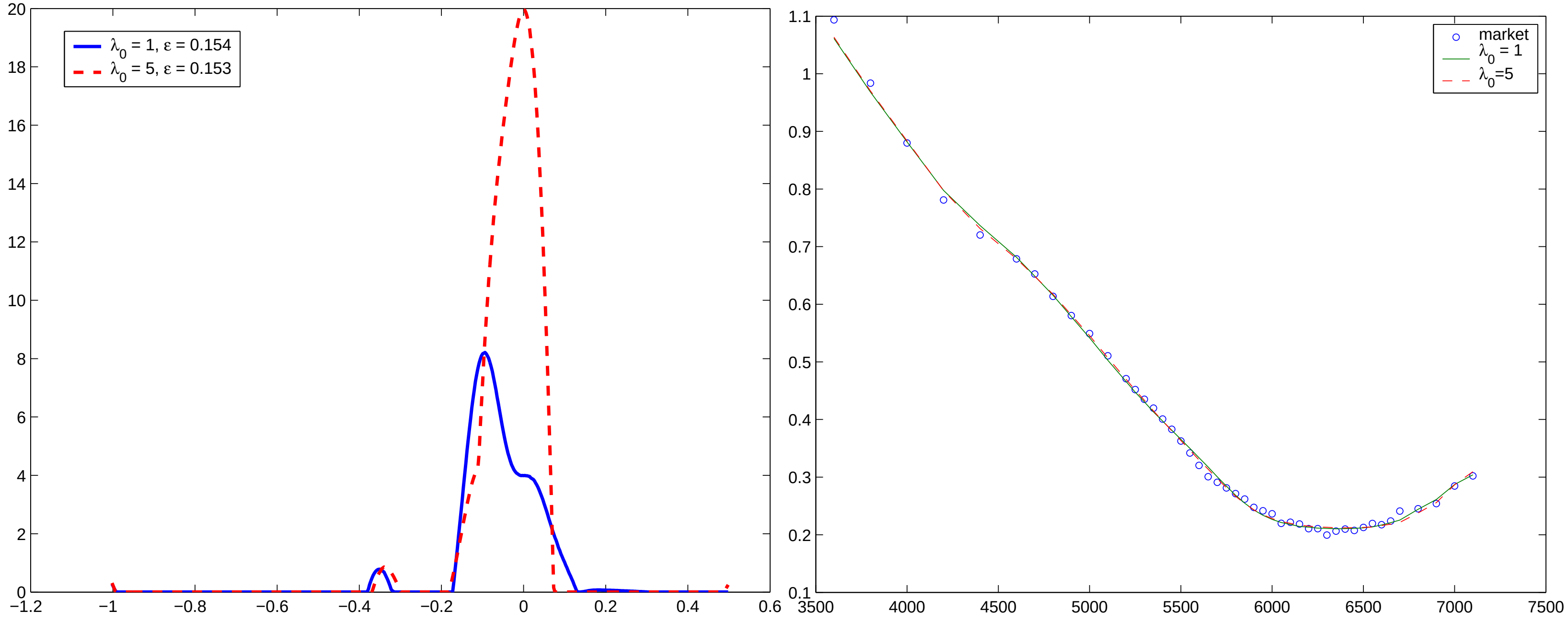


Figure 1: Left: Lévy measures calibrated by least squares, using Merton models with different intensities as initializers. Right: implied volatility smiles corresponding to these measures.

Lack of identification: parametric case

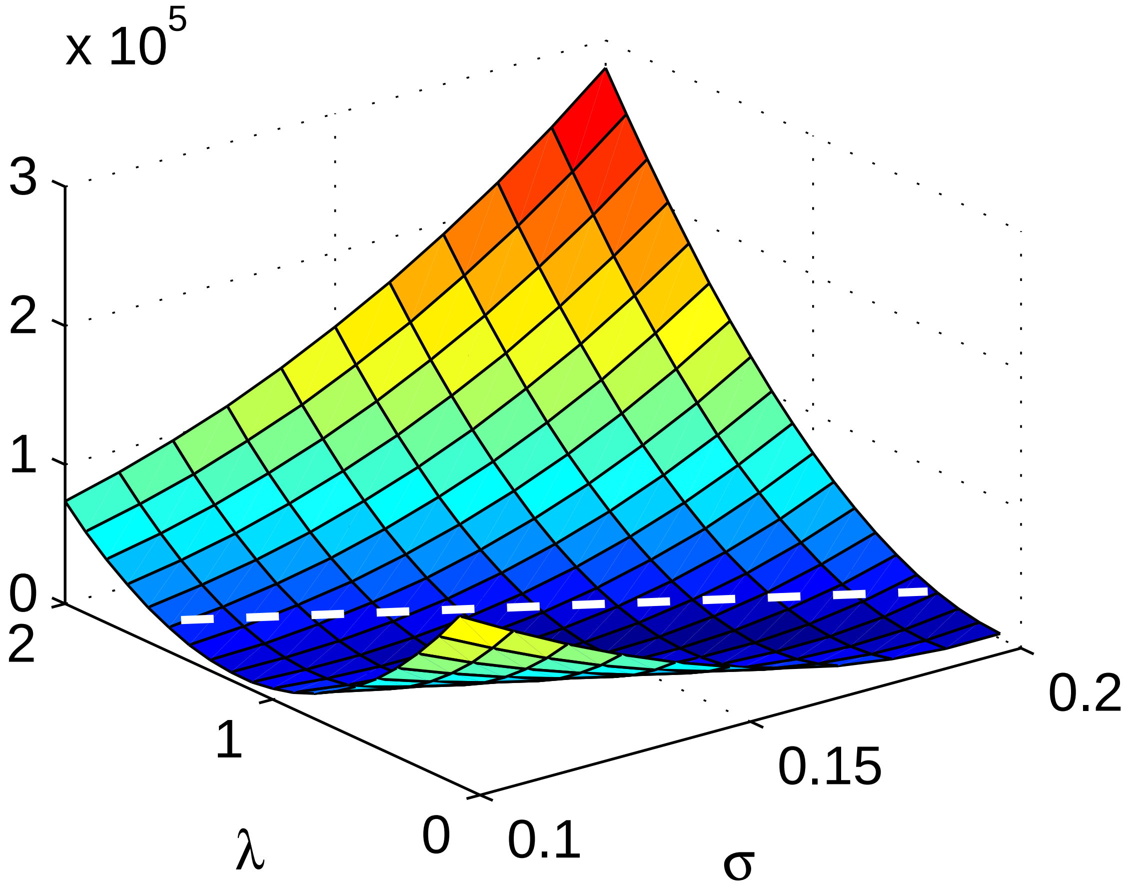
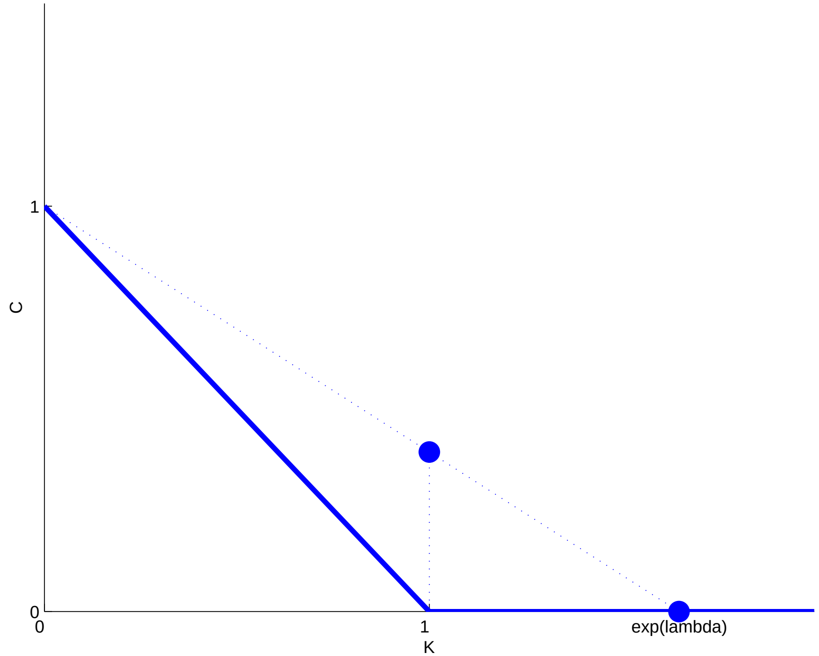


