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Risk Management and Model Specifications Issues in Finance

A General Equilibrium Model of the Term Structure of Interest Rates under Regime-switching Risk

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Outline

Introduction

• The aggregate economy is characterized by periodic shifts between distinct regimes of the business cycle

Hamilton (1989), Filardo (1994) and Diebold and Rudebusch (1996) among others.

• Markov regime-switching models of the short-term interest rate

Hamilton (1988), Garcia and Perron (1996), Gary (1996), Ang and Bakeart (1998).

• Dynamic term structure models under regime shifts:

Naik and Lee (1997), and Bansal and Zhou (2002).

• Dai and Singleton (2003): **the risk of regime shifts is not priced.**

Objectives

• Develop a dynamic term structure model under the systematic risk of regime shifts in a general equilibrium setting similar to that in CIR.

• How do regime shifts affect bond returns? How important is the regime-switching risk premium?

Main results

• We show that the regime-switching risk can be priced in a similar way as in the case of jump risk (e.g. Ahn and Thompson, 1988).

• We show that regime switching introduces a new source of time-variation in bond risk premiums, which is associated with the systematic risk of periodic shifts in bond prices due to regime changes.

• Closed-form approximate solution for the entire yield curve is obtained for an affine model.

• Empirical evidence suggests that the market price of the regime-shift risk is not only statistically significant, but also economically important.

The Equilibrium Model

The Underlying Economy: Single good and a large number of infinitely lived and identical consumers similarly to that in CIR (1985a,b).

Modeling Regime Switching:
– *Hidden Markov Model:* Elliott & Mamon (2002), Elliott & Wilson (2002).
– *Conditional Markov Chain:* Bielecki & Rutkowski (2000), Yin & Zhang (1998)
– *Marked Point Process:* Landen (2000). Suppose N regimes.
Mark Space: $E = \{z = (i, j) : i \in \{1, ..., N\}, j \in \{1, 2, ..., N\}, i \neq j\}$.
Marked point process: for $A \in E$, $m(t, A)$ counts the cumulative number of regime shifts that belong to A during $(0, t]$ with *stochastic intensity kernel*:

$$\gamma_m(dt, dz) = h(z, x(t-))\mathbf{I}\{s(t-) = i\}\epsilon_z(dz)dt,$$

$h(z, x(t-))$: the regime-shift intensity at $z = (i, j)$,
$\epsilon_z(A)$: the Dirac measure at point z, ($\epsilon_z(A) = 1$ if $z \in A$ and 0, otherwise).
• $\gamma_m(dt, dz)$ is the conditional probability of a shift from regime i to regime j during $[t, t + dt)$ given $x(t-)$ and $s(t-) = i$.

Two State Variables: (s,x)

- The Regime: $s(t)$

$$ds(t) = \int_E \zeta(z) m(dt, dz)$$

where $\zeta(z) = \zeta((i,j)) = j - i$ with the compensator $\gamma_s(t)dt = \int_E \zeta(z)\gamma_m(dt, dz)$. For example, if there is a regime shift from i to j occurred at time t, the above equation implies $s_t = (j - i) + s_{t-} = j$.

- The other usual state variable: $(\mu_x$ and σ_x depend on $x(t), s(t))$

$$dx(t) = \mu_x dt + \sigma_x dB_t$$

Investment Opportunities

- Production:

$$dy = y\mu_y dt + y\sigma_y dB_t + \int_E y\delta_y(z) m(dt, dz)$$

where $\delta_y(z)$ is the percentage change in y due to a regime shift,

- Competitive market for a default-free pure discount bond with the price:

$$dF = F\mu_F dt + F\sigma_F dB_t + \int_E F\delta_F(z) m(dt, dz)$$

- Competitive market for (local) risk-free lending and borrowing at $r(t)$.

1.3 The Consumer's Objective Function

- Maximize consumer's lifetime expected utility

$$E_0\left[\int_0^\infty e^{-\rho t}U(c(t))dt\right]$$

where $c(t)$ is the flow of consumption.

- Subject to the budget constraint

$$dw = w\mu_w dt + w\sigma_w dB_t + \int_E w\delta_w(z)m(dt, dz)$$

where

$$\begin{aligned} w\mu_w &= w[\phi_1(\mu_y - r) + \phi_2(\mu_F - r) + r] - c \\ w\sigma_w &= w[\phi_1\sigma_y + \phi_2\sigma_F] \\ w\delta_w(z) &= w[\phi_1\delta_y(z) + \phi_2\delta_F(z)] \end{aligned}$$

$w(t)$ is the her wealth at time t,

ϕ_1 is the proportion of her wealth invested in the physical production,

ϕ_2 is the proportion of her wealth invested in the discount bond.

The Equilibrium Short Rate

- Define: $J(w(t), s(t), x(t)) = \sup\ E_t \left[\int_t^\infty e^{-\rho(\tau-t)} U(c(\tau)) d\tau\right]$,

where the "sup" is over the admissible feedback control of (c, ϕ_1, ϕ_2).

