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Risk Management and Model Specifications Issues in Finance

A General Equilibrium Model
of the Term Structure of Interest Rates
under Regime-switching Risk

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Introduction

- The aggregate economy is characterized by periodic shifts between distinct regimes of the business cycle
- Markov regime-switching models of the short-term interest rate
- Dynamic term structure models under regime shifts:
  Naik and Lee (1997), and Bansal and Zhou (2002).
- Dai and Singleton (2003): the risk of regime shifts is not priced.
Objectives
- Develop a dynamic term structure model under the systematic risk of regime shifts in a general equilibrium setting similar to that in CIR.
- How do regime shifts affect bond returns? How important is the regime-switching risk premium?

Main results
- We show that the regime-switching risk can be priced in a similar way as in the case of jump risk (e.g. Ahn and Thompson, 1988).
- We show that regime switching introduces a new source of time-variation in bond risk premiums, which is associated with the systematic risk of periodic shifts in bond prices due to regime changes.
- Closed-form approximate solution for the entire yield curve is obtained for an affine model.
- Empirical evidence suggests that the market price of the regime-shift risk is not only statistically significant, but also economically important.
The Equilibrium Model

The Underlying Economy: Single good and a large number of infinitely lived and identical consumers similarly to that in CIR (1985a,b).

Modeling Regime Switching:


Mark Space: $E = \{z = (i, j) : i \in \{1,\ldots,N\}, j \in \{1,2,\ldots,N\}, i \neq j\}$.

Marked point process: for $A \in E$, $m(t, A)$ counts the cumulative number of regime shifts that belong to $A$ during $(0, t]$ with stochastic intensity kernel:

$$\gamma_m(dt, dz) = h(z, x(t\!-\!)\mathbf{I}\{s(t\!-\!) = i\}\epsilon_z(dz)dt,$$

$h(z, x(t\!-\!))$: the regime-shift intensity at $z = (i, j)$,

$\epsilon_z(A)$: the Dirac measure at point $z$, ($\epsilon_z(A) = 1$ if $z \in A$ and 0, otherwise).

- $\gamma_m(dt, dz)$ is the conditional probability of a shift from regime $i$ to regime $j$ during $[t, t + dt)$ given $x(t\!-\!)$ and $s(t\!-\!) = i$. 

Two State Variables: \((s,x)\)

- The Regime: \(s(t)\)

\[
ds(t) = \int_E \zeta(z)m(dt, dz)
\]

where \(\zeta(z) = \zeta((i,j)) = j - i\) with the compensator \(\gamma_s(t)dt = \int_E \zeta(z)\gamma_m(dt, dz)\).

For example, if there is a regime shift from \(i\) to \(j\) occurred at time \(t\), the above equation implies \(s_t = (j - i) + s_{t^-} = j\).

- The other usual state variable: \((\mu_x \text{ and } \sigma_x \text{ depend on } x(t), s(t))\)

\[
dx(t) = \mu_x dt + \sigma_x dB_t
\]

Investment Opportunities

- Production:

\[
dy = y\mu_y dt + y\sigma_y dB_t + \int_E y\delta_y(z)m(dt, dz)
\]

where \(\delta_y(z)\) is the percentage change in \(y\) due to a regime shift,

- Competitive market for a default-free pure discount bond with the price:

\[
dF = F\mu_F dt + F\sigma_F dB_t + \int_E F\delta_F(z)m(dt, dz)
\]

- Competitive market for (local) risk-free lending and borrowing at \(r(t)\).
1.3 The Consumer’s Objective Function

- Maximize consumer’s lifetime expected utility

\[ E_0 \left[ \int_0^\infty e^{-\rho t} U(c(t)) dt \right] \]

where \( c(t) \) is the flow of consumption.

- Subject to the budget constraint

\[ dw = w\mu_w dt + w\sigma_w dB_t + \int_E w\delta_w(z) m(dt, dz) \]

where

\[ w\mu_w = w[\phi_1(\mu_y - r) + \phi_2(\mu_F - r) + r] - c \]
\[ w\sigma_w = w[\phi_1\sigma_y + \phi_2\sigma_F] \]
\[ w\delta_w(z) = w[\phi_1\delta_y(z) + \phi_2\delta_F(z)] \]

\( w(t) \) is the her wealth at time \( t \),
\( \phi_1 \) is the proportion of her wealth invested in the physical production,
\( \phi_2 \) is the proportion of her wealth invested in the discount bond.
The Equilibrium Short Rate

- Define: $J(w(t), s(t), x(t)) = \sup E_t \left[ \int_t^\infty e^{-\rho(\tau-t)} U(c(\tau))d\tau \right]$, where the “sup” is over the admissible feedback control of $(c, \phi_1, \phi_2)$.