Figure 2: Error surface for Merton model: ill-posedness of the calibration problem may lead to a flat valley in error landscape

Example of data set for which the problem (LS) has no solution

$$\begin{cases} C_M(T=1, K=1) = 1 - e^{-\lambda} \\ C_M(T=1, K=e^{\lambda}) = 0. \end{cases}$$



From absence of arbitrage:

$$C_M(K) = (1 - Ke^{-\lambda})^+.$$

$$\Rightarrow \quad P[S_1 \leq K] = e^{-\lambda} 1_{K \leq e^{\lambda}}.$$

$\Rightarrow$ No Lévy process is compatible with these prices

However, for the process $X_t^n := -nN_t + \lambda t(1 - e^{-n})$,

$$\lim_{n \to \infty} E[(e^{X_t^n} - K)^+] = (1 - Ke^{-\lambda t})^+.$$

Existence of solution of least squares calibration

Theorem 1. *Let $\mathcal{M}_B$ denote the set of all risk-neutral probabilities Q such that (X, Q) is a Lévy process satisfying $|\Delta X_t| \leq B$ for all t. If there exists $Q_0 \in \mathcal{M}_B$ with*

$$\|C^{Q_0} - C_M\|_w < \sqrt{w_0}(S_0 - C_M(T_0, K_0)),$$

then the problem (LS) has a solution in $\mathcal{M}_B$.

Proof:

- The pricing error functional $\|C_M - C^Q\|_w^2$, is weakly continuous with respect to $Q \in \mathcal{M}_B$.
- The set of risk-neutral Lévy processes $Q \in \mathcal{M}_B$ satisfying $C^Q(T, K) \leq C < S_0$ is relatively weakly compact in $\mathcal{M}_B$.

Continuity of solutions with respect to the data

Definition 1. *The solutions of a calibration problem are said to depend continuously on input data at the point C_M if for every sequence of data sheets $\{C_M^n\}_{n\geq 0}$ such that $\|C_M^n - C_M\|_w \xrightarrow[n\to\infty]{} 0$, if Q_n is a solution of the calibration problem with data C_M^n then*

1. *$\{Q_n\}_{n\geq 1}$ has a weakly convergent subsequence $\{Q_m\}_{m\geq 1}$.*
2. *The limit Q of every convergent subsequence of $\{Q_n\}_{n\geq 1}$ is a solution of the calibration problem with data C_M.*

Non-convexity of the pricing error functional

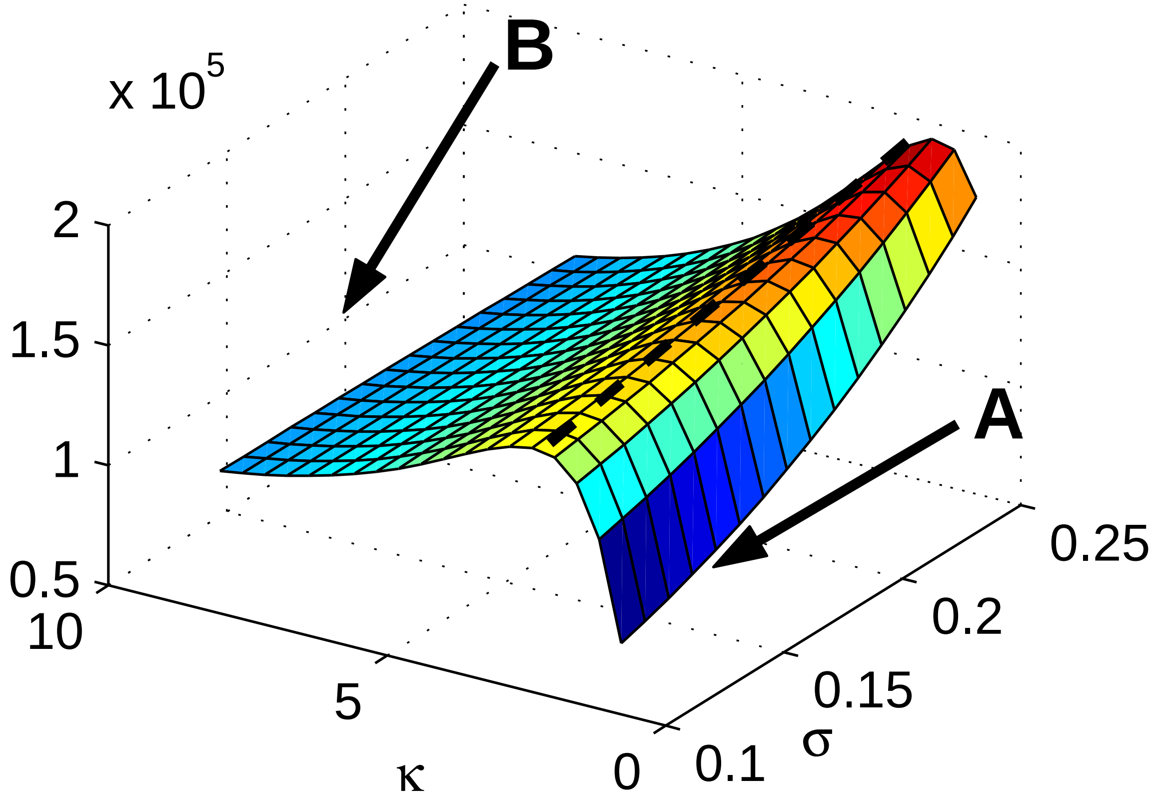


Figure 3: Error surface for variance gamma model: the error function may have distinct local minima

Selection using relative entropy

$\mathcal{Q}^{LS}$ will typically contain many elements $\Rightarrow$ choose the ones with smallest relative entropy with respect to a prior $P \in \mathcal{M}_L$:

$$\mathcal{Q}^{MELS} = \{Q^* \in \mathcal{Q}^{LS} : I(Q^*|P) = \inf_{Q \in \mathcal{Q}^{LS}} I(Q|P)\}, \qquad (MELS)$$

where

$$I(Q|P) := \begin{cases} \int \frac{dQ}{dP} \log \frac{dQ}{dP} dP, & Q \ll P, \\ +\infty, & \text{otherwise.} \end{cases}$$

- Convex non-negative functional of Q for fixed P, equal to zero iff $\frac{dQ}{dP} = 1$ a.s.
- Easy to express in terms of Lévy measures and to compute
- Well-known in the literature, many results are available (large deviations, information theory, finance, convex duality results)

The logic of regularization: case of unique minimum entropy solution

- Suppose that for some (hypothetical) exact data C_M, there exists a unique solution to problem $(MELS)$, denoted by $C^+(C_M)$.
- Only noisy option prices C_M^δ, with $\|C_M^\delta - C_M\| < \delta$, are known
- $C^+(C_M^\delta)$ is not a good approximation of $C^+(C_M)$ because C^+ is not continuous and $C^+(C_M^\delta)$ may not even exist

The solution: *regularize* C^+, construct a family of continuous mappings R_α: for every $\{\delta_n\}_{n\geq 1}$ with $\delta_n \to 0$,

$$R_{\alpha_n}(C_M^{\delta_n}) \to C^+(C_M),$$

where α_n is chosen appropriately depending on δ_n and $C_M^{\delta_n}$.

