1. Basics of Projective Geometry

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IMA Tutorial on Geometric Design:
Geometries for CAGD
Minneapolis, April 19, 2001

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1.1. The real projective plane

Origin of projective geometry: development of rules of perspective drawing; parallel lines (if not parallel to image plane) are mapped to intersecting lines.

→ view parallelity as form of intersection

→ projective extension of Euclidean space.

First: projective extension of Euclidean plane

Add \textit{ideal point} or \textit{point at infinity} to each line $L$, such that parallel lines share the same ideal point.

Two lines are parallel $\iff$ intersect at infinity

\textit{ideal line} $\omega$: consists of all ideal points
Homogeneous coordinates

Embed Euclidean plane $E^2$ (Cartesian coordinates $x, y$) in $\mathbb{R}^3$ (coordinates $x_0, x_1, x_2$) via

$$\mathbf{x} = (x, y) \mapsto \mathbf{X} = (1, x, y).$$
1-dim. subspace spanned by $x = (1, x, y)$:

$$x \mathbb{R} = \{(\lambda, \lambda x, \lambda y), \lambda \in \mathbb{R}\}.$$

Any triple $(\lambda, \lambda x, \lambda y), \lambda \neq 0$ defines the subspace and the corresponding point in $E^2$: **homogeneous Cartesian coordinates** of point $x = x \mathbb{R}$ (write also $(x_0 : x_1 : x_2)$).

Conversion to inhomogeneous Cartesian coordinates:

$$x = \frac{x_1}{x_0}, \quad y = \frac{x_2}{x_0}.$$ 

Ideal point of line $L \subset E^2$ parallel to $(l_1, l_2)$ has homogeneous coordinates

$$(0, l_1, l_2).$$

Point $(x_0 : x_1 : x_2)$ is ideal $\iff x_0 = 0$

One-to-one correspondence between 1-dimensional subspaces of $\mathbb{R}^3$ and points of the extended plane, called **real projective plane** $P^2$. 
Lines

Lines of $P^2$: represented by 2-dim. subspaces of $\mathbb{R}^3$.

Parametric representation of line $L$ spanned by points $a \mathbb{R}, b \mathbb{R}$:

$$x \mathbb{R} = (\lambda_0 a + \lambda_1 b) \mathbb{R}, \quad \lambda_0, \lambda_1 \in \mathbb{R}.$$  

This 2-dim. subspace may also be written as

$$u_0 x_0 + u_1 x_1 + u_2 x_2 = 0. \quad (*)$$

$(u_0, u_1, u_2) = u$: **homogeneous line coordinates** 
(write $L = \mathbb{R} u$)

$(*)$ with canonical scalar product:

$$u \cdot x = 0.$$

*Incidence relation* of line $\mathbb{R} u$ and point $x \mathbb{R}$.

$\implies$ Connecting line $L = \mathbb{R} u$ of points $a \mathbb{R}$ and $b \mathbb{R}$:

$$u = a \times b.$$
Duality

Incidence condition is symmetric: remains unchanged if point and line coordinates are interchanged. ⇝

Def.: 'Line' and 'point' are dual to each other in $P^2$. If in a statement about points and lines and their incidences every occurrence is replaced by its dual, this new statement is called the dual of the original one.

**Principle of Duality:** If a true statement about geometric objects in $P^2$ employs only 'points', 'lines' and 'incidence', its dual is also true, and vice versa.
Examples:

- range of points dual to pencil of lines
- connecting line dual to point of intersection
- Point and incident line (line element): self-dual figure.

Left: Pappos’ theorem. Right: its dual
1.2. Projective $n$-space

Extend Euclidean $n$-space $E^n$ analogously to extension of plane by adding ideal points, one for each class of parallel lines.

Homogeneous coordinates $(x_0 : x_1 : \ldots : x_n)$.

For proper points, inhomogeneous coordinates are recovered as

$$\left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \ldots, \frac{x_n}{x_0}\right).$$

Ideal point of line $L \subset E^n$, parallel to $(l_1, \ldots, l_n)$ has homogeneous coordinates

$$(0 : l_1 : \ldots : l_n).$$

Ideal points lie in *ideal hyperplane* $\omega : x_0 = 0$.

