Instantaneous Motions - Applications to Problems in Reverse Engineering and 3D Inspection

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One parameter motions

Trajectory of a point $x$:

$$x_0(t) = u(t) + A(t) \cdot x$$

with $A^T \cdot A \equiv E$.

Velocity vector field of $x$:

$$\dot{x_0}(t) = \dot{u}(t) + \dot{A}(t) \cdot x$$

Representation in the same system, e.g. moving system $\Sigma$:

$$v(x) = A^T \cdot \dot{x_0} = A^T \cdot \dot{u} + A^T \dot{A} \cdot x$$
One parameter motions

\[ \mathbf{v}(\mathbf{x}) = \underbrace{A^T \cdot \dot{\mathbf{u}}}_{\mathbf{c}} + \underbrace{A^T \dot{A}}_{\mathbf{S}} \cdot \mathbf{x} \]

With \( A^T \cdot A \equiv \mathbf{E} \) we have \( A^T \cdot \dot{A} + \dot{A}^T \cdot A \equiv \mathbf{0} \)

\[ \iff \mathbf{S} + \mathbf{S}^T \equiv \mathbf{0} \quad \Rightarrow \mathbf{S} \text{ skew symmetric} \]

with \( \mathbf{c} := \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \) there is \( \mathbf{S} \cdot \mathbf{x} = \mathbf{c} \times \mathbf{x} \)

⇒ velocity vector \( \mathbf{v}(\mathbf{x}) \) of \( \mathbf{x} \):

\[ \mathbf{v}(\mathbf{x}) = \overline{\mathbf{c}} + \mathbf{c} \times \mathbf{x} \]

Linear vector field!
Discussion of the geometry of the velocity vector field

Special case 1: \( c = \bar{0} \)

\[ \mathbf{v}(x) = \bar{c} \quad \text{instantaneous translation} \]

Special case 2: \( \bar{c} = o \)

\[ \mathbf{v}(x) = \mathbf{c} \times x \]

For all points of the line \( A : x = \lambda \mathbf{c} \) there is \( \mathbf{v}(x) = \bar{0} \).

instantaneous rotation about axis \( A \)
Discussion of the geometry of the velocity vector field

Special case 3 (geometrically equivalent to case 2):
\[ \mathbf{v}(\mathbf{x}) = \mathbf{\bar{c}} + \mathbf{c} \times \mathbf{x} \]
mit
\[ \mathbf{c} \cdot \mathbf{\bar{c}} = 0. \]

Instantaneous rotation

Rotation axis \( A \) has direction vector \( \mathbf{c} \) and passes through points \( \mathbf{p} \) with \( \mathbf{\bar{c}} = \mathbf{c} \times \mathbf{p} \).

( \( \mathbf{\bar{c}} \) … moment vector of the axis \( A \))

General case:
Composition of an instantaneous rotation and an instantaneous translation

… instantaneous screw motion
Discussion of the geometry of the velocity vector field

One parameter motions with a **constant** (time independent) velocity vector field:

- **translation** with constant velocity
- **uniform rotation** about an axis
- **uniform screw motion**
Uniform screw motion

Composition of a rotation about an axis $A$ and a proportional translation parallel to $A$.

$$\begin{align*}
x(t) &= x_0 \cos(t) - y_0 \sin(t) \\
y(t) &= x_0 \sin(t) + y_0 \cos(t) \\
z(t) &= z_0 + pt
\end{align*}$$

$p \ldots$ pitch

A curve generates a helical surface.

$p = 0$: pure rotation, generates rotational surface.

'$p = \infty$': pure translation, generates cylindrical surface.
Problem description:

Usually, in CAD/CAM systems are used to design and produce technical objects.

Conversely, reverse engineering deals with the reconstruction of a computer model from an existing object.

Different 3D scanning tools give a large number of data points (with measurement errors) on the surface of the technical object.
Goal:

Automatic detection of simple surfaces in the reconstruction process the CAD model. This is important for a precise CAD representation!