The logic of regularization: general case

- Suppose that for the exact data C_M, $\mathcal{Q}^{MELS}(C_M) \neq \emptyset$.
- Only noisy option prices C_M^δ, with $\|C_M^\delta - C_M\| < \delta$, are known
- $\mathcal{Q}^{MELS}(C_M^\delta)$ may be very different from $\mathcal{Q}^{MELS}(C_M)$ and may even be empty

The solution: construct a family of calibration problems $(R_\alpha)_{\alpha>0}$ such that for every $\{\delta_n\}_{n\geq 1}$ with $\delta_n \to 0$, if for every n, Q_n is a solution of (R_{α_n}) with data $C_M^{\delta_n}$, then

- $\{Q_n\}_{n\geq 1}$ has a convergent subsequence and
- the limit of every convergent subsequence is a minimum entropy least squares solution with data C_M.

The regularized calibration problem

To regularize problem $(MELS)$, we replace it with:

$$\mathcal{Q}^R = \arg\inf_{Q \in \mathcal{M}_L} \{\|C_M - C^Q\|_w^2 + \alpha I(Q|P)\} \qquad (R_\alpha)$$

$\Rightarrow$ the entropy is used for *penalization* rather than selection.

- The problem (R_α) has at least one solution.
- The solutions of (R_α) are continuous with respect to C_M.
- The solutions of (R_α) converge to solutions of $(MELS)$ as $\delta \to 0$, if α is chosen appropriately.
- The problem (R_α) can be discretized by taking a discretized prior, and the solutions of discretized problem converge to those of the continuous one.

Assumptions

We suppose that the prior Lévy process, $P = P(\sigma^P, \nu^P, \gamma^P)$ satisfies the following assumptions:

1. $\sigma^P > 0$ and $\nu^P(\mathbb{R}) < \infty$ (jump-diffusion).
2. $E^P[e^{X_t}] = 1$ (the probability P is risk-neutral).
3. The support of ν^P is bounded from above:

$$\nu^P([x_0, \infty)) = 0 \text{ for some } x_0 > 0.$$

Conditions of absolute continuity

Proposition 1 (Jacod & Shiryaev (2003), Theorem IV.4.39). *Let* $(X_t)_{t\geq 0}$ *be a real-valued Lévy process on* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, Q)$ *and on* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ *with characteristic triplets* $(\sigma^Q, \nu^Q, \gamma^Q)$ *and* $(\sigma^P, \nu^P, \gamma^P)$*. Then* $Q|_{\mathcal{F}_t} \ll P|_{\mathcal{F}_t}$ *for all* t *if and only if*

1. $\sigma^Q = \sigma^P := \sigma$.
2. $\nu^Q \ll \nu^P$ *with* $\phi := \frac{d\nu^Q}{d\nu^P}$.
3. $\int_{-\infty}^{\infty} (\sqrt{\phi(x)} - 1)^2 \nu^P(dx) < \infty$.
4. *If* $\sigma = 0$ *then* $\gamma^Q - \gamma^P = \int_{|x|\leq 1} x(\phi(x) - 1)\nu^P(dx)$.

Under the above assumptions, $Q|_{\mathcal{F}_t} \ll P|_{\mathcal{F}_t}$ for all t iff $\sigma^Q = \sigma^P$, $\nu^Q(\mathbb{R}) < \infty$, and $\nu^Q \ll \nu^P$.

Computing relative entropy of Lévy processes

Theorem 2. *Suppose that the conditions 1–4 of Proposition 1 are satisfied. Then*

$$I(Q|P) = I(Q|_{\mathcal{F}_T}|P|_{\mathcal{F}_T}) = T\int_{\mathbb{R}}\left(\frac{d\nu^Q}{d\nu^P}\log\frac{d\nu^Q}{d\nu^P} + 1 - \frac{d\nu^Q}{d\nu^P}\right)\nu^P(dx)$$
$$+\frac{T}{2\sigma^2}\left\{\gamma^Q - \gamma^P - \int_{-1}^{1} x(\nu^Q-\nu^P)(dx)\right\}^2 1_{\sigma\neq 0}$$

Remark. If Q and P are risk-neutral measures, the second term becomes

$$\frac{T}{2\sigma^2}\left\{\int_{-\infty}^{\infty}(e^x-1)(\nu^Q-\nu^P)(dx)\right\}^2$$

$I(Q,P)$ becomes a convex non-negative functional of ν^Q for fixed ν^P, equal to zero iff $\frac{d\nu^Q}{d\nu^P}=1$ ν^P-a.e.

Regularized problem in terms of density of Lévy measures

$$J(Q) = \|C_M - C^Q\|_w^2 + \alpha I(Q|P)$$

Since $J(P) < \infty$, we can suppose w.l.o.g. that $Q \ll P$.
$\Rightarrow Q$ can be parametrized by $\phi := \frac{d\nu^Q}{d\nu^P}$.

$$\text{Find } \phi^* \in L^1(\nu^P),\ \phi^* \geq 0 \quad : \quad J(\phi^*) = \inf_{\phi \in L^1(\nu^P),\ \phi \geq 0} J(\phi),$$

where

$$J(\phi) = \|C_M - C^{Q(\phi)}\|_w^2 + \alpha(H(\phi) + I(\phi))$$

and

$$H(\phi) = \frac{1}{2\sigma^2}\left(\int (e^x - 1)(\phi(x) - 1)\nu^P(dx)\right)^2,$$

$$I(\phi) = \int (\phi(x)\log\phi(x) + 1 - \phi(x))\nu^P(dx).$$

Regularized problem: existence of solution

Proposition 2. *For all $\alpha > 0$, the regularized calibration problem (R_α) has at least one solution $Q_\alpha \in \mathcal{M}_L$.*

Proof.

- $H(\phi)$ and $\|C_M - C^{Q(\phi)}\|_w^2$ are weakly continuous in ϕ.

$$\psi^Q(u) = -\frac{\sigma^2}{2}(u^2 + iu) + \int_{\mathbb{R}} (e^{iux} - 1 - iu - iue^x)\phi(x)\nu^P(dx)$$

- From Teboule and Vajda (1993), $I(\phi)$ is weakly lower semicontinuous and for every $r > 0$,

$$B_r := \{\phi \in L^1(\nu^P) : \phi \geq 0 \text{ and } I(\phi) \leq r\}$$

is weakly compact.

- $J(1) = \|C_M - C^P\|_w^2 \leq 4S_0^2 < \infty \Rightarrow \phi^* \in B_{J(1)}$. $\square$

Regularized problem: continuity of solution with respect to data

Proposition 3. *Let* $\{C_M^n\}_{n\geq 0}$ *be such that* $\|C_M^n - C_M\|_w \xrightarrow[n\to\infty]{} 0$ *and let* ϕ_n *be a solution of the regularized problem* (R_α) *for some* $\alpha > 0$ *with data* C_M^n. *Then*

1. $\{\phi_n\}$ *has a weakly convergent subsequence*

2. *The limit of every convergent subsequence of* $\{\phi_n\}$ *is a solution of the regularized problem with data* C_M.

Regularized problem: convergence

Theorem 3. *Suppose that there exist a solution Q of problem (LS) with data C_M such that $I(Q|P) < \infty$.*

If $\|C_M - C^Q\|_w = 0$, suppose that $\alpha(\delta) \to 0$ and $\frac{\delta^2}{\alpha(\delta)} \to 0$ as $\delta \to 0$. Otherwise, suppose that $\alpha(\delta) \to 0$ and $\frac{\delta}{\alpha(\delta)} \to 0$ as $\delta \to 0$.

