MPFA Discretization on Quadrilateral Grids

Ivar Aavatsmark, Geir Terje Eigestad, Runhild Aae Klausen, Jan Martin Nordbotten

Center for Integrated Petroleum Research
University of Bergen

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Outline

- Motivation
- MPFA: The O-method
- Convergence
- Monotonicity
- Preconditioning of the linear solver
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Reservoir Simulation
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Conservation equations

$$\int_{\Omega} \frac{\partial}{\partial t} (\phi S_w \rho_w) \ d\tau + \int_{\partial \Omega} \rho_w \mathbf{v}_w \cdot \mathbf{n} \ d\sigma = \int_{\Omega} Q_w \ d\sigma$$

$$\int_{\Omega} \frac{\partial}{\partial t} (\phi (1 - S_w) \rho_o) \ d\tau + \int_{\partial \Omega} \rho_o \mathbf{v}_o \cdot \mathbf{n} \ d\sigma = \int_{\Omega} Q_o \ d\sigma$$
Reservoir Simulation

Conservation equations

\[
\int_{\Omega} \frac{\partial}{\partial t}(\phi S_w \rho_w) \, d\tau + \int_{\partial \Omega} \rho_w \mathbf{v}_w \cdot \mathbf{n} \, d\sigma = \int_{\Omega} Q_w \, d\sigma
\]

\[
\int_{\Omega} \frac{\partial}{\partial t}(\phi (1 - S_w) \rho_o) \, d\tau + \int_{\partial \Omega} \rho_o \mathbf{v}_o \cdot \mathbf{n} \, d\sigma = \int_{\Omega} Q_o \, d\sigma
\]

\[
\mathbf{v}_w = -\frac{k_{rw}}{\mu_w} \mathbf{K} (\text{grad} \, p_w - \rho_w g \, \text{grad} \, D)
\]

\[
\mathbf{v}_o = -\frac{k_{ro}}{\mu_o} \mathbf{K} (\text{grad} \, p_o - \rho_o g \, \text{grad} \, D)
\]

Darcy’s law
Equation with elliptic character
Equation with elliptic character

Assume for simplicity incompressible flow.
Equation with elliptic character

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Elliptic equation:

\[- \text{div}[(\lambda_o + \lambda_w) \text{grad} p_o - (\lambda_o \rho_o + \lambda_w \rho_w) \text{grad} D - \lambda_w K \text{grad} p_c] = \frac{Q_w}{\rho_w} + \frac{Q_o}{\rho_o}\]
Equation with elliptic character

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Elliptic equation:

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Elliptic equation:

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Compressible flow
⇒ Parabolic equation with small accumulation term
Equation with hyperbolic character
Equation with hyperbolic character

\[ \phi \frac{\partial S_w}{\partial t} + \mathbf{v} \cdot \text{grad} \, f_w + [K(\rho_w - \rho_o)g \text{ grad } D] \cdot \text{grad } h + \text{div}(hK \text{ grad } p_c) \]

\[ = (1 - f_w) \frac{Q_w}{\rho_w} - f_w \frac{Q_o}{\rho_o} \]
Equation with hyperbolic character

$$\phi \frac{\partial S_w}{\partial t} + \mathbf{v} \cdot \text{grad} f_w + [K(\rho_w - \rho_o)g \text{grad } D] \cdot \text{grad } h + \text{div}(hK \text{ grad } p_c)$$

$$= (1 - f_w) \frac{Q_w}{\rho_w} - f_w \frac{Q_o}{\rho_o}$$
Equation with hyperbolic character

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Equation with a strongly nonlinear convection term
Equation properties
Equation properties

Coupled set of partial differential equations
Equation properties

Coupled set of partial differential equations

- Almost elliptic part requires *implicit* discretization
Equation properties

Coupled set of partial differential equations

- Almost elliptic part requires implicit discretization

- Almost hyperbolic part with strong nonlinearity requires locally conservative and monotone discretization
Equation properties

Coupled set of partial differential equations

- Almost elliptic part requires **implicit** discretization

- Almost hyperbolic part with strong nonlinearity requires **locally conservative** and **monotone** discretization

- A monotone discretization may be achieved by upstream weighting of an **explicit flux**.
Medium
Medium

Permeability in lateral grid
Medium

Permeability in lateral grid

Vertical cross section of a grid
Medium
The discretization should be valid for nonorthogonal quadrilaterals which are almost parallelograms.
Medium

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- The discretization should be valid for strongly heterogenous media
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- The discretization should be valid for nonorthogonal quadrilaterals which are almost parallelograms
- The discretization should be valid for strongly heterogenous media
- The discretization should be valid for arbitrary anisotropy of the permeability
Model equation
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$$\text{div } q = f, \quad q = -K \text{ grad } u \quad \text{in } \Omega$$
Model equation

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- \( K \) should be positive definite. It may be strongly discontinuous with a strong eigenvalue ratio and arbitrary principal directions.
Model equation

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- The discretization must be valid for \( u \in H^1(\Omega) \) and \( q \in H(\text{div}, \Omega) \).
**Model equation**

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- The discretization must be valid for \( u \in H^1(\Omega) \) and \( q \in H(\text{div}, \Omega) \).