- CIR’s Notations: $\text{Var}(w_c) = (w\sigma_w)^2$, $\text{Var}(x) = \sigma_x^2$, $\text{Cov}(w_c, x) = (w\sigma_w)\sigma_x$. And $\Delta_s f = f(s(t)) - f(s(t-))$ for any function $f(\cdot)$ that depends on $s(t)$.

- Parato Optimality implies no trading at equilibrium, so $\phi_1 = 1$ and $\phi_2 = 0$.

**Proposition 1** The equilibrium short-term interest rate is given by

$$r = \mu_y^* - \left( \frac{J_{ww}}{J_w} \right) \frac{\text{Var}(w_c)}{w} - \left( \frac{J_{wx}}{J_x} \right) \frac{\text{Cov}(w_c, x)}{w} - \int_E \left( \frac{\Delta_s J_w}{J_w} \right) \frac{\Delta_s w}{w} \gamma_m(dz)$$

where

$$\mu_y^* = \mu_y + \int_E \delta_y(z) \gamma_m(dz), \quad \text{and} \quad \gamma_m(dz) = h(z, x(t-))I\{s(t-) = i\} \epsilon_z(dz).$$

**Remark** Usually (for example, under log utility), $\left( \frac{\Delta_s J_w}{J_w} \right) \frac{\Delta_s w}{w} > 0$. So, the impact of a systematic regime-switching risk is to lower the equilibrium short-term interest rate, as that of jump risk in Ahn and Thompson (1988).
The Equilibrium Bond Returns

**Proposition 2** Let $\mu^*_F dt = E_{t-} \left( \frac{dF(t)}{F(t-)} \right)$ be the instantaneous expected rate of return of the discount bond. At equilibrium,

$$\mu^*_F - r = \left[ \left( -\frac{J_{ww}}{J_w} \right) Var(w) + \left( -\frac{J_{wx}}{J_w} \right) Cov(w, x) \right] \frac{F_w}{F}$$

$$+ \left[ \left( -\frac{J_{ww}}{J_w} \right) Cov(w, x) + \left( -\frac{J_{wx}}{J_w} \right) Var(x) \right] \frac{F_x}{F}$$

$$+ \int_E \left( -\frac{\Delta_s J_w}{J_w} \right) \frac{\Delta_s F}{F} \gamma_m(dz)$$

**Remark 1** Rewrite the above equation as

$$\mu^*_F - r = -Cov \left( \frac{dJ^c_w}{J_w}, \frac{dF^c}{F} \right) - \int_E \frac{\Delta_s J_w}{J_w} \frac{\Delta_s F}{F} \gamma_m(dz)$$

where

$$Cov \left( \frac{dJ^c_w}{J_w}, \frac{dF^c}{F} \right) = \left( \frac{\sigma J_w}{J_w} \right) \left( \frac{\sigma F}{F} \right) = \left( \frac{J_{ww}w\sigma_w + J_{wx}\sigma_x}{J_w} \right) \left( \frac{(w\sigma_w)F_w + \sigma_x F_x}{F} \right)$$
Regime-Switching Risk Premium

Remark 2  When \( J_{wx} = 0 \) and \( F_w = 0 \) (for example, under log utility),
\[
\mu_F^* - r = \left( -\frac{J_{ww}}{J_w} \right) w\sigma_w \frac{\sigma_x F_x}{F} + \int_E \left( -\frac{\Delta_s J_w}{J_w} \right) \frac{\Delta_s F}{F} \gamma_m(dz)
\]

- The first term is the instantaneous diffusion risk premium.
  \( \frac{\sigma_x F_x}{F} \) is the volatility of the bond return due to diffusions in \( x(t) \).
  \( \left( -\frac{J_{ww}}{J_w} \right) w\sigma_w \) measures the extra rate of return per unit of such volatility and is referred as the market price of risk.

- The second term is analogously defined as the instantaneous regime-switching risk premium.
  \( \frac{\Delta_s F}{F} \) is the percentage change in bond price due to regime shifts
  \( \left( -\frac{\Delta_s J_w}{J_w} \right) \) measures the excess bond return per unit of such changes.
Simplification under Log-Utility for Default-free Bond

- Under Log-Utility, \( J_{wx} = 0, J_w = \frac{1}{\rho_w}, \) and \( J_{ww} = -\frac{1}{\rho_w}. \)
- “Default-free” implies \( F_w = 0, F_{ww} = 0 \) and \( F_{wx} = 0. \)

Proposition 3 Under \( U(c) = \log(c) \) as CIR. The equilibrium short rate is

\[
r = \mu_y - \sigma_y^2 + \int_E \lambda_s(z)\gamma_m(dz)
\]