Let $\{\delta_k\}_{k\geq 0}$ be such that $\delta_k \to 0$, $\{C_M^{\delta_k}\}_{k\geq 1}$ be a sequence of data sets such that $\|C_M^{\delta_k} - C_M\|_w \leq \delta_k$ and let ϕ^{δ_k} be a solution of problem $(R_{\alpha(\delta_k)})$ with data $C_M^{\delta_k}$. Then

1. *There exists a solution of $(MELS)$ with data C_M.*
2. *$\{\phi^{\delta_k}\}_{k\geq 0}$ has a weakly convergent subsequence.*
3. *The limit of every convergent subsequence of $\{\phi^{\delta_k}\}$ is a solution of $(MELS)$ with data C_M.*
4. *If such a solution ϕ^+ is unique then $\lim_{\delta\to 0} \phi^\delta = \phi^+$.*

Numerical implementation: computing prices of call options

• European call/put options can be priced using the fast Fourier transform (Carr & Madan)

$$C^Q(T,K) = e^{-rT}E^Q[(S_T - K)^+] = e^{-rT}\int_{-\infty}^{\infty}(e^{rT+x} - e^k)^+ q_T(x)dx$$

• Call price is not integrable $\Rightarrow$ one cannot compute its Fourier transform directly.

• Possible solution: construct an integrable function by subtracting the intrinsic value:

$$z_T(k) = e^{-rT}E[(e^{rT+X_T} - e^k)^+] - (1 - e^{k-rT})^+$$

$$\zeta_T(v) = \int_{-\infty}^{+\infty} e^{ivk} z_T(k)dk = e^{ivrT}\frac{\Phi_T(v-i) - 1}{iv(1+iv)}$$

Numerical implementation: discretizing the Lévy measure

- The calibration problem can be discretized by taking a discrete prior:

$$\nu^P(dx) = \sum_{k=1}^{N} a_k \delta_{\{x_0 + k\Delta\}}(dx),$$

In this case, the calibrated measure is also discrete:

$$\nu^Q(dx) = \sum_{k=1}^{N} b_k \delta_{\{x_0 + k\Delta\}}(dx),$$

- The grid must be uniform in order to use the FFT to compute $\Phi_T(u)$, the characteristic function of X_T.
- An explicit representation of the gradient of $J(Q)$ (minimization functional) allows to use a gradient based optimization method to solve the minimization problem.

Calibration with entropic regularization

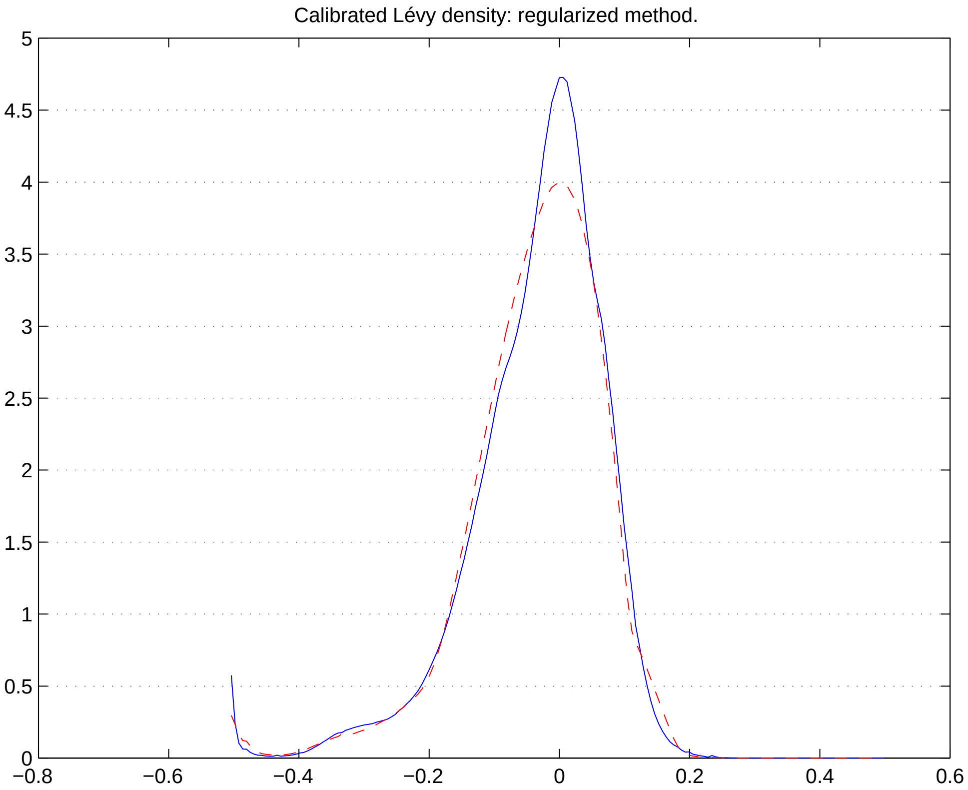


Figure 4: Lévy measure calibrated by entropic regularization. DAX options, 10 May 2001. Maturity 1 month. Again, Merton models with different intensities $\lambda_1 = 1$ and $\lambda_2 = 5$ are used as initializers

Stability of solutions with respect to small changes of the prior measure

Theorem 4. *Let* ν^P, $\{\nu_n^P\}_{n\geq 1}$ *be positive finite measures on* $(-\infty, x_0]$ *such that* $\nu_n^P \Rightarrow \nu^P$, *and for each* n *let* ν_n^Q *be a solution of problem* (R_α) *with prior* $P(\sigma, \nu_n^P)$ *and some* $\alpha > 0$. *Then*

1. *There exists a weakly convergent subsequence* $\{\nu_{n_k}^Q\} \subseteq \{\nu_n^Q\}$.
2. *The limit* ν^Q *of every convergent subsequence of* $\{\nu_n^Q\}_{n\geq 1}$ *is a solution of* (R_α) *with prior* $P(\sigma, \nu^P)$.

Numerical implementation: choice of the prior measure

- Based on historical estimation.
- Based on the calibrated measure of the day before.
- Based on the user's views, taking a parametric model with reasonable values of parameters.
- From the same dataset, using precalibration: first calibrate a simple parametric model (i.e. Merton's model) using least squares and then use it as prior.