- CIR's Notations: $Var(w^c) = (w\sigma_w)^2$, $Var(x) = \sigma_x^2$, $Cov(w^c, x) = (w\sigma_w)\sigma_x$.

And $\Delta_s f = f(s(t)) - f(s(t-))$ for any function $f(\cdot)$ that depends on $s(t)$.

- Parato Optimality implies no trading at equilibrium, so $\phi_1 = 1$ and $\phi_2 = 0$.

Proposition 1 The *equilibrium short-term interest rate* is given by

$$r = \mu_y^* - \left(-\frac{J_{ww}}{J_w}\right)\frac{Var(w^c)}{w} - \left(-\frac{J_{wx}}{J_x}\right)\frac{Cov(w^c, x)}{w} - \int_E \left(-\frac{\Delta_s J_w}{J_w}\right)\frac{\Delta_s w}{w}\gamma_m(dz)$$

where

$$\mu_y^* = \mu_y + \int_E \delta_y(z)\gamma_m(dz), \quad \text{and} \quad \gamma_m(dz) = h(z, x(t-))\mathbf{I}\{s(t-) = i\}\epsilon_z(dz).$$

Remark Usually(for example, under log utility), $\left(-\frac{\Delta_s J_w}{J_w}\right)\frac{\Delta_s w}{w} > 0$. So, the impact of a systematic regime-switching risk is to lower the equilibrium short-term interest rate, as that of jump risk in Ahn and Thompson (1988).

The Equilibrium Bond Returns

Proposition 2 Let $\mu_F^* dt = E_{t-}\left(\frac{dF(t)}{F(t-)}\right)$ be the instantaneous expected rate of return of the discount bond. At equilibrium,

$$\mu_F^* - r = \left[\left(-\frac{J_{ww}}{J_w}\right) Var(w) + \left(-\frac{J_{wx}}{J_w}\right) Cov(w,x)\right] \frac{F_w}{F} + \left[\left(-\frac{J_{ww}}{J_w}\right) Cov(w,x) + \left(-\frac{J_{wx}}{J_w}\right) Var(x)\right] \frac{F_x}{F} + \int_E \left(-\frac{\Delta_s J_w}{J_w}\right) \frac{\Delta_s F}{F} \gamma_m(dz)$$

Remark 1 Rewrite the above equation as

$$\mu_F^* - r = -Cov\left(\frac{dJ_w^c}{J_w}, \frac{dF^c}{F}\right) - \int_E \frac{\Delta_s J_w}{J_w} \frac{\Delta_s F}{F} \gamma_m(dz)$$

where

$$Cov\left(\frac{dJ_w^c}{J_w}, \frac{dF^c}{F}\right) = \left(\frac{\sigma_{J_w}}{J_w}\right)\left(\frac{\sigma_F F}{F}\right) = \left(\frac{J_{ww} w\sigma_w + J_{wx}\sigma_x}{J_w}\right)\left(\frac{(w\sigma_w)F_w + \sigma_x F_x}{F}\right)$$

Regime-Switching Risk Premium

Remark 2 When $J_{wx} = 0$ and $F_w = 0$ (for example, under log utility),

$$\mu_F^* - r = \left(-\frac{J_{ww}}{J_w}\right) w\sigma_w \frac{\sigma_x F_x}{F} + \int_E \left(-\frac{\Delta_s J_w}{J_w}\right) \frac{\Delta_s F}{F} \gamma_m(dz)$$

• The first term is the instantaneous *diffusion risk premium.*

—$\frac{\sigma_x F_x}{F}$ is the volatility of the bond return due to diffusions in $x(t)$.

—$\left(-\frac{J_{ww}}{J_w}\right) w\sigma_w$ measures the extra rate of return per unit of such volatility and is referred as the market price of risk.

• The second term is analogously defined as the instantaneous *regime-switching risk premium.*

—$\frac{\Delta_s F}{F}$ is the percentage change in bond price due to regime shifts

—$\left(-\frac{\Delta_s J_w}{J_w}\right)$ measures the excess bond return per unit of such changes.

Simplification under Log-Utility for Default-free Bond

- Under Log-Utility, $J_{wx} = 0$, $J_w = \frac{1}{\rho w}$, and $J_{ww} = -\frac{1}{\rho w^2}$.
- "Default-free" implies $F_w = 0$, $F_{ww} = 0$ and $F_{wx} = 0$.