One-to-one correspondence between *1-dimensional subspaces* of $\mathbb{R}^{n+1}$ and *points* of the extended space $P^n$, called *$n$-dimensional real projective space*. 
Projective subspaces

$(k + 1)$-dim. linear subspaces of $\mathbb{R}^{n+1}$ define $k$-dim. projective subspaces of $P^n$:

\[ k = -1: \text{ empty subspace} \]
\[ k = 0: \text{ point} \]
\[ k = 1: \text{ line} \]
\[ k = 2: \text{ plane} \]
\[ k = n-1: \text{ hyperplane.} \]

Def.: points $P_0 = p_0 \mathbb{R}, \ldots, P_k = p_k \mathbb{R}$ are projectively independent $\iff$ vectors $p_0, \ldots, p_k$ are linearly independent.

Indep. points $P_0 = p_0 \mathbb{R}, \ldots, P_k = p_k \mathbb{R}$ span proj. $k$-space $P_0 \vee \ldots \vee P_k$; parameterization with $k+1$ homogeneous parameters $\lambda_0, \ldots, \lambda_k \in \mathbb{R}$:

\[ x \mathbb{R} = (\lambda_0 p_0 + \ldots + \lambda_k p_k) \mathbb{R}. \]

Represent hyperplane by one linear equation,

\[ u_0 x_0 + \ldots + u_n x_n = 0 \quad \text{or} \quad u \cdot x = 0. \quad (*) \]

$(u_0: \ldots: u_n)$ are homogeneous hyperplane coordinates. $(*)$ is incidence relation of hyperplane $Ru$ and point $x \mathbb{R}$. 
Duality

Projective span $U \vee V$ (intersection $U \cap V$) of projective subspaces $U, V$ is determined by linear hull (intersection, resp.) of corresponding linear subspaces of $\mathbb{R}^{n+1}$.

Def.: If a statement involving $k$-dim. subspaces, proj. span, intersection and inclusion of subspaces in $P^n$ is modified by replacing these items by $n-k-1$-dim. subspaces, intersection, proj. span, and reverse inclusion of subspaces, this new statement is called dual to the original one.

**Principle of Duality:** A statement which fulfills the criteria of the above definition is true if and only if its dual is.

(Proof by orthogonality relation (*))

Duality in $P^3$: Points are dual to planes, lines are dual to lines, a range of points is dual to a pencil of planes,...
1.3. Projective maps

Projective geometry of $P^n$ studies properties invariant under projective maps. These are induced by linear maps $f$ in $\mathbb{R}^{n+1}$,

$$f: x \mapsto x' = A \cdot x,$$

with a regular $(n+1) \times (n+1)$ matrix $A$. $f$ determines a projective map $\phi: P^n \to P^n$ by

$$\phi: x \mapsto (A \cdot x) \mathbb{R}.$$

$\phi$ is one-to-one on the set of $k$-dim. subspaces of $P^n$, for all $k = 0, \ldots, n-1$.

Def.: A set of $n+2$ points of $P^n$ is called a fundamental set if every subset of $n+1$ points is proj. independent.

Theorem: If $P_0, \ldots, P_{n+1}$ and $P'_0, \ldots, P'_{n+1}$ are fundamental sets in $P^n$, there exists a unique projective map $P^n \to P^n$ which maps $P_i$ to $P'_i$ for $i = 0, \ldots, n+1$. 
Sketch of proof: Represent $P_i$ as $b_i \mathbb{R}$ with $b_{n+1} = \sum_{i=0}^{n} b_i$. Same for $P'_i$ and define $f$ by $b_i \mapsto b'_i$, $i = 0, \ldots, n$.

A fundamental set defines a projective coordinate system as follows: Fundamental set $(P_0, \ldots, P_n, E)$ can be written as

$$(b_0 \mathbb{R}, \ldots, b_n \mathbb{R}, e \mathbb{R} = (b_0 + \ldots + b_n) \mathbb{R}).$$

Points $b_i \mathbb{R}$ are called fundamental points, $e \mathbb{R}$ is called unit point. All vectors $p \in \mathbb{R}^{n+1}$ can be expressed as $p = x_0 b_0 + \ldots + x_n b_n$.

$(x_0, \ldots, x_n)$ are called projective coordinates of point $p \mathbb{R}$ with respect to the projective frame $(P_0, \ldots, P_n; E)$. To indicate homogeneity, we write also $(x_0 : \ldots : x_n)$.