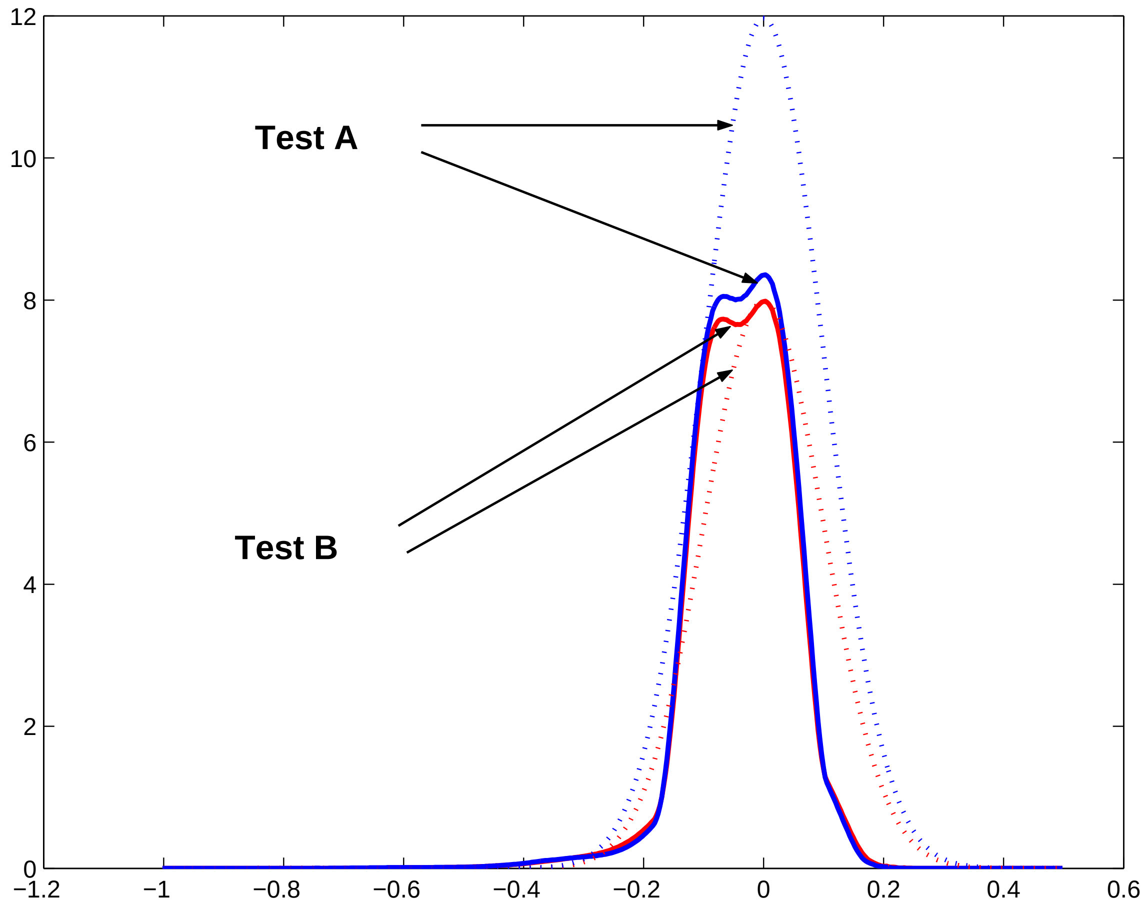


Figure 5: Sensitivity of calibrated measures to minor variations in the parameters of the prior model

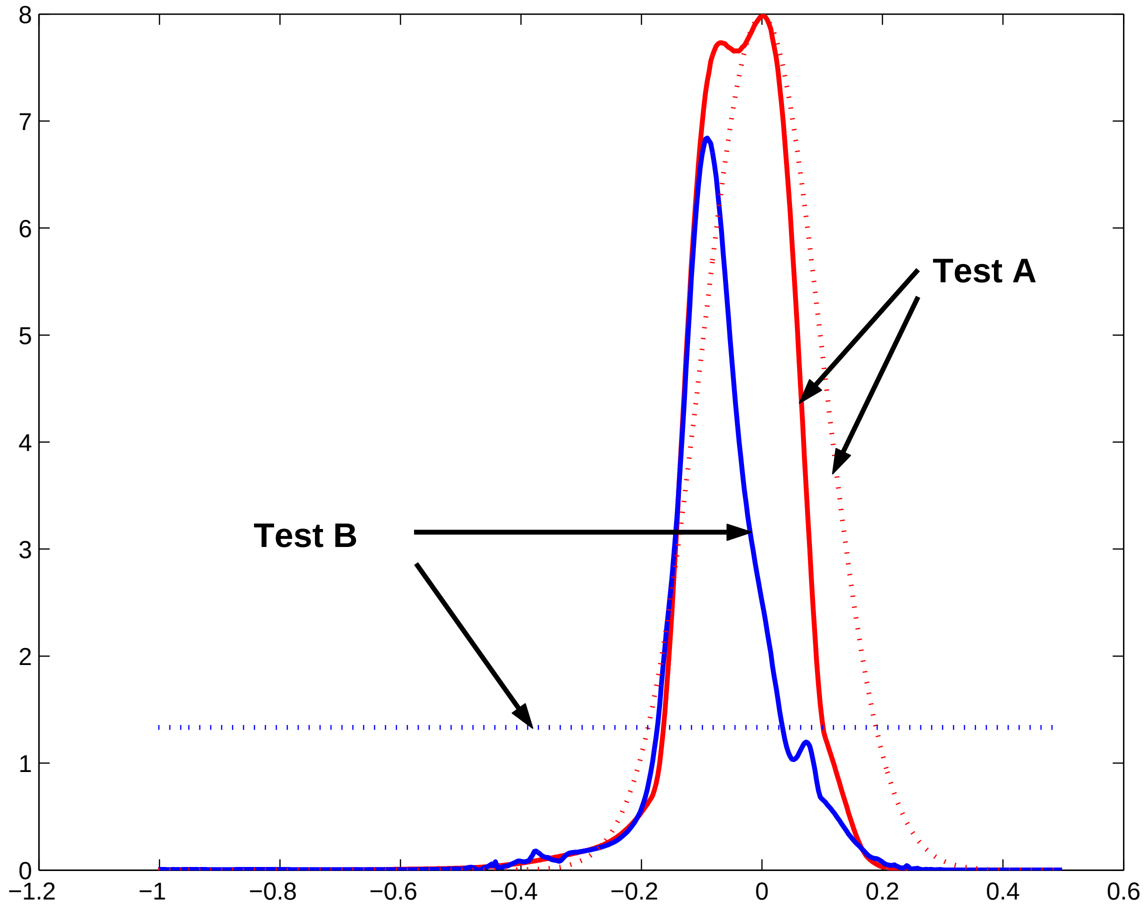


Figure 6: Sensitivity of calibrated measures to qualitative change of the prior model

Tests on simulated data

Model 1: Kou's model (compound Poisson)

$$\nu(x) = \lambda[p\alpha_1 e^{-\alpha_1 x} 1_{x>0} + (1-p)\alpha_2 e^{-\alpha_2 |x|} 1_{x<0}]$$

Option prices were computed for 21 equidistant strikes, ranging from 6 to 14 (the money being at 10).

The Merton's model (with symmetric Gaussian jumps) was used as prior.

Tests on simulated data

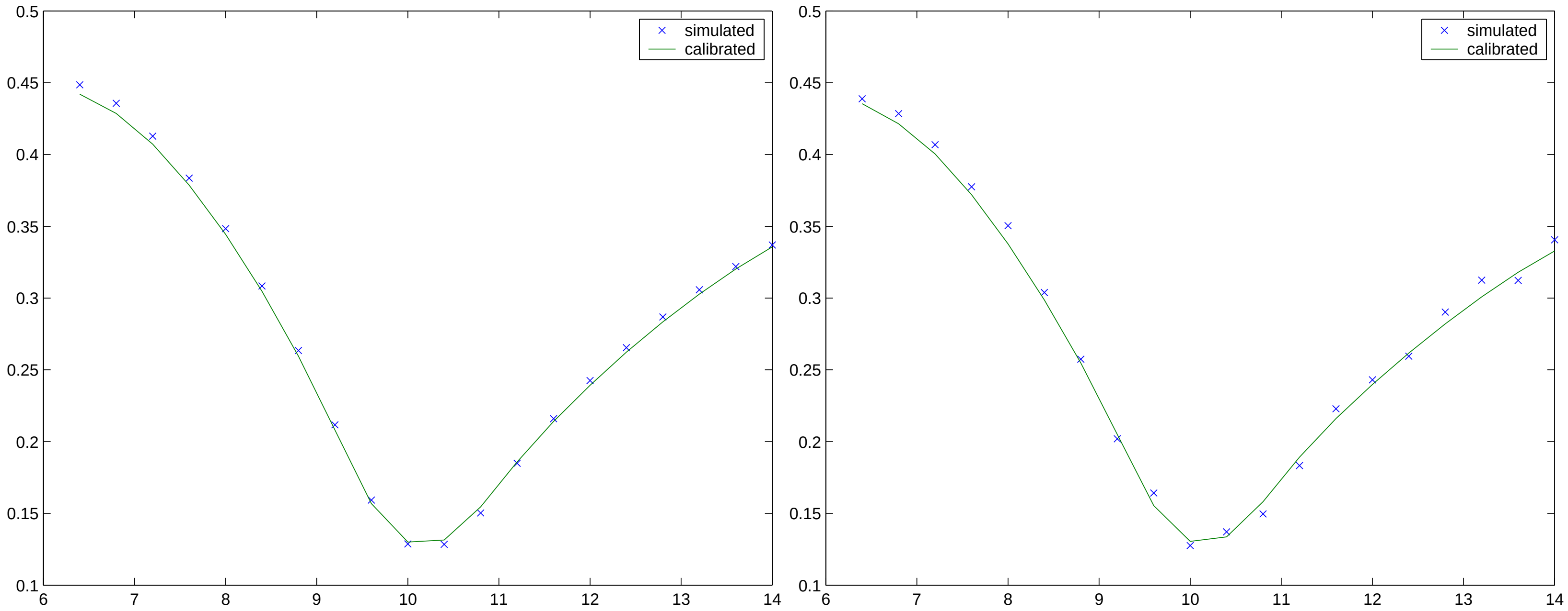


Figure 7: Calibration quality for Kou's jump diffusion model. Left: no perturbations. Right: 1% perturbations were added to the data