Proposition 3 Under $U(c) = log(c)$ as CIR. The equilibrium short rate is

$$r = \mu_y - \sigma_y^2 + \int_E \lambda_s(z)\gamma_m(dz)$$

The price of a default-free pure discount bond $F(t, x(t), s(t), T)$ becomes,

$$F_t + (\mu_x - \sigma_y \sigma_x)F_x + \frac{1}{2}\sigma_x^2 F_{xx} + \int_E \Delta_s F(1 - \lambda_s(z))\gamma_m(dz) = rF$$

for each $s \in \{1, 2, \cdots, N\}$, with the boundary condition: $F(T, x, s, T) = 1$, and where $\Delta_s F = F(t, x(t), s(t-) + \zeta(z), T) - F(t-, x(t-), s(t-), T)$, and $\lambda_s(z) = \frac{\delta_y(z)}{1+\delta_y(z)}$. Also, $F(t, x(t), s(t), T) = E^Q\Big[\exp\big(-\int_t^T r(s)ds\big)\Big]$.

A Term Structure with R-S Risk for Estimation

Further assume: $\mu_x = a_0(s) + a_1(s)x, \sigma_x = \sqrt{\sigma(s)x}, h(z, x(t-)) = e^{\eta_s(z)}, \sigma_y = \theta_x(s)\sqrt{\sigma(s)x}, \mu_y = x + \theta_x^2(s)\sigma(s)x - \int_E \lambda_s(z)\gamma_m(dz)$, and $\lambda_s(z) = 1 - e^{\theta_s(z)}$. Then, $r = x$. That is,

$$dr(t) = (a_0(s) + a_1(s))rdt + \sqrt{\sigma(s)r(t)}dB_t.$$

Proposition 4 With the above assumptions, $F(t,\tau) = e^{A(\tau,s_t)+B(\tau,s_t)r_t}$: the approximate price at time t of a default-free pure discount bond with maturity τ; $R(t,\tau) = -\frac{A(\tau,s_t)}{\tau} - \frac{B(\tau,s_t)r_t}{\tau}$: the τ-period interest rate, where $A(\tau,s)$ and $B(\tau,s)$ satisfy

$$\begin{aligned} -\frac{\partial B(\tau,s)}{\partial\tau} &+ [a_1(s) - \theta_x(s)\sigma(s)]B(\tau,s) + \frac{1}{2}\sigma(s)B^2(\tau,s) \\ &+ \int_E \left(e^{\Delta_s A}\Delta_s B\right) e^{\eta_s(z)+\theta_s(z)}\mathbf{1}(s=i)\epsilon_z(dz) = 1 \end{aligned} \tag{1}$$

and

$$-\frac{\partial A(\tau,s)}{\partial\tau} + a_0(s)B(\tau,s) + \int_E \left(e^{\Delta_s A} - 1\right) e^{\eta_s(z)+\theta_s(z)}\mathbf{1}(s=i)\epsilon_z(dz) = 0 \tag{2}$$

with boundary conditions $A(0,s) = 0$ and $B(0,s) = 0$, and $s = 1, 2$.

Empirical Results

Data Monthly interest rates from June 1964 to November 2000 obtained from CRSP. Eight series of interest rates with maturities 1m, 3m, 6m, 1-5 yrs. (6m, 5yr) are chosen to fit Models 1,2,3 for statistical analysis.
All eight series are used to estimated the implied regimes.

Model 1: CIR has four parameters $(a_0, a_1, \sigma, \theta_x)$.
Model 2: CIR with two regimes but no R-S risk, has ten parameters $(a_0(1), a_1(1), \sigma(1), \theta_x(1), a_0(2), a_1(2), \sigma(2), \theta_x(2), \eta_s(1,2), \eta_s(2,1))$.
Model 3: CIR with two regimes and R-S risk, has twelve parameters $(a_0(1), a_1(1), \sigma(1), \theta_x(1), a_0(2), a_1(2), \sigma(2), \theta_x(2), \theta_s(1), \theta_s(2), \eta_s(1,2), \eta_s(2,1))$.

Econometric Methodology *Efficient Method of Moments* (EMM):
Bansal, Gallant and Tauchen (1995) and Gallant and Tauchen (1996, 2001).
Application to term structure under regime shifts: Bansal and Zhou (2002).

Efficient Method of Moments

Step 1: Projection Using quasi maximum likelihood to project the observed data (6m,5yr) to an auxilary model close to the true data generating process.
- Gallant and Tauchen (2001) suggests a SNP model based on Hermite polynomial expansion as a convenient general purpose auxiliary model.
- The dimension of this auxiliary model is selected by minimizing BIC.
- The score function are used as moment conditions to compute a chi-square criterion function.

Step 2: Simulation Simulate (6m,5yr) according to the wanted stationary model to evaluate the expected value of the score and compute a chi-squared criterion function.

Step 3: Optimization A nonlinear optimizer is used to find the parameter setting that minimizes the criterion.

Advantage: If the auxiliary model is a close approximation of the true one, then EMM is asymptotic efficient, close to the efficiency of ML.

Conclusions and Future Works

- Affine regime-switching jump diffusion term structure model
- Regime-switching risk on interest rate derivatives.
- Optimal portfolio choice under regime-switching risk.
- Monetary policy regimes.
- Structural relation between business cycles and the yield curve.

- Paper is available at `http://mendota.umkc.edu/paper-term.html`