The price of a default-free pure discount bond \( F(t, x(t), s(t), T) \) becomes,

\[
F_t + (\mu_x - \sigma_y \sigma_x) F_x + \frac{1}{2} \sigma_x^2 F_{xx} + \int_E \Delta_s F(1 - \lambda_s(z))\gamma_m(dz) = rF
\]

for each \( s \in \{1, 2, \cdots, N\} \), with the boundary condition: \( F(T, x, s, T) = 1 \), and

\[
\Delta_s F = F(t, x(t), s(t-), \zeta(z, T) - F(t-, x(t-), s(t-), T), \text{and}
\lambda_s(z) = \frac{\delta_y(z)}{1 + \delta_y(z)}. \quad \text{Also, } F(t, x(t), s(t), T) = E^Q \left[ \exp \left( - \int_t^T r(s) ds \right) \right].
\]
A Term Structure with R-S Risk for Estimation

Further assume: \( \mu_x = a_0(s) + a_1(s)x, \sigma_x = \sqrt{\sigma(s)x}, h(z, x(t^-)) = e^{\eta_s(z)}, \sigma_y = \theta_x(s)\sqrt{\sigma(s)x}, \mu_y = x + \theta_x^2(s)\sigma(s)x - \int_E \lambda_s(z)\gamma_m(dz), \) and \( \lambda_s(z) = 1 - e^{\theta_s(z)}. \) Then, \( r = x. \) That is,

\[
dr(t) = (a_0(s) + a_1(s))rdt + \sqrt{\sigma(s)r(t)}dB_t.
\]

**Proposition 4**  With the above assumptions, \( F(t, \tau) = e^{A(\tau, s_t) + B(\tau, s_t)r_t}: \) the approximate price at time \( t \) of a default-free pure discount bond with maturity \( \tau; \)

\[
R(t, \tau) = -\frac{A(\tau, s_t)}{\tau} - \frac{B(\tau, s_t)r_t}{\tau}: \) the \( \tau \)-period interest rate, where \( A(\tau, s) \) and \( B(\tau, s) \) satisfy

\[
-\frac{\partial B(\tau, s)}{\partial \tau} + [a_1(s) - \theta_x(s)\sigma(s)]B(\tau, s) + \frac{1}{2}\sigma(s)B^2(\tau, s)
\]

\[
+ \int_E \left( e^{\Delta_s A}\Delta_s B \right) e^{\eta_s(z) + \theta_s(z)}1(s = i)\epsilon_z(dz) = 1
\]

(1)

and

\[
-\frac{\partial A(\tau, s)}{\partial \tau} + a_0(s)B(\tau, s) + \int_E \left( e^{\Delta_s A} - 1 \right) e^{\eta_s(z) + \theta_s(z)}1(s = i)\epsilon_z(dz) = 0
\]

(2)

with boundary conditions \( A(0, s) = 0 \) and \( B(0, s) = 0, \) and \( s = 1, 2. \)
Empirical Results

Data  Monthly interest rates from June 1964 to November 2000 obtained from CRSP. Eight series of interest rates with maturities 1m, 3m, 6m, 1-5 yrs. (6m, 5yr) are chosen to fit Models 1,2,3 for statistical analysis. All eight series are used to estimated the implied regimes.

Model 1: CIR has four parameters $(a_0, a_1, \sigma, \theta_x)$.
Model 2: CIR with two regimes but no R-S risk, has ten parameters $(a_0(1), a_1(1), \sigma(1), \theta_x(1), a_0(2), a_1(2), \sigma(2), \theta_x(2), \eta_s(1,2), \eta_s(2,1))$.
Model 3: CIR with two regimes and R-S risk, has twelve parameters $(a_0(1), a_1(1), \sigma(1), \theta_x(1), a_0(2), a_1(2), \sigma(2), \theta_x(2), \theta_s(1), \theta_s(2), \eta_s(1,2), \eta_s(2,1))$.

Efficient Method of Moments

Step 1: Projection Using quasi maximum likelihood to project the observed data (6m,5yr) to an auxiliary model close to the true data generating process.
- Gallant and Tauchen (2001) suggests a SNP model based on Hermite polynomial expansion as a convenient general purpose auxiliary model.
- The dimension of this auxiliary model is selected by minimizing BIC.
- The score function are used as moment conditions to compute a chi-square criterion function.

Step 2: Simulation Simulate (6m,5yr) according to the wanted stationary model to evaluate the expected value of the score and compute a chi-squared criterion function.

Step 3: Optimization A nonlinear optimizer is used to find the parameter setting that minimizes the criterion.

Advantage: If the auxiliary model is a close approximation of the true one, then EMM is asymptotic efficient, close to the efficiency of ML.
Conclusions and Future Works

- Affine regime-switching jump diffusion term structure model
- Regime-switching risk on interest rate derivatives.
- Optimal portfolio choice under regime-switching risk.
- Monetary policy regimes.
- Structural relation between business cycles and the yield curve.

- Paper is available at http://mendota.umkc.edu/paper-term.html