Tests on simulated data

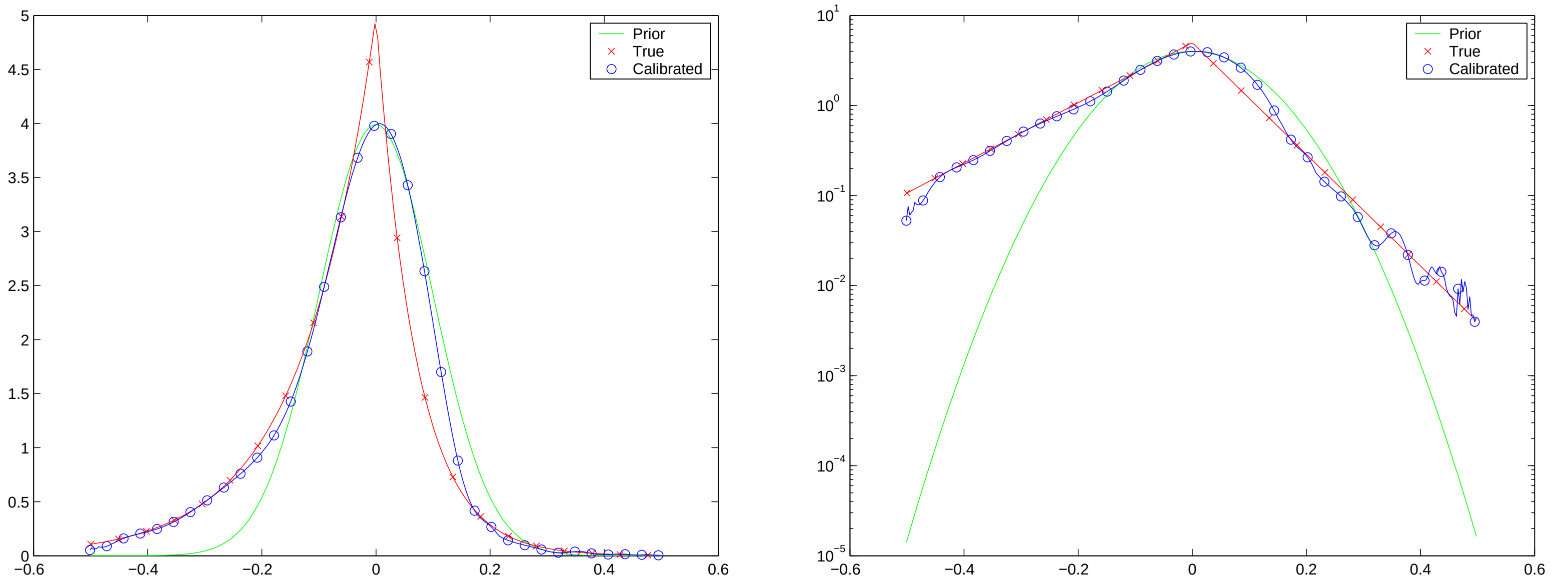


Figure 8: Lévy measure calibrated to option prices simulated from Kou's jump diffusion model with $\sigma_0 = 10\%$. No perturbations were added.

Tests on simulated data

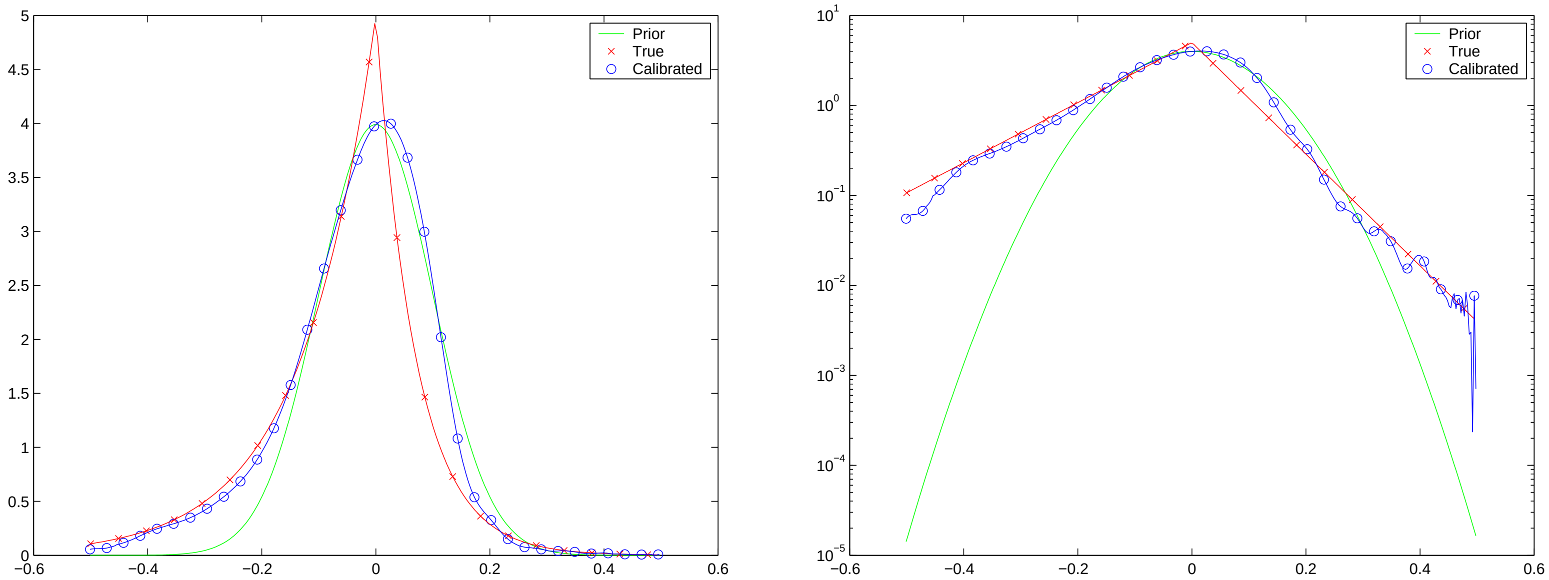


Figure 9: Lévy measure calibrated to option prices simulated from Kou's jump diffusion model with $\sigma_0 = 10\%$. 1% perturbations were added to the data.