Fundamental points and unit point have coordinates

$$(1 : 0 : \ldots : 0), \ldots, (0 : \ldots : 0 : 1), \ (1 : 1 : \ldots : 1).$$

Cartesian coordinate system $\Sigma$ defines proj. coord. system: $P_0$ is origin of $\Sigma$, $E$ is unit point of $\Sigma$, $P_1, \ldots, P_n$ are ideal points of coordinate axes.
Example in $P^2$

Homogeneous Cartesian System $(B_0, B_1, B_2; E)$ is mapped under projective map $\phi$ onto proj. system $(B'_0, B'_1, B'_2; E')$. Image of ideal line $\omega = B_1B_2$ is $B'_1B'_2$. Image of parabola $c : y = x^2$, whose homog. equation is

$$x_1^2 - x_0x_2 = 0,$$

is a conic with the same equation in the image frame $(B'_0, \ldots, E')$. 
1.4. Curves

Curve in $P^n$ given by parameterization in homogeneous coordinates:

$$c(t) = (c_0(t), c_1(t), \ldots, c_n(t)) \neq (0, \ldots, 0) \text{ for all } t \in I.$$ 

Multiplication of $c(t)$ with scalar function $\rho(t)$ ('renormalization') yields parameterization $c^*(t) = \rho(t)c(t)$ of the same curve: view curve in $P^n$ as cone in $\mathbb{R}^{n+1}$.

Advantage of homog. coordinates in CAGD: rational curve has polynomial homog. representation.
Application of renormalization: degree elevation of a rational curve $c(t)$, obtained by multiplication with linear function $at + b \implies$ degree elevation for rational curves not unique.

**Curve tangents and singularities**

First derivative $\dot{c}(t)$ (if $\neq 0$) determines a POINT $c^1(t) = \dot{c}(t)\mathbb{R}$.

Curve is regular at point $c(t)$ if $c^1(t) \neq c(t)$. Then the curve tangent there is the line $c(t) \vee c^1(t)$ (view tangent plane of cone in $\mathbb{R}^{n+1}$). At a singular point, $c(t) = c^1(t)$.

Regular curve in $\mathbb{R}^{n+1}$ representing a curve with singularity in $P^n$. 
Osculating spaces

For sufficiently differentiable curve \( c(t) = c(t) \mathbb{R} \), the linear subspaces

\[ [c(t)], \ [c(t), \dot{c}(t)], \ [c(t), \dot{c}(t), \ddot{c}(t)], \ldots \]

define projective subspaces

\[ c(t), \ c(t) \vee c^1(t), \ c(t) \vee c^1(t) \vee c^2(t), \ldots, \]

where

\[ c(t) = c(t) \mathbb{R}, \ c^1(t) = \dot{c}(t) \mathbb{R}, \ c^2(t) = \ddot{c}(t) \mathbb{R}, \ldots. \]

The projective subspace \( c(t) \vee \ldots \vee c^k(t) \) is called osculating (sub)space of order \( k \) (dimension \( \leq k \)). Sequence of dimensions of osculating spaces characterizes special curve points:

- 0, 1 \hspace{1em} regular point
- 0, 1, ..., \( n \) \hspace{1em} point of main type
- 0, 1, 1 \hspace{1em} inflection point (ordinary if 0,1,1,2)
- 0, 1, 1, 1 \hspace{1em} flat point
- 0, 1, 2, 2 \hspace{1em} point with stationary osculating plane
- 0, 0 \hspace{1em} cusp (ordinary if 0,0,1)
Example

Semicubic parabola $y^2 = x^3$ has homogeneous parameterization $t \mapsto (1,t^2,t^3)$. \(\implies\) ordinary cusp at $t = 0$). Behaviour at infinity? Reparameterization $t = 1/u \ (u \neq 0)$ yields $u \mapsto (1,1/u^2,1/u^3)$. Continuous extension by

$$c(t) = (u^3,u,1), \ u \in \mathbb{R}.$$ 

Point $c(0)$ is ideal point. It is an inflection point because $c^1(u) = (3u^2,1,0)\mathbb{R}$, $c^2(u) = (6u,0,0)\mathbb{R}$ and $\dim(c(0) \vee c^1(0)) = \dim(c(0) \vee c^1(0) \vee c^2(0)) = 1$. Figure shows projectively equivalent curve (also projectively equivalent to $y = x^3$).
Contact order

Def.: Two curves \(c(t)\mathbb{R}, d(u)\mathbb{R}\) are said to have contact of order \(k\) at parameter values \(t_0, u_0\) if after a suitable regular parameter transform \(u = u(t)\) with \(u_0 = u(t_0)\) and re-normalization the derivatives of order \(0, 1, \ldots, k\) agree.