- The discretization should be locally conservative with explicit flux.

- The discretization must yield convergence for both \( u \) and \( q \).
Matrix properties
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Assume that with homogeneous Dirichlet conditions the discretization yields a system of equations $Au = b$. 
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- \( A^{-1} \) should be monotone, i.e., \( A^{-1} \geq O \) (not required).
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- The method should be $L^2$-stable, i.e., $u^TAu \geq \alpha u^TDu$ with $D = \text{diag}(V_i)$.
- $A^{-1}$ should be monotone, i.e., $A^{-1} \geq O$ (not required).

$$u(x) = \int_{\Omega} g(x, \xi) f(\xi) d\tau_\xi$$ where $g(x, \xi) \geq 0$, $g$ Green's function
Control volume formulation
Control volume formulation

\[ \int_{\partial \Omega_i} q \cdot n \, d\sigma = \int_{\Omega_i} f \, d\tau. \]
Control volume formulation

\[ \int_{\partial \Omega_i} q \cdot n \, d\sigma = \int_{\Omega_i} f \, d\tau. \]

\[ q = \int_S q \cdot n \, d\sigma, \]
One-dimensional flow
One-dimensional flow

\[ \frac{\Delta x_i}{\Delta x_{i+1}} \]

\[
\begin{array}{c|c|c}
\Delta x_i & \Delta x_{i+1} \\
\hline
x_i & x_{i+1/2} & x_{i+1}
\end{array}
\]

\[ \frac{\Delta x_i}{\Delta x_{i+1}} \]
One-dimensional flow

\[ u_i - \bar{u}_{i+1/2} = q \frac{x_{i+1/2} - x_i}{k_i} = q \frac{\Delta x_i}{2k_i} \]

\[ \bar{u}_{i+1/2} - u_{i+1} = q \frac{x_{i+1} - x_{i+1/2}}{k_{i+1}} = q \frac{\Delta x_{i+1}}{2k_{i+1}} \]
One-dimensional flow

\[
\begin{align*}
\Delta x_i & \quad \Delta x_{i+1} \\
x_i & \quad x_{i+1/2} \quad x_{i+1}
\end{align*}
\]

\[
\begin{align*}
\bar{u}_{i+1/2} - u_{i+1} &= q \frac{x_{i+1} - x_{i+1/2}}{k_{i+1}} = q \frac{\Delta x_{i+1}}{2k_{i+1}} \\
\bar{u}_{i+1/2} - u_i &= q \frac{x_{i+1/2} - x_i}{k_i} = q \frac{\Delta x_i}{2k_i} \\
u_i - u_{i+1} &= q_{i+1/2} \frac{1}{2} \left( \frac{\Delta x_i}{k_i} + \frac{\Delta x_{i+1}}{k_{i+1}} \right)
\end{align*}
\]
One-dimensional flow

\[
\Delta x_i \quad \Delta x_{i+1}
\]

\[
x_i \quad x_{i+1/2} \quad x_{i+1}
\]

\[
u_i - \bar{u}_{i+1/2} = q \frac{x_{i+1/2} - x_i}{k_i} = q \frac{\Delta x_i}{2k_i}
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\[
u_i - u_{i+1} = q_{i+1/2} \frac{1}{2} \left( \frac{\Delta x_i}{k_i} + \frac{\Delta x_{i+1}}{k_{i+1}} \right)
\]

Principles:
One-dimensional flow

\[
\Delta x_i, \quad \Delta x_{i+1}
\]

\[
x_i \quad x_{i+1/2} \quad x_{i+1}
\]

\[
u_i - \tilde{u}_{i+1/2} = q \frac{x_{i+1/2} - x_i}{k_i} = q \frac{\Delta x_i}{2k_i}
\]

\[
\tilde{u}_{i+1/2} - u_{i+1} = q \frac{x_{i+1} - x_{i+1/2}}{k_{i+1}} = q \frac{\Delta x_{i+1}}{2k_{i+1}}
\]

\[
u_i - u_{i+1} = q_i + 1/2 \frac{1}{2} \left( \frac{\Delta x_i}{k_i} + \frac{\Delta x_{i+1}}{k_{i+1}} \right)
\]