Empirical results (one maturity)

- The calibrated Lévy measures are strongly asymmetric: the distribution of jump sizes is skewed towards negative values.
- Most of the calibrated measures are bimodal with one mode (at zero) corresponding to regular activity and the second one corresponding to a large negative jump
- A small intensity of jumps λ can be sufficient for explaining the shape of the implied volatility: empirically $\lambda \sim 1$

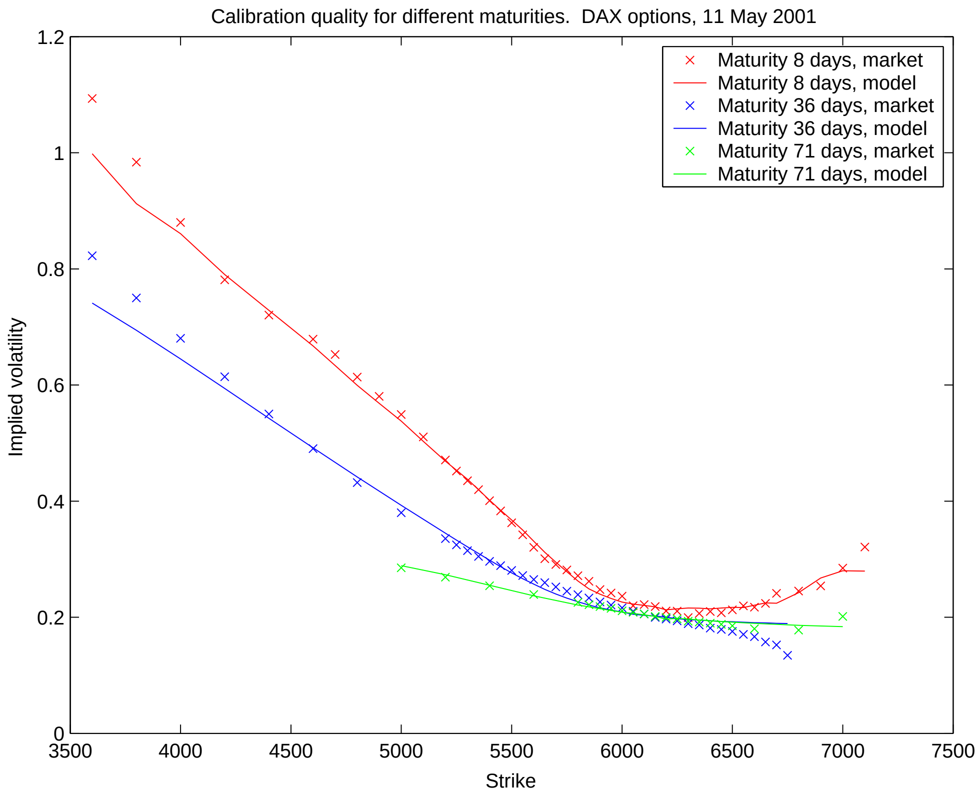


Figure 10: Calibration quality for different maturities: market implied volatilities for DAX options against model implied volatilities. Each maturity has been calibrated separately.

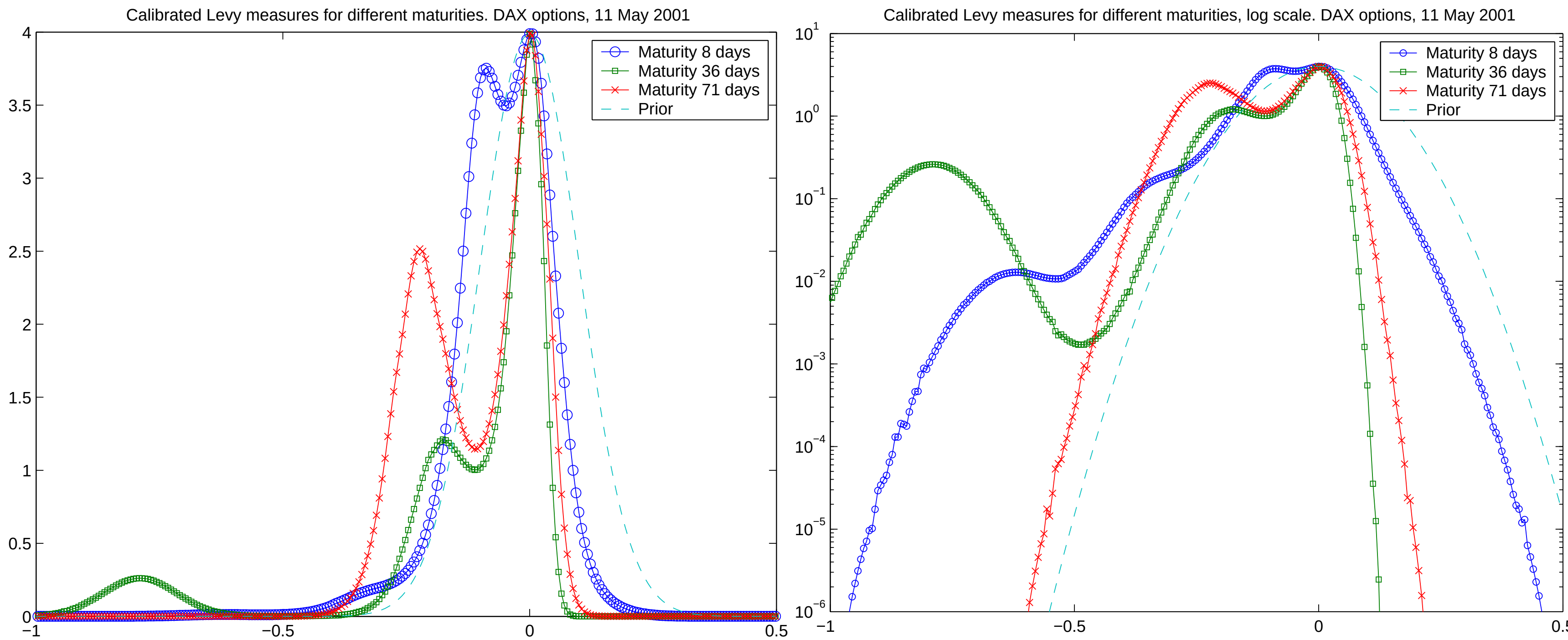


Figure 11: Lévy measures calibrated to DAX option prices for three different maturieis, linear and logarithmic scale.

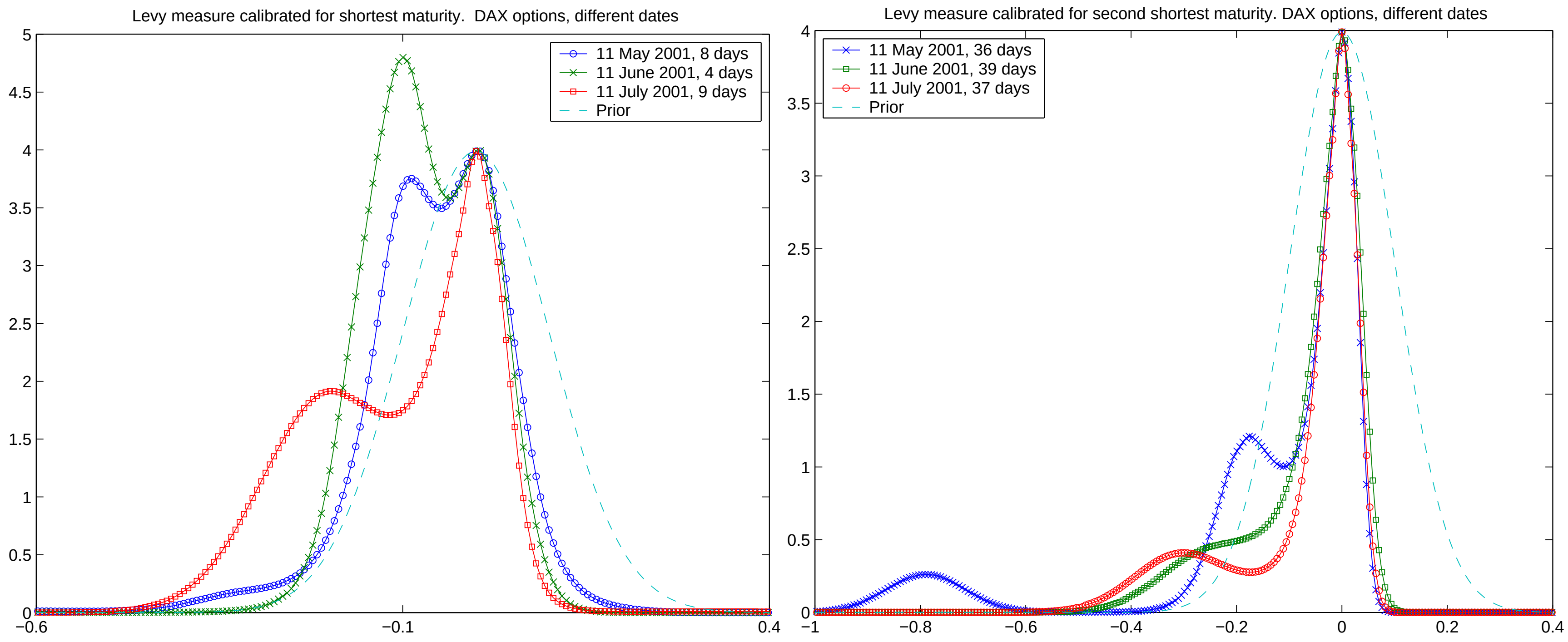


Figure 12: Stability of calibration over calendar dates. Lévy measures have been calibrated at different dates for shortest (left) and second shortest (right) maturity.