The order of contact of two curves is the maximum \(k\) such that the two curves have contact of order \(k\) there.

This definition is projectively invariant (even invariant with respect to \(C^k\) diffeomorphisms).

Example: Curve (higher parabola)

\[
y = x^n, \quad (n \geq 3),
\]

with inhomogeneous parametrization \((t, t^n)\) has contact of order \(n - 1\) with the \(x\)-axis \((t, 0)\) at the origin \((0, 0)\).
Example (contd.) \((0,0)\) is an inflection point according to its definition, but side change of the tangent only for \(n\) odd. Visualize contact of order \(k\) as limit of \(k+1\) intersection points.

Local behavior at higher inflection points

Osculating circle at vertex (left) has contact of order 3 with ellipse and at a general point (right) has contact of order 2.
Surfaces in $\mathbb{P}^3$

Homogeneous parameterization

$$(u,v) \mapsto s(u,v) = (s_0(u,v), \ldots, s_3(u,v)), \quad (u,v) \in D \subset \mathbb{R}^2.$$  

Projective subspace spanned by $s(u,v) = s(u,v)\mathbb{R}$ and the 'partial derivative points'

$$s_u(u,v) = \frac{\partial}{\partial u}s(u,v)\mathbb{R}, \quad s_v(u,v) = \frac{\partial}{\partial v}s(u,v)\mathbb{R},$$

is called tangent space at $s(u,v)$. If its dimension equals 2 (tangent plane) the surface point is called regular.

Analogously as for curves: contact order between surfaces at a point. Contact order $k$ between curve $c$ and surface $s$: on the surface $s$ there exists a curve $\tilde{c}$ which has contact of order $k$ with $c$.

Example: asymptotic tangents (at parabolic or hyperbolic surface points) have contact of order $\geq 2$ with the surface.
Dual curves in $P^2$

Curve $c(t) = c(t) \mathbb{R}$ described as family of its tangents (dual curve),

$$c^*(t) = \mathbb{R}u(t) = \mathbb{R}(c(t) \times \dot{c}(t)).$$

The iterated dual $c^{**}$ is calculated as

$$c^{**} = (c^* \times \dot{c}^*) \mathbb{R} = (c \times \dot{c} \times \dot{c} \times \dot{c} + c \times \dot{c}) \mathbb{R}$$

$$= \det(c, \dot{c}, \ddot{c})c \mathbb{R} = c \mathbb{R} = c.$$  

The fact that iterated duality gives the original curve again leads to the notion of curve as a self-dual object, which can be described either as family of points or family of lines (i.e., tangents).

Remark: Dual parameterization to a rational parameterization of degree $n$ is rational of degree $\leq 2n - 2$ ($\Rightarrow$ dual Bézier form).
Example

The curve
\[ c(t) = c(t) \mathbb{R} = (1, t^m, t^n) \mathbb{R}, \quad (m < n) \]
has the line representation
\[ c^*(t) = \mathbb{R}c(t) \times \dot{c}(t) = \mathbb{R}((n-m)t^{m+n-1}, -nt^{n-1}, mt^{m-1}). \]
Renormalization yields the equivalent form
\[ c^*(t) = \mathbb{R}((n-m)t^n, -nt^{n-m}, m). \]

\((m, n) = (1, 3)\) gives a curve with an inflection at \( t = 0 \). Dual representation is \((2t^3, -3t^2, 1)\); it has a cusp at \( t = 0 \) (if we view it as point set; see Figure). (\( \implies \) algorithms of J. Hoschek for detection of inflection points at hand of cusps of their duals).

Inflection and cusp are dual to each other. Left: a line intersects in three points. Right: A point is incident with three tangents.
References