Simple classes of surfaces include:

- planes,
- cylinders & cones of revolution,
- spheres,
- tori,
- general rotational surfaces,
- helical surfaces.
Reconstruction of helical surfaces

Given: data points $d_i$

1. step: estimation of surface normals $n_i$

2. step: calculate an appropriate motion with the velocity vector field $v(x) = \overline{c} + c \times x$.

$$v(d_i) \cdot n_i = \overline{c} \cdot n_i + c \cdot \overline{n}_i \quad \text{with} \quad \overline{n}_i = d_i \times n_i.$$ 

Minimization of $\sum \cos^2 \alpha_i$ is nonlinear in $(c, \overline{c}) =: C$.

instead: minimize $\sum_{i=1}^{N} (\overline{c} \cdot n_i + c \cdot \overline{n}_i)^2 = C^T \cdot N \cdot C$

with side condition $1 = \|c\|^2 = C^T \cdot D \cdot C, \quad D = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$
Generalized eigenvalues

Lagrange: \((\mathbf{N} - \lambda \mathbf{D}) \cdot \mathbf{C} = 0, \quad \mathbf{C}^T \cdot \mathbf{D} \cdot \mathbf{C} = 1\).

\[ \text{det}(\mathbf{N} - \lambda \mathbf{D}) = 0, \quad \text{cubic in } \lambda. \]

For a gen. eigenvalue \(\lambda\) and gen. eigenvector \(\mathbf{C}\) we have

\[ F(\mathbf{C}) = \mathbf{C}^T \cdot \mathbf{N} \cdot \mathbf{C} = \lambda \mathbf{C}^T \cdot \mathbf{D} \cdot \mathbf{C} = \lambda. \]

Therefore, the solution of \(F(\mathbf{C}) = \min\) is the generalized eigenvector \(\mathbf{C}\) to the smallest generalized eigenvalue.

From the solution vector \(\mathbf{C} = (\mathbf{c}, \overline{\mathbf{c}}) \in \mathbb{R}^6\) the axis \(\mathbf{A} = (\mathbf{a}, \overline{\mathbf{a}})\) and the pitch \(p\) of the ‘best fitting’ screw motion is calculated:

\[ \mathbf{a} = \frac{\mathbf{c}}{\|\mathbf{c}\|}, \quad \overline{\mathbf{a}} = \frac{\overline{\mathbf{c}} - \rho \mathbf{c}}{\|\mathbf{c}\|}, \quad p = \frac{\mathbf{c} \cdot \overline{\mathbf{c}}}{\mathbf{c}^2}. \]
Reconstruction process

- Estimation of surface normals in the data points.
- Computation of the one parameter motion whose velocity vector field fits the estimated normals best.
- Reconstruction of the profile curve by moving the data into an appropriate plane.
- Reconstruction of the surface by applying the one parameter motion to the profile curve.
Curve reconstruction from unorganized data points

- Compute minimal spanning tree of the complete graph by Delauny triangulation.
- Apply ‘moving least squares’ algorithm (see next slide).
- Spline curve through selected ordered data points.
Moving least squares

- Choose a point $p$ and its neighborhood (distance $\leq H$) in the ‘minimum spanning tree’.
- Fit parabola (weighted point data).
- Project $p$ onto parabola.
Example: rotational surface

- Data points and estimated surface normal vectors
- Data points and computed axis of rotation
- Data points, rotated into a plane, and approximating curve
- Reconstructed surface of revolution
Example: tubular surface

- Data points and estimated surface normal vectors

- Reconstruction of spine curve with locally approximating tori

- Reconstructed tubular surface
Example: developable surface

- Developable surface
- Disturbed data points of the original surface
- Computation of locally best-fitting cones of revolution by region-growing
- Reconstruction of a $G^1$-surface composed of segments of cones of revolution (right circular cones)
Example: developable surface
3D Inspection

Problem description:
Determine a one parameter motion that matches a point cloud to a surface (or another point cloud) as good as possible.

Motivation:
• Inspection of technical objects / quality checks,
• 3D matching, e.g. joining different 3D laser scanner images to a 3D object.
3D inspection

\( x_i \ldots \) data points

determine distance \( d_i \) to surface and surface normal \( n_i \) in the corresponding surface point.

velocity vector field \( v(x_i) \) of a one parameter motion:

\[
v(x) = \overline{c} + c \times x.
\]

minimize \( \sum (d_i + n_i \cdot (\overline{c} + c \times x_i))^2 \)

quadratic in \( c, \overline{c} \)!
Industrial inspection

CAD-model of a technical object
Industrial inspection

Image of the existing object
Industrial inspection

Cloud of data points from the surface of the existing object
Industrial inspection

Point cloud with transparent surface
Industrial inspection

Input to the matching algorithm
Industrial inspection

Output of the matching algorithm
Industrial inspection

Color coding of deviations
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