Principles: continuous potential
One-dimensional flow

\[
\begin{align*}
\Delta x_i & \quad \Delta x_{i+1} \\
\bullet & \quad \bullet
\end{align*}
\]

\[
u_i - \bar{u}_{i+1/2} = q \frac{x_{i+1/2} - x_i}{k_i} = q \frac{\Delta x_i}{2k_i},
\]

\[
\bar{u}_{i+1/2} - u_{i+1} = q \frac{x_{i+1} - x_{i+1/2}}{k_{i+1}} = q \frac{\Delta x_{i+1}}{2k_{i+1}}.
\]

\[
u_i - u_{i+1} = q_{i+1/2} \frac{1}{2} \left( \frac{\Delta x_i}{k_i} + \frac{\Delta x_{i+1}}{k_{i+1}} \right)
\]

Principles: continuous potential and continuous flux.
One-dimensional flow

\[ u_i - \bar{u}_{i+1/2} = q \frac{x_{i+1/2} - x_i}{k_i} = q \frac{\Delta x_i}{2k_i} \]
\[ \bar{u}_{i+1/2} - u_{i+1} = q \frac{x_{i+1} - x_{i+1/2}}{k_{i+1}} = q \frac{\Delta x_{i+1}}{2k_{i+1}} \]
\[ u_i - u_{i+1} = q_{i+1/2} \frac{1}{2} \left( \frac{\Delta x_i}{k_i} + \frac{\Delta x_{i+1}}{k_{i+1}} \right) \]

Principles: continuous potential and continuous flux. The flux is determined by the interaction of linear potentials in two cells.
Two-dimensional flow
Two-dimensional flow

Cells with common corner
Two-dimensional flow

Cells with common corner

Interaction region
Two-dimensional flow

Cells with common corner

Interaction region

Determine the flux through the half-edges from the interaction
Two-dimensional flow

Cells with common corner

Interaction region

Determine the flux through the half-edges from the interaction of linear potentials in the four cells.
Two-dimensional flow

Cells with common corner

Interaction region

Determine the flux through the half-edges from the interaction of linear potentials in the four cells. Require continuous potential at $\bar{x}_i$.
Two-dimensional flow

Cells with common corner

Interaction region

Determine the flux through the half-edges from the interaction of linear potentials in the four cells. Require continuous potential at \( \vec{x}_i \) and continuous flux through the half-edges.
Two-dimensional flow

Determine the flux through the half-edges from the interaction of linear potentials in the four cells. Require continuous potential at \( \bar{x}_i \) and continuous flux through the half-edges. This method is called the O-method.
O-method
O-method

Interaction region
O-method

Linear potential in 4 cells: $4 \cdot 3 = 12$ unknowns

Interaction region
**O-method**

Linear potential in 4 cells: \(4 \cdot 3 = 12\) unknowns

Potential continuity at \(\bar{x}_i\): 4 conditions
O-method

Linear potential in 4 cells: $4 \cdot 3 = 12$ unknowns

Potential continuity at $\bar{x}_i$: 4 conditions

Flux continuity at the edges: 4 conditions
O-method

Linear potential in 4 cells: $4 \cdot 3 = 12$ unknowns

Potential continuity at $\bar{x}_i$: 4 conditions

Flux continuity at the edges: 4 conditions

Potential values at cell centers: 4 conditions
O-method

Linear potential in 4 cells: $4 \cdot 3 = 12$ unknowns

Potential continuity at $\bar{x}_i$: 4 conditions

Flux continuity at the edges: 4 conditions

Potential values at cell centers: 4 conditions

12 equations with 12 unknowns.
Flux expression in each cell
Flux expression in each cell
Flux expression in each cell

\[ \mathbf{q}_1^{(k)} \mathbf{q}_2^{(k)} = -G_k \begin{bmatrix} \bar{u}_1 - u_k \\ \bar{u}_2 - u_k \end{bmatrix}, \]
Flux expression in each cell

\[
\begin{bmatrix}
q_1^{(k)} \\
q_2^{(k)}
\end{bmatrix} = -G_k \begin{bmatrix}
\bar{u}_1 - u_k \\
\bar{u}_2 - u_k
\end{bmatrix}, \quad G_k = \frac{1}{2F_k} \begin{bmatrix}
\Gamma_1 n_1^T K_k \nu_1^{(k)} & \Gamma_1 n_1^T K_k \nu_2^{(k)} \\
\Gamma_2 n_2^T K_k \nu_1^{(k)} & \Gamma_2 n_2^T K_k \nu_2^{(k)}
\end{bmatrix}
\]
Flux equations in an interaction region

Cells with common corner

\[ q_1 = q_1^{(1)} = q_1^{(2)} \]
\[ q_2 = q_2^{(4)} = q_2^{(3)} \]
\[ q_3 = q_3^{(3)} = q_3^{(1)} \]
\[ q_4 = q_4^{(2)} = q_4^{(4)} \]
Flux equations in an interaction region
Flux equations in an interaction region

\[ q_1 = -g^{(1)}_{1,1}(\bar{u}_1 - u_1) - g^{(1)}_{1,2}(\bar{u}_3 - u_1) = g^{(2)}_{1,1}(\bar{u}_1 - u_2) - g^{(2)}_{1,2}(\bar{u}_4 - u_2) \]

\[ q_2 = g^{(4)}_{1,1}(\bar{u}_2 - u_4) + g^{(4)}_{1,2}(\bar{u}_4 - u_4) = -g^{(3)}_{1,1}(\bar{u}_2 - u_3) + g^{(3)}_{1,2}(\bar{u}_3 - u_3) \]

\[ q_3 = -g^{(3)}_{2,1}(\bar{u}_2 - u_3) + g^{(3)}_{2,2}(\bar{u}_3 - u_3) = -g^{(1)}_{2,1}(\bar{u}_1 - u_1) - g^{(1)}_{2,2}(\bar{u}_3 - u_1) \]

\[ q_4 = g^{(2)}_{2,1}(\bar{u}_1 - u_2) - g^{(2)}_{2,2}(\bar{u}_4 - u_2) = g^{(4)}_{2,1}(\bar{u}_2 - u_4) + g^{(4)}_{2,2}(\bar{u}_4 - u_4) \]
Flux equations in an interaction region

\[ q_1 = -g_{1,1}^{(1)}(\tilde{u}_1 - u_1) - g_{1,2}^{(1)}(\tilde{u}_3 - u_1) = g_{1,1}^{(2)}(\tilde{u}_1 - u_2) - g_{1,2}^{(2)}(\tilde{u}_4 - u_2) \]

\[ q_2 = g_{1,1}^{(4)}(\tilde{u}_2 - u_4) + g_{1,2}^{(4)}(\tilde{u}_4 - u_4) = -g_{1,1}^{(3)}(\tilde{u}_2 - u_3) + g_{1,2}^{(3)}(\tilde{u}_3 - u_3) \]

\[ q_3 = -g_{2,1}^{(3)}(\tilde{u}_2 - u_3) + g_{2,2}^{(3)}(\tilde{u}_3 - u_3) = -g_{2,1}^{(1)}(\tilde{u}_1 - u_1) - g_{2,2}^{(1)}(\tilde{u}_3 - u_1) \]

\[ q_4 = g_{2,1}^{(2)}(\tilde{u}_1 - u_2) - g_{2,2}^{(2)}(\tilde{u}_4 - u_2) = g_{2,1}^{(4)}(\tilde{u}_2 - u_4) + g_{2,2}^{(4)}(\tilde{u}_4 - u_4) \]

\[ \tilde{q} = [q_1, q_2, q_3, q_4]^T \quad u = [u_1, u_2, u_3, u_4]^T \quad v = [\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{u}_4]^T \]
Flux equations in an interaction region

\[ q_1 = -g_{1,1}^{(1)}(\tilde{u}_1 - u_1) - g_{1,2}^{(1)}(\tilde{u}_3 - u_1) = g_{1,1}^{(2)}(\tilde{u}_1 - u_2) - g_{1,2}^{(2)}(\tilde{u}_4 - u_2) \]

\[ q_2 = g_{1,1}^{(4)}(\tilde{u}_2 - u_4) + g_{1,2}^{(4)}(\tilde{u}_4 - u_4) = -g_{1,1}^{(3)}(\tilde{u}_2 - u_3) + g_{1,2}^{(3)}(\tilde{u}_3 - u_3) \]

\[ q_3 = -g_{2,1}^{(3)}(\tilde{u}_2 - u_3) + g_{2,2}^{(3)}(\tilde{u}_3 - u_3) = -g_{2,1}^{(1)}(\tilde{u}_1 - u_1) - g_{2,2}^{(1)}(\tilde{u}_3 - u_1) \]

\[ q_4 = g_{2,1}^{(2)}(\tilde{u}_1 - u_2) - g_{2,2}^{(2)}(\tilde{u}_4 - u_2) = g_{2,1}^{(4)}(\tilde{u}_2 - u_4) + g_{2,2}^{(4)}(\tilde{u}_4 - u_4) \]

\[ \tilde{q} = [q_1, q_2, q_3, q_4]^T \quad u = [u_1, u_2, u_3, u_4]^T \quad v = [\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{u}_4]^T \]

\[ \tilde{q} = Cv + Fu, \]
Flux equations in an interaction region

\[ q_1 = -g_{1,1}^{(1)}(\bar{u}_1 - u_1) - g_{1,2}^{(1)}(\bar{u}_3 - u_1) = g_{1,1}^{(2)}(\bar{u}_1 - u_2) - g_{1,2}^{(2)}(\bar{u}_4 - u_2) \]

\[ q_2 = g_{1,1}^{(3)}(\bar{u}_2 - u_4) + g_{1,2}^{(3)}(\bar{u}_4 - u_4) = -g_{1,1}^{(3)}(\bar{u}_2 - u_3) + g_{1,2}^{(3)}(\bar{u}_3 - u_3) \]

\[ q_3 = -g_{2,1}^{(3)}(\bar{u}_2 - u_3) + g_{2,2}^{(3)}(\bar{u}_3 - u_3) = -g_{2,1}^{(1)}(\bar{u}_1 - u_1) - g_{2,2}^{(1)}(\bar{u}_3 - u_1) \]

\[ q_4 = g_{2,1}^{(2)}(\bar{u}_1 - u_2) - g_{2,2}^{(2)}(\bar{u}_4 - u_2) = g_{2,1}^{(4)}(\bar{u}_2 - u_4) + g_{2,2}^{(4)}(\bar{u}_4 - u_4) \]

\[ \tilde{q} = [q_1, q_2, q_3, q_4]^T \quad u = [u_1, u_2, u_3, u_4]^T \quad v = [\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4]^T \]

\[ \tilde{q} = Cv + Fu, \quad Av = Bu \]
Flux equations in an interaction region

\[
q_1 = -g_{1,1}^{(1)}(\bar{u}_1 - u_1) - g_{1,2}^{(1)}(\bar{u}_3 - u_1) = g_{1,1}^{(2)}(\bar{u}_1 - u_2) - g_{1,2}^{(2)}(\bar{u}_4 - u_2)
\]

\[
q_2 = g_{1,1}^{(4)}(\bar{u}_2 - u_4) + g_{1,2}^{(4)}(\bar{u}_4 - u_4) = -g_{1,1}^{(1)}(\bar{u}_2 - u_3) + g_{1,2}^{(3)}(\bar{u}_3 - u_3)
\]

\[
q_3 = -g_{2,1}^{(3)}(\bar{u}_2 - u_3) + g_{2,2}^{(3)}(\bar{u}_3 - u_3) = -g_{2,1}^{(1)}(\bar{u}_1 - u_1) - g_{2,2}^{(1)}(\bar{u}_3 - u_1)
\]

\[
q_4 = g_{2,1}^{(2)}(\bar{u}_1 - u_2) - g_{2,2}^{(2)}(\bar{u}_4 - u_2) = g_{2,1}^{(4)}(\bar{u}_2 - u_4) + g_{2,2}^{(4)}(\bar{u}_4 - u_4)
\]

\[
\tilde{q} = [q_1, q_2, q_3, q_4]^T \quad u = [u_1, u_2, u_3, u_4]^T \quad v = [\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4]^T
\]

\[
\tilde{q} = Cv + Fu, \quad Av = Bu \quad \Rightarrow \quad \tilde{q} = (CA^{-1}B + F)u
\]
Flux expression
Flux expression

\[ q_i = \sum_{j=1}^{4} t_{i,j} u_j \quad \text{where} \quad \sum_{j=1}^{4} t_{i,j} = 0 \]
Flux expression

Flux stencil

Cells with common corner

$q_i = \sum_{j=1}^{4} t_{i,j} u_j$ where $\sum_{j=1}^{4} t_{i,j} = 0$

$q_i = \sum_{j=1}^{6} t_{i,j} u_j$ where $\sum_{j=1}^{6} t_{i,j} = 0$
Flux expression

Cells with common corner

Flux stencil

Multipoint flux approximation (MPFA)

\[
q_i = \sum_{j=1}^{4} t_{i,j} u_j \quad \text{where} \quad \sum_{j=1}^{4} t_{i,j} = 0
\]

\[
q_i = \sum_{j=1}^{6} t_{i,j} u_j \quad \text{where} \quad \sum_{j=1}^{6} t_{i,j} = 0
\]
Stencils
Stencils

\[ \begin{array}{c c}
 4 & 3 \\
 1 & 2 \\
 5 & 6 \\
\end{array} \]

\(i\)-flux stencil

\[ \begin{array}{c c c}
 3 & 2 & 6 \\
 4 & 1 & 5 \\
\end{array} \]

\(j\)-flux stencil
Stencils

$i$-flux stencil

$j$-flux stencil

Cell stencil
Parallelogram cells
Parallelogram cells
Parallelogram cells

\[ G_k = \frac{1}{V_k} \left[ a_1^{(k)} \ a_2^{(k)} \right]^T K_k \left[ a_1^{(k)} \ a_2^{(k)} \right] \]

\[ G_k = \frac{1}{|\det J_k|} J_k^T K_k J_k \]

Congruence transformation
⇒ \( G_k \) positive definite
Parallelogram cells

\[ G_k = \frac{1}{V_k} \left[ \begin{array}{c} a_1^{(k)} \\ a_2^{(k)} \end{array} \right]^T K_k \left[ \begin{array}{c} a_1^{(k)} \\ a_2^{(k)} \end{array} \right] \]

\[ G_k = \frac{1}{|\det J_k|} J_k^T K_k J_k \]

Congruence transformation
⇒ \( G_k \) positive definite

Symmetry and positive definiteness of \( G_k \) is crucial for proving that the coefficient matrix in the system of equations is symmetric and positive definite.
Parallelogram cells
Parallelogram cells
Parallelogram cells

\[ G_k = \frac{1}{\Delta \xi_k \Delta \eta_k} D_k H_k D_k, \]
Parallelogram cells

\[ G_k = \frac{1}{\Delta \xi_k \Delta \eta_k} D_k H_k D_k, \]

\[ H_k = \frac{1}{\det[n_1, n_2]} \begin{bmatrix} n_1^T K_k n_1 & n_1^T K_k n_2 \\ n_2^T K_k n_1 & n_2^T K_k n_2 \end{bmatrix}, \quad D_k = \text{diag}(\Delta \eta_k, \Delta \xi_k). \]
Parallelogram cells

\[ H_k = \frac{1}{\det[n_1, n_2]} \begin{bmatrix} n_1^T K_k n_1 & n_1^T K_k n_2 \\ n_2^T K_k n_1 & n_2^T K_k n_2 \end{bmatrix}, \quad D_k = \text{diag}(\Delta \eta_k, \Delta \xi_k). \]

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\( H_k \) contains anisotropy and grid skewness. \( D_k \) contains cell dimensions.
Parallelogram cells

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\( H_k \) contains anisotropy and grid skrewness. \( D_k \) contains cell dimensions.

If \( n_i^T K n_j = 0, \ i \neq j \), then \( H_k \) and \( G_k \) are diagonal, and the multipoint flux reduces to a two-point flux. Such a grid is called \( K \)-orthogonal.
O-method
O-method

- $L^2$-stability can be proved for the O-method on parallelogram grids.
O-method

- $L^2$-stability can be proved for the O-method on parallelogram grids.

- The O-method extends to three dimensions. The interaction region contains 8 cells. The flux stencil has 18 cells, and the cell stencil has 27 cells.
Results — pressure
Results — pressure
Results — pressure
Results — pressure
Results — saturation
Results — saturation

TPFA

P P I
Results — saturation

TPFA

MPFA
Related methods — U-method
Related methods — U-method

Interaction region
Related methods — U-method

Linear potential in 4 cells: $4 \cdot 3 = 12$ unknowns

Interaction region
Related methods — U-method

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Weak potential continuity at central edge: 1 condition

Interaction region
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Linear potential in 4 cells: \(4 \cdot 3 = 12\) unknowns

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Potential values at cell centers: 4 conditions
Related methods — U-method

Linear potential in 4 cells: \(4 \cdot 3 = 12\) unknowns

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Non-symmetric matrix of coefficients.
Related methods — U-method

Linear potential in 4 cells: $4 \cdot 3 = 12$ unknowns

Weak potential continuity at central edge: 1 condition

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Flux continuity at 3 edges: 3 conditions

Potential values at cell centers: 4 conditions

Non-symmetric matrix of coefficients.
Looser hinging might be advantageous by strong heterogeneity.
Related methods — Edwards & Rogers
Related methods — Edwards & Rogers

Interaction region
Related methods — Edwards & Rogers

Linear potential in 4 cells: \( 4 \cdot 3 = 12 \) unknowns

Interaction region
Related methods — Edwards & Rogers

Linear potential in 4 cells: $4 \cdot 3 = 12$ unknowns

Potential continuity at $\bar{x}_i$: 4 conditions

Interaction region
Related methods — Edwards & Rogers

Interaction region

Linear potential in 4 cells: $4 \cdot 3 = 12$ unknowns

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Interaction region

Linear potential in 4 cells: \( 4 \cdot 3 = 12 \) unknowns

Potential continuity at \( \bar{x}_i \): 4 conditions

Flux continuity at 4 edges: 4 conditions

Potential values at cell centers: 4 conditions
Related methods — Edwards & Rogers

Linear potential in 4 cells: $4 \cdot 3 = 12$ unknowns

Potential continuity at $\bar{x}_i$: 4 conditions

Flux continuity at 4 edges: 4 conditions

Potential values at cell centers: 4 conditions

Does not reduce to five-point stencil for $K$-orthogonal grids.
Expanded mixed finite element method
Expanded mixed finite element method

\[ q = -Kg, \quad g = \text{grad} \ u, \quad \text{div} \ q = f \]
Expanded mixed finite element method

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- Multiply with test functions, \( h, p, v \). Assume parallelogram grid.
Expanded mixed finite element method

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- Apply \( \text{RT}_0 \) functions. \( g \) must be constant on each edge, but is not continuous.
Expanded mixed finite element method

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- Multiply with test functions, \( h, p, v \). Assume parallelogram grid.

- Apply \( \text{RT}_0 \) functions. \( g \) must be constant on each edge, but is not continuous.

- Klausen proved convergence with trapezoidal quadrature: If \( u \in H^2(\Omega) \) and \( q \in (H^2(\Omega))^d \), then

\[
\|u_h - u\|_{L^2(\Omega)} + \|\mathbf{q}_h - \mathbf{q}\|_{H(\text{div}, \Omega)} \leq Ch
\]
Convergence for O-method on parallelogram grids
Convergence for O-method on parallelogram grids

Recall the flux expressions

\[
\begin{bmatrix}
    q_1^{(k)} \\
    q_2^{(k)}
\end{bmatrix} = -G_k \left[ \bar{u}_1 - u_k \right], \quad G_k = \frac{1}{\Delta \xi_k \Delta \eta_k} D_k H_k D_k,
\]

\[
G_k = 1 \Delta \xi_k \Delta \eta_k \quad D_k H_k D_k,
\]
Convergence for O-method on parallelogram grids

Recall the flux expressions

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\begin{bmatrix}
q_1^{(k)} \\
q_2^{(k)}
\end{bmatrix}
= -G_k \begin{bmatrix}
\bar{u}_1 - u_k \\
\bar{u}_2 - u_k
\end{bmatrix}, \quad G_k = \frac{1}{\Delta \xi_k \Delta \eta_k} D_k H_k D_k,
\]

\[
H_k = \frac{1}{\det[n_1, n_2]} \begin{bmatrix}
n_1^T K_k n_1 & n_1^T K_k n_2 \\
n_2^T K_k n_1 & n_2^T K_k n_2
\end{bmatrix}, \quad D_k = \text{diag}(\Delta \eta_k, \Delta \xi_k).
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Convergence for O-method on parallelogram grids

Recall the flux expressions

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q_1^{(k)} \\
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\end{bmatrix} = -G_k \begin{bmatrix}
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\bar{u}_2 - u_k
\end{bmatrix}, \quad G_k = \frac{1}{\Delta \xi_k \Delta \eta_k} D_k H_k D_k,
\]

\[
H_k = \frac{1}{\det[n_1, n_2]} \begin{bmatrix}
K_k n_1 n_1^T & K_k n_1 n_2^T \\
K_k n_1 n_1^T & K_k n_2 n_2^T
\end{bmatrix}, \quad D_k = \text{diag}(\Delta \eta_k, \Delta \xi_k).
\]

Klausen showed that these fluxes can be expressed by the fluxes from the Expanded Mixed Finite Element Method with trapezoidal quadrature.
Relation between EMFEM and MPFA O-method
Relation between EMFEM and MPFA O-method

\[ Q = \frac{1}{2 \det H} \begin{bmatrix}
\alpha & \alpha & \delta & \delta & \beta & -\beta & \beta & -\beta \\
\delta & \delta & \alpha & \alpha & -\beta & \beta & -\beta & \beta \\
-\gamma & \gamma & -\gamma & \gamma & \alpha & \alpha & \delta & \delta \\
\gamma & -\gamma & \gamma & -\gamma & \delta & \delta & \alpha & \alpha
\end{bmatrix} \]
Relation between EMFEM and MPFA O-method

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Q = \frac{1}{2 \text{det } H} \begin{bmatrix}
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\delta & \delta & \alpha & \alpha & -\beta & \beta & -\beta & \beta \\
-\gamma & \gamma & -\gamma & \gamma & \alpha & \alpha & \delta & \delta \\
\gamma & -\gamma & \gamma & -\gamma & \delta & \delta & \alpha & \alpha \\
\end{bmatrix}
\]

\[
\alpha = 2h_{11}h_{22} - h_{12}^2, \quad \beta = h_{11}h_{12} \\
\gamma = h_{12}h_{22}, \quad \delta = h_{12}^2
\]
Relation between EMFEM and MPFA O-method

\[ Q = \frac{1}{2 \det H} \begin{bmatrix} \alpha & \alpha & \delta & \delta & \beta & -\beta & \beta & -\beta \\ \delta & \delta & \alpha & \alpha & -\beta & \beta & -\beta & \beta \\ -\gamma & \gamma & -\gamma & \gamma & \alpha & \alpha & \delta & \delta \\ \gamma & -\gamma & \gamma & -\gamma & \delta & \delta & \alpha & \alpha \end{bmatrix} \]

\[ \alpha = 2h_{11}h_{22} - h_{12}^2, \quad \beta = h_{11}h_{12}, \quad \gamma = h_{12}h_{22}, \quad \delta = h_{12}^2 \]
Convergence
Convergence

Hence, if $u \in H^2(\Omega)$ and $q \in (H^2(\Omega))^d$, then for the O-method on parallelogram grids:

$$\|u_h - u\|_{L^2(\Omega)} + \|q_h - q\|_{H(\text{div},\Omega)} \leq Ch$$
Convergence

Hence, if $u \in H^2(\Omega)$ and $q \in (H^2(\Omega))^d$, then for the O-method on parallelogram grids:

$$\|u_h - u\|_{L^2(\Omega)} + \|q_h - q\|_{H(\text{div},\Omega)} \leq C h$$

The proof can be extended to arbitrary quadrilaterals using the Piola mapping.
Convergence of nonregular solutions
Convergence of nonregular solutions

Potential $u \sim r^\alpha$
Convergence of nonregular solutions

Potential $u \sim r^\alpha$

Error $\|u_h - u\|_{L^2}$

$h - \text{exp} \alpha$
Monotonicity
Monotonicity

Isotropic

$A^{-1} \geq O$
Monotonicity

Isotropic

\[ A^{-1} \geq O \]

Anisotropy ratio 1 : 1000, \( \theta = \pi / 6 \).

\[ A^{-1} \nless O \]
Monotonicity
Monotonicity

\[ Au = b \]

\[ A^{-1} \geq O \]
Monotonicity

\[ Au = b \]
\[ A^{-1} \geq O \]

Homogeneous medium
Uniform grid
Enhanced monotonicity
Enhanced monotonicity

Z-stencil $\sim$ U-stencil
Enhanced monotonicity

Flux continuity on 3 edges

Z-stencil $\sim$ U-stencil
Enhanced monotonicity

Flux continuity on 3 edges

Weak potential continuity at central edge

Z-stencil $\sim$ U-stencil
Enhanced monotonicity

Flux continuity on 3 edges
Weak potential continuity at central edge
Full potential continuity at two edges
Enhanced monotonicity

Flux continuity on 3 edges
Weak potential continuity at central edge
Full potential continuity at two edges

Z-stencil ~ U-stencil

O- and Z-methods may be combined based on proximity to a corner.
Enhanced monotonicity

Flux continuity on 3 edges

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Full potential continuity at two edges

O- and Z-methods may be combined based on proximity to a corner.

Combined O&Z increased the monotone cases by 18% compared to the O-method in a large set of grids in a layered medium.
Enhanced monotonicity

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O- and Z-methods may be combined based on proximity to a corner.

Combined O&Z increased the monotone cases by 18% compared to the O-method in a large set of grids in a layered medium.

Proximity is important for monotonicity.
Iterative solution — Edwards splitting
Iterative solution — Edwards splitting

\[ Au = b \]
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\[ Au = b \]

\[ Bu^{(k+1)} + (A - B)u^{(k)} = b \]
Iterative solution — Edwards splitting

\[ Au = b \]

\[ Bu^{(k+1)} + (A - B)u^{(k)} = b \]

\[ u^{(k+1)} = (I - B^{-1}A)u^{(k)} + B^{-1}b \]
Iterative solution — Edwards splitting

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\[ \rho(I - B^{-1}A) < 1 \]
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\( A \) is the multipoint flux-approximation matrix of coefficients.
Iterative solution — Edwards splitting

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\( A \) is the multipoint flux-approximation matrix of coefficients.  

\( B \) is obtained by removing the off-diagonal elements in \( G_k \), i.e., discretizing as if the grid was \( K \)-ortogonal.
Iterative solution — single-phase flow
Iterative solution — single-phase flow

Spectral radius $\rho$ depends on
Iterative solution — single-phase flow

Spectral radius $\rho$ depends on

- orientation of the principal axes of $K$
Iterative solution — single-phase flow

Spectral radius $\rho$ depends on

- orientation of the principal axes of $K$
- grid skewness
Iterative solution — single-phase flow

Spectral radius $\rho$ depends on

- orientation of the principal axes of $K$
- grid skewness
- anisotropy ratio
Iterative solution — single-phase flow

Spectral radius $\rho$ depends on

- orientation of the principal axes of $K$
- grid skewness
- anisotropy ratio

Spectral radius $\rho$ depends only weakly on
Iterative solution — single-phase flow

Spectral radius $\rho$ depends on

- orientation of the principal axes of $K$
- grid skewness
- anisotropy ratio

Spectral radius $\rho$ depends only weakly on

- grid aspect ratio
Iterative solution — single-phase flow

Spectral radius $\rho$ depends on

- orientation of the principal axes of $K$
- grid skewness
- anisotropy ratio

Spectral radius $\rho$ depends only weakly on

- grid aspect ratio
- heterogeneity
Iterative solution — single-phase flow

Spectral radius $\rho$ depends on
- orientation of the principal axes of $K$
- grid skewness
- anisotropy ratio

Spectral radius $\rho$ depends only weakly on
- grid aspect ratio
- heterogeneity

For moderate skewness angles ($< 20^\circ$) and moderate anisotropy ratio ($< 100$) the spectral radius satisfies $\rho \lesssim 0.6$. This ensures fast convergence in the iteration.
Conclusions

• MPFA methods are well suited for reservoir simulation on non-$K$-orthogonal grids.

• The O-method gives convergence of order $h$ in potential and flow density.

• The O-method is conditionally monotone. The monotonicity range may be enhanced by combination of MPFA methods based on proximity.

• Fast convergence of the linear solver can be achieved by preconditioning with the Edwards splitting.
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