Empirical results (several maturities)

Lévy processes explain the smile for one maturity but perform poorly for several maturities at the same time

- The implied volatility smile flattens too fast
- In Lévy models at the money implied volatility increases with time while the market ATM implied volatility changes little

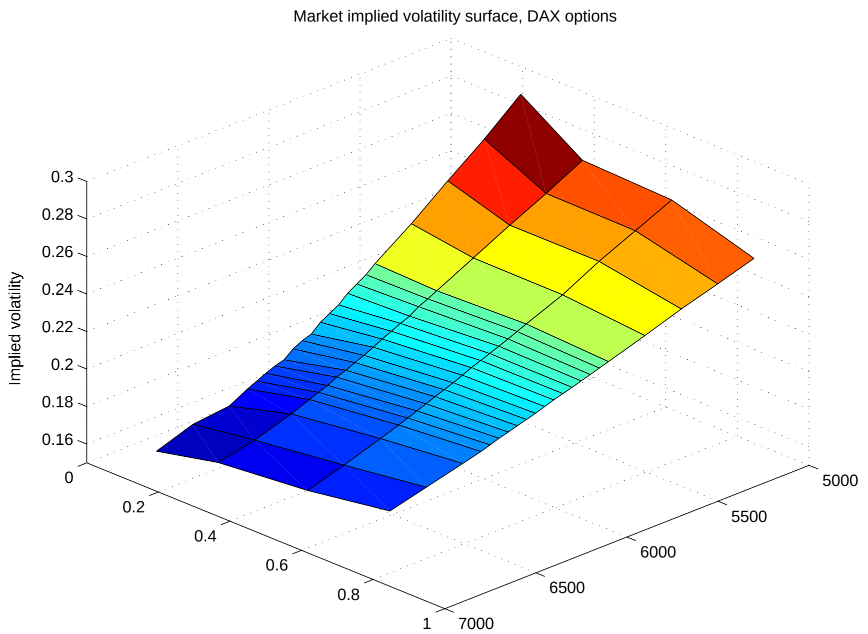


Figure 13: Market implied volatility surface

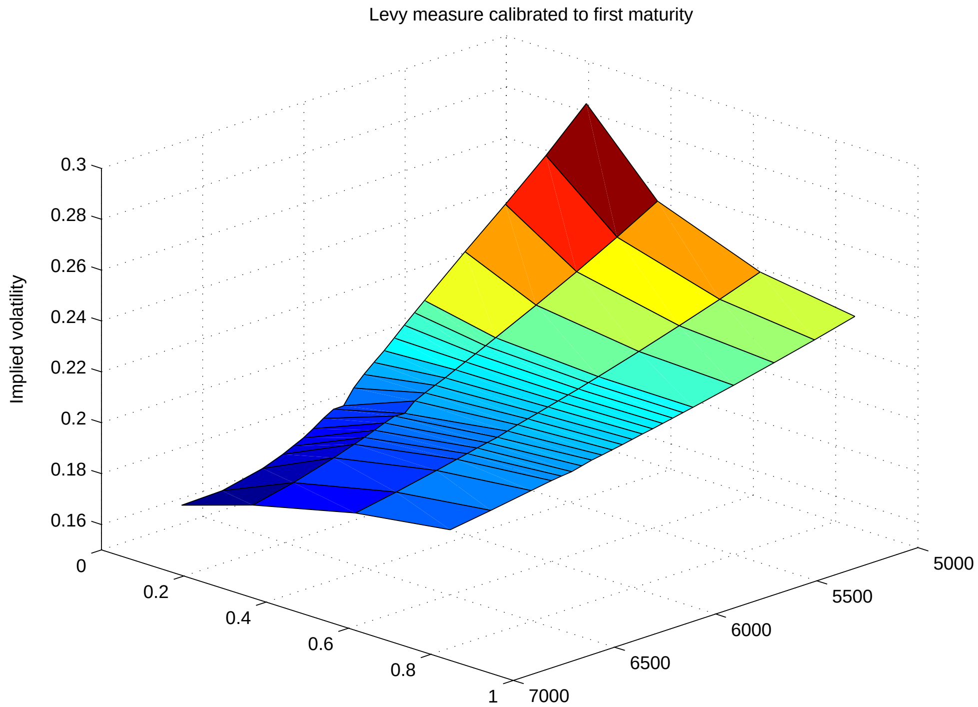


Figure 14: Implied volatilities for all maturities were computed, using the Lévy measure, calibrated to the first maturity

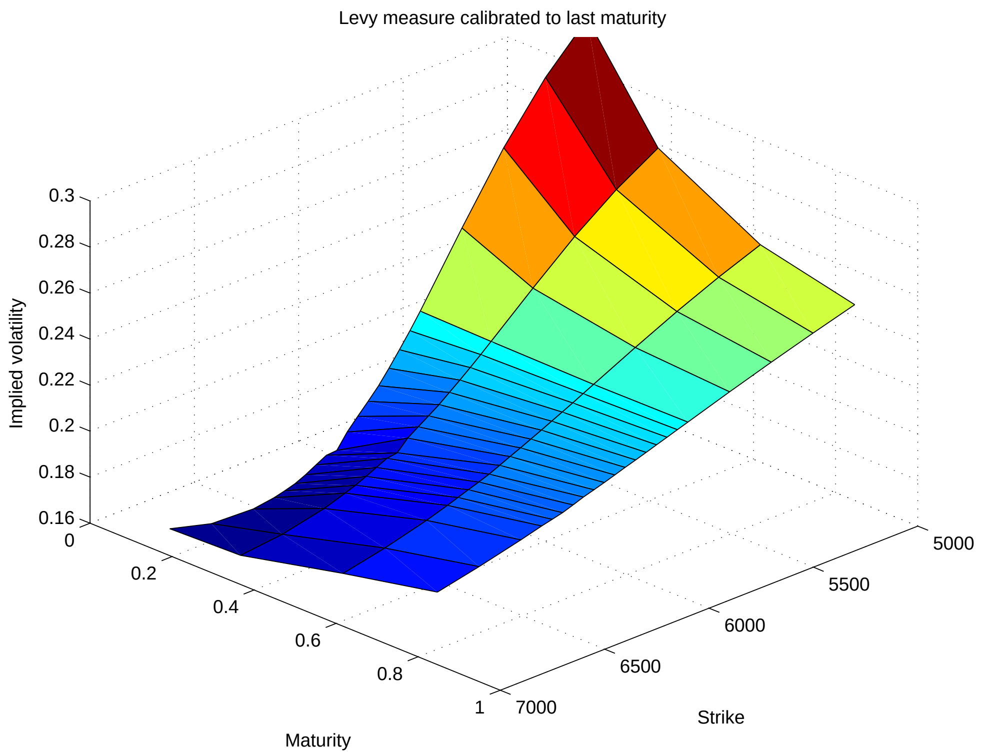


Figure 15: Implied volatilities for all maturities were computed, using the Lévy measure, calibrated to the